TAYLOR SERIES MANIPULATIONS

THE GENERAL FORM FOR A TAYLOR SERIES OF A SMOOTH FUNCTION $f(x)$ ABOUT $x = x_0$ IS

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n . \qquad (*)$$

WHERE $f^{(n)}(x_0)$ DENOTES n^{th}-derivative OF $f(x)$ AT $x = x_0$ (WITH $f^{(0)}(x_0) \equiv f(x_0)$)

THE SPECIAL CASE OF $(*)$ WHERE $x_0 = 0$ IS CALLED A Mac LAURIN SERIES. THERE ARE FOUR BASIC FUNCTIONS THAT YOU SHOULD KNOW THE MacLAURIN SERIES FOR:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n} (-1)^n}{(2n)!} \qquad (3)$$

THESE THREE TAYLOR SERIES CONVERGE FOR ALL x, i.e. $R = \infty$.

FINALLY, RECALL THE GEOMETRIC SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n , \quad \text{VALID FOR } |x| < 1 , \quad (4).$$

WE CAN DERIVE MANY OTHER SERIES EXPANSIONS, SUCH AS $\log(1-x)$ AND $\arctan x$ BY MANIPULATING (integrating) the geometric series, using substitution ETC..

REMARK THE DERIVATION OF (1) - (3) IS SIMPLE. FOR (2), WE LET $f(x) = \sin x$ AND CALCULATE $f'(x) = \cos x$, $f'' = -\sin x$, $f''' = -\cos x$, $f^{(iv)} = \sin x$ WHICH THEN REPEATS.

HENCE $f(0)=0$ AND $f^{(2n)}(0)=0$ FOR $n=1,2,3,\ldots$

THEN $f'(0)=f^{(iv)}(0)=f^{(9)}(0)=1$

$f'''(0)=f^{(7)}(0)=f^{(11)}(0)=-1$

SO $$f(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\ldots\ldots$$

NOW USE RATIO TEST WITH $a_n=\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ SO THAT

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}\,\frac{(2n+1)!}{x^{2n+1}}\right|=\frac{|x|^2}{(2n+3)(2n+2)}$$

NOW FOR ANY FIXED $|x|$ WE HAVE $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0$.

THUS THE Maclaurin series FOR $\sin x$ converges FOR ALL x.

THE FINAL USEFUL APPROXIMATION IS THE LEADING TERM AS $x\to 0$ FOR $f(x)=(1+x)^p$ WITH p ANY REAL NUMBER. SINCE $f(0)=0$ AND $f'(0)=p$, THE TANGENT LINE APPROXIMATION IS

$$(1+x)^p\cong 1+px+\ldots \quad \text{AS } x\to 0.$$

IN SUMMARY, OUR KEY RESULTS THAT SHOULD BE COMMITTED TO MEMORY ARE

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \quad \text{VALID FOR ALL } x$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\ldots \quad \text{VALID FOR ALL } x$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\ldots \quad \text{VALID FOR ALL } x$$

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots \quad \text{VALID FOR } |x|<1$$

$$(1+x)^p\cong 1+px+\ldots \quad \text{FOR } x\to 0.$$

THESE SERIES CAN BE INTEGRATED AND DIFFERENTIATED IN THEIR DOMAINS

OF CONVERGENCE. WE WILL CONSIDER A SERIES OF EXAMPLES TO ILLUSTRATE THE USE OF SUCH SERIES.

EXAMPLE 1 LET $f(x) = \log(1+2x^2)$ FOR $|x| < 1/\sqrt{2}$.

(i) FIND Maclaurin series FOR $f(x)$.

(ii) calculate $\lim\limits_{x\to 0} \dfrac{\log(1+2x^2)}{3x^2}$.

(iii) calculate $f^{(8)}(0)$.

SOLUTION

(i) WE RECALL $\dfrac{1}{1-y} = 1 + y + y^2 + \dots$ FOR $|y|<1$.

INTEGRATE WRT y AND SET INTEGRATION CONSTANT TO ZERO:

$$-\log(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \dots$$

SO $$\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} \dots \quad \text{FOR } |y|<1.$$

NOW PUT $y = -2x^2$. THEN FOR $2|x^2|<1$, OR $|x|<1/\sqrt{2}$

$$\log(1+2x^2) = -(-2x^2) - \frac{(-2x^2)^2}{2} - \frac{(-2x^2)^3}{3} - \frac{(-2x^2)^4}{4} \dots$$

SO $$\log(1+2x^2) = 2x^2 - \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3} - \frac{(2x^2)^4}{4} \dots \quad (*)$$

$$\log(1+2x^2) = \sum_{n=1}^{\infty} \frac{(2x^2)^n}{n}(-1)^{n+1} = \sum_{n=1}^{\infty} \frac{2^n(-1)^{n+1}}{n} x^{2n} \quad \text{FOR } |x|<1/\sqrt{2}$$

(ii) WE HAVE $\log(1+2x^2) = 2x^2 - 2x^4 + \dots$ FROM $(*)$

THUS $$\lim_{x\to 0} \frac{\log(1+2x^2)}{x^2} = \lim_{x\to 0} \frac{2x^2 - 2x^4 + \dots}{x^2} = \lim_{x\to 0}(2 - 2x^2 + \dots) = 2.$$

(iii) LET $f(x) = \log(1+2x^2)$. FORMULA $(*)$ YIELDS

$$(+) \qquad f(x) = 2x^2 - (2x^2)^2/2 + (2x^2)^3/3 - (2x^2)^4/4 + \dots \quad \text{FOR } |x|<1/\sqrt{2}.$$

NOW A GENERAL MACLAURIN SERIES HAS FORM

$$f(x) = f(0) + f'(0)\,x + f^{(2)}(0)\frac{x^2}{2!} + \dots + f^{(8)}(0)\frac{x^8}{8!} + \dots \qquad (++)$$

MATCHING THE x^8 TERMS IN $(+)$ AND $(++)$ WE GET

$$\frac{f^{(8)}(0)}{8!} = -\frac{2^4}{4} = -\frac{16}{4} = -4.$$

SOLVING FOR $f^{(8)}(0)$ WE GET $f^{(8)}(0) = -4 \cdot 8!$.

EXAMPLE 2 CALCULATE $\displaystyle\lim_{x \to 0} \frac{\cos(x^2) - (1 - x^4/2)}{x^8}$.

SOLUTION WE HAVE $\cos y = 1 - \dfrac{y^2}{2!} + \dfrac{y^4}{4!} - \dfrac{y^6}{6!} \dots$ VALID FOR ALL y.

SET $y = x^2$, SO $\cos(x^2) = 1 - \dfrac{(x^2)^2}{2!} + \dfrac{(x^2)^4}{4!} - \dfrac{(x^2)^6}{6!} + \dots$

REARRANGING GIVES $\cos(x^2) - 1 + \dfrac{x^4}{2} = \dfrac{x^8}{4!} - \dfrac{x^{12}}{6!} \dots$

DIVIDING BY x^8 AND TAKING THE LIMIT, WE GET

$$\frac{\cos(x^2) - 1 + x^4/2}{x^8} = \frac{1}{4!} - \frac{1}{6!}x^4 + \dots$$

THUS $\displaystyle\lim_{x \to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} = \frac{1}{4!}$.

EXAMPLE 3 LET $f(x) = e^{-x^2}$. USE MACLAURIN SERIES TO CALCULATE $f^{(8)}(0)$.

SOLUTION RECALL $e^y = 1 + y + \dfrac{y^2}{2!} + \dfrac{y^3}{3!} + \dfrac{y^4}{4!} + \dots$ VALID FOR ALL y.

NOW SET $y = -x^2$ TO GET $e^{-x^2} = 1 - x^2 + \dfrac{(-x^2)^2}{2!} + \dfrac{(-x^2)^3}{3!} + \dfrac{(-x^2)^4}{4!} + \dots$

THIS BECOMES

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

NOW LET $f(x) = e^{-x^2}$. A GENERAL MACLAURIN SERIES HAS

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(8)}(0)\frac{x^8}{8!} + \dots$$

COMPARING THE x^8 TERMS WE GET $\dfrac{f^{(8)}(0)}{8!} = \dfrac{1}{4!}$.

THUS $$f^{(8)}(0) = \frac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 .$$

<u>EXAMPLE 4</u> LET $f(x) = x^3 \sin(x^2)$. IN THE MACLAURIN SERIES $f(x) = \sum_{n=0}^{\infty} c_n x^n$ IDENTIFY THE COEFFICIENTS c_5, c_9, c_{11}, c_{13}, c_{17}.

<u>SOLUTION</u> $$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$$

SO $$\sin(x^2) = (x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \frac{1}{7!}(x^2)^7 + \dots$$

MULTIPLY BY x^3: WE DO SOME ALGEBRA TO OBTAIN:

$$f(x) = x^3 \sin(x^2) = x^3 \left[x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \dots \right]$$

$$f(x) = x^5 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{13} - \frac{1}{7!}x^{17} + \dots$$

COMPARING WITH $f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_5 x^5 + \dots + c_9 x^9 + \dots + c_{13} x^{13} + \dots + c_{17} x^{17} + \dots$

WE IDENTIFY

$$c_5 = 1, \quad c_9 = -\frac{1}{3!}, \quad c_{13} = \frac{1}{5!}, \quad c_{11} = 0, \quad c_{17} = -\frac{1}{7!}.$$

EXAMPLE 5 $f^{(V)}(0)$ FOR $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ WHERE $f^{(V)}(0)$ INDICATES FIFTH DERIVATIVE OF $f(x)$ AT $x=0$.

SOLUTION WE WILL NOT CALCULATE $f^{(V)}(0)$ DIRECTLY BY DIFFERENTIATING FIVE TIMES. INSTEAD WE USE AN APPROACH THAT CALCULATES THE TAYLOR SERIES, FROM WHICH WE IDENTIFY $f^{(V)}(0)$.

RECALL $\frac{1}{1-y} = 1+y+y^2+y^3+y^4+y^5$ FOR $|y|<1$.

SO $-\log(1-y) = y + y^2/2 + y^3/3 + y^4/4 + y^5/5 + \cdots$

$$\log(1-y) = -y - y^2/2 - y^3/3 - y^4/4 - y^5/5 + \cdots \qquad (1)$$

NOW REPLACE y BY $-y$ IN THIS FORMULA

$$\log(1+y) = -(-y) - (-y)^2/2 - (-y)^3/3 - (-y)^4/4 - (-y)^5/5$$

$$\log(1+y) = y - y^2/2 + y^3/3 - y^4/4 + y^5/5 + \cdots \qquad (2)$$

NOW $\log(1+y) - \log(1-y) = \log\left(\frac{1+y}{1-y}\right) = \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} + \cdots\right) - \left(-y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} + \cdots\right)$

THIS GIVES:

$$\log\left(\frac{1+y}{1-y}\right) = 2y + 2\left(y^3/3\right) + 2\left(y^5/5\right) + 2\left(\frac{y^7}{7}\right) + \cdots$$ ALL EVEN TERMS CANCEL.

REPLACE y BY x:

$$f(x) \equiv \log\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$ VALID FOR $|x|<1$.

WE CONCLUDE BY COMPARING WITH Maclaurin series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!}$

THAT $\frac{f^{(5)}(0)}{5!} = \frac{2}{5} \longrightarrow f^{(5)}(0) = \frac{2}{5}\cdot 5! = 2\cdot[1\cdot2\cdot3\cdot4] = 48.$

SO $f^{(5)}(0) = 48$.

EXAMPLE 6 DEFINE $F'(X) = \cos(2X^2)$. ASSUME THAT $F(0) = 0$.

FIND Maclaurin SERIES OF $F(X)$ UP TO AND INCLUDING TERMS OF ORDER X^{13}.

SOLUTION WE RECALL THAT FOR ALL y

$$\cos y = 1 - y^2/2! + y^4/4! - y^6/6! + \cdots$$

NOW SET $y = 2x^2$. THEN

$$\cos(2x^2) = 1 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \frac{(2x^2)^6}{6!} + \cdots$$

USE $4! = 24$, $6! = 720$, TO GET AFTER EVALUATING

$$\cos(2x^2) = 1 - 2x^4 + \frac{2}{3}x^8 - \frac{4}{45}x^{12} + \cdots$$

NOW WE INTEGRATE WITH $F(0) = 0$; REPLACING X BY DUMMY VARIABLE t:

$$F(x) = \int_0^x \left(1 - 2t^4 + \frac{2}{3}t^8 - \frac{4}{45}t^{12} + \cdots\right) dt$$

SO $$F(x) = x - \frac{2x^5}{5} + \frac{2x^9}{27} - \frac{4}{45}\left(\frac{1}{13}\right)x^{13} + \cdots$$

EXAMPLE 7 DEFINE $F'(x) = \arctan x^2$ WITH $F(0) = 0$.

FIND A FULL Maclaurin SERIES FOR $F(X)$ THAT IS VALID IN $|x| < 1$.

SOLUTION WE START WITH THE GEOMETRIC SERIES WHICH IS ONE OF THE SERIES WE MUST REMEMBER.

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{VALID FOR } |y| < 1.$$

NOW REPLACE $y = -t^2$. $\rightarrow$ $$\frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-t^2)^n = \sum_{n=0}^{\infty}(-1)^n t^{2n}.$$

FOR $|t^2| < 1 \rightarrow |t| < 1$.

NOW INTEGRATE ON t AND USE $\arctan t = 0$ WHEN $t = 0$.

WE GET
$$\arctan t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}, \quad \text{VALID FOR } |t|<1.$$

NOW REPLACE t BY x^2. WE GET RECALLING DEFINITION OF $F'(x)$:
$$F'(x) \equiv \arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)}. \quad \left(\text{VALID FOR: } |x^2|<1\right)$$

FINALLY INTEGRATE IN X AND SET INTEGRATION CONSTANT TO ZERO SINCE WE WANT $F(0)=0$. SINCE $|x^2|<1 \rightarrow |x|<1$,

THIS GIVES
$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)} \quad \text{IN} \quad |x| \leq 1$$

AS THE FULL MacLaurin SERIES.

<u>EXAMPLE 8</u> CALCULATE $S \equiv \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$ BY SUMMING THE SERIES.

<u>SOLUTION</u> MOTIVATED BY $n!$ TERM IN DENOMINATOR WE TRY TO IDENTIFY THE MacLaurin SERIES FROM OUR LIST OF 5 THAT BEST MATCHES THE SUM. WE TRY TO "FIT" THE EXPONENTIAL.
$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}. \qquad \text{USE} \quad (-1)^n x^{4n} = (-x^4)^n$$

NOW WRITE S AS
$$S = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!}. \quad \text{THUS} \quad y = -x^4$$

UPON COMPARISON. THIS YIELDS
$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}.$$

EXAMPLE 9 DEFINE $F(X)$ BY $F'(X) = \frac{\sin X}{X}$ WITH $F(0) = 0$.

CALCULATE A MACLAURIN SERIES FOR $F(X)$.

SOLUTION WE CAN WRITE $F(X) = \int_0^X \frac{\sin y}{y}\,dy$ FOR THEN $F(0) = 0$

AND BY FTC, $F'(X) = \frac{\sin X}{X}$.

WE HAVE $\frac{\sin y}{y} = \frac{1}{y}\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots \right] = \frac{1}{y} \sum_{n=0}^{\infty} \frac{y^{2n+1}(-1)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{y^{2n}(-1)^n}{(2n+1)!}$

THIS CONVERGES FOR ALL VALUES OF y.
WE INTEGRATE TO GET

$$F(x) = \int_0^X \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n+1)!}\,dy = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^X y^{2n}\,dy$$

SO $$F(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{y^{2n+1}}{2n+1} \Big|_0^X \right) = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n+1}}{(2n+1)\,(2n+1)!}$$

WRITING OUT A FEW TERMS GIVES

$$F(X) = X - \frac{X^3}{3 \cdot 3!} + \frac{X^5}{5 \cdot 5!} - \frac{X^7}{7 \cdot 7!} \dots \quad \text{VALID FOR ALL } X.$$

EXAMPLE 10 CALCULATE $\lim_{X \to \infty} \sqrt{X}\left[\sqrt{X+4} - \sqrt{X+1} \right]$.

SOLUTION WE CALCULATE $\sqrt{X+4} - \sqrt{X+1}$ FOR LARGE X USING

OUR FORMULA $(1+y)^p \cong 1 + py + \dots$ FOR SMALL y.

WE WRITE $$\sqrt{X+4} - \sqrt{X+1} = \sqrt{X(1+4/X)} - \sqrt{X(1+1/X)} \quad \text{FOR } X > 0$$

$$= \sqrt{X}\left((1+4/X)^{1/2} - (1+1/X)^{1/2} \right)$$ (let $y = 4/X$ AND $p = 1/2$)

SO $$\sqrt{X+4} - \sqrt{X+1} \cong \sqrt{X}\left(1 + \frac{2}{X} + \dots - \left(1 + \frac{1}{2X} + \dots \right) \right) \cong \sqrt{X}\left(\frac{3}{2X} + \dots \right)$$

MULTIPLY BY $\sqrt{X}$: THUS $\sqrt{X}\left[\sqrt{X+4} - \sqrt{X+1} \right] \cong X\left(\frac{3}{2X} + \dots \right)$ FOR Large X.

THEREFORE

$$\lim_{x\to\infty} \sqrt{x}\left[\sqrt{x+4}-\sqrt{x+1}\right] = \lim_{x\to\infty} x\left(3/2x\right) = 3/2.$$

ANOTHER WAY RECALL $\sqrt{a}-\sqrt{b} = (\sqrt{a}-\sqrt{b})\dfrac{(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$.

SO USE $\sqrt{x+4}-\sqrt{x+1} = (\sqrt{x+4}-\sqrt{x+1})\dfrac{(\sqrt{x+4}+\sqrt{x+1})}{(\sqrt{x+4}+\sqrt{x+1})} = \dfrac{(x+4)-(x+1)}{\sqrt{x+4}+\sqrt{x+1}}$

THUS, $\sqrt{x}\left[\sqrt{x+4}-\sqrt{x+1}\right] = \sqrt{x}\left[\dfrac{x+4-(x+1)}{\sqrt{x+4}+\sqrt{x+1}}\right] = \dfrac{3\sqrt{x}}{\sqrt{x+4}+\sqrt{x+1}}$.

NOW FOR LARGE x, $\dfrac{3\sqrt{x}}{\sqrt{x+4}+\sqrt{x+1}} \cong \dfrac{3\sqrt{x}}{\sqrt{x}+\sqrt{x}} \to \dfrac{3}{2}$ AS $x\to\infty$.

WE CONCLUDE THAT

$$\lim_{x\to\infty} \sqrt{x}\left[\sqrt{x+4}-\sqrt{x+1}\right] = 3/2.$$

EXAMPLE 11 CALCULATE $L \equiv \lim_{x\to\infty} x^2\left[\log(1+x^2) - 2\log x\right]$. [log = NATURAL log OR $\ln$]

SOLUTION WRITE $\log(1+x^2) = \log\left[x^2\left(1+\frac{1}{x^2}\right)\right] = \log x^2 + \log\left(1+\frac{1}{x^2}\right)$

USING $\log(AB) = \log A + \log B$. WE CALCULATE

$$\log(1+x^2) - 2\log x = \log x^2 + \log\left(1+\frac{1}{x^2}\right) - 2\log x = 2\log x + \log\left(1+\frac{1}{x^2}\right) - 2\log x$$
$$= \log\left(1+\frac{1}{x^2}\right).$$

NOW RECALL $\dfrac{1}{1-y} = 1+y+\ldots \to$ SO $\to \log(1-y) = -y+\ldots \to \log\left(1+\dfrac{1}{x^2}\right) \cong +\dfrac{1}{x^2}$ FOR x large.
FOR y SMALL
(let $y = -1/x^2$)

WE CONCLUDE:

$$L = \lim_{x\to\infty} x^2\left[\log(1+x^2) - 2\log x\right] = \lim_{x\to\infty} x^2 \cdot \log(1+1/x^2) = \lim_{x\to\infty} x^2\left(\frac{1}{x^2}+\ldots\right) = 1.$$

Maclaurin series and improper integrals

NOW COMBINE IDEAS ON Maclaurin SERIES TO INVESTIGATE CONVERGENCE OF IMPROPER INTEGRALS.

EXAMPLE 12 DEFINE I AS THE IMPROPER INTEGRAL

$$I = \int_0^\infty \left(1 - \frac{x}{\sqrt{x^2+1}}\right) dx.$$

EXPLAIN WHY I IS FINITE AND CALCULATE IT EXPLICITLY.

SOLUTION LET $f(x) = 1 - \dfrac{x}{\sqrt{x^2+1}}$. ESTIMATE $f(x)$ AS $x \to +\infty$

USING $(1+y)^p \simeq 1 + py + \dots$ FOR SMALL y.

WRITE $f(x) = 1 - \dfrac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$ FOR $x > 0$

$$= 1 - \left(1 + \frac{1}{x^2}\right)^{-1/2} \qquad (\text{NOW SET } p = -1/2,\ y = 1/x^2)$$

$$\simeq 1 - \left(1 - \frac{1}{2x^2} + \dots\right) \qquad \text{for large } x$$

SO $f(x) \simeq \dfrac{1}{2x^2} + \dots$ AS $x \to \infty$.

THUS $f(x)$ HAS SUFFICIENT DECAY AS $x \to \infty$ FOR CONVERGENCE OF IMPROPER INTEGRAL (RECALL $\int_1^\infty \frac{1}{x^p}\,dx$ IS FINITE IF AND ONLY IF $p > 1$).

NOW CALCULATE

$$I = \lim_{L\to\infty} \int_0^{\infty} \left(1 - \frac{x}{\sqrt{x^2+1}}\right) dx = \lim_{L\to\infty} \left[\left(x - (x^2+1)^{1/2}\right)\Big|_0^L\right]$$

↑ USE $u = x^2+1$ SUBSTITUTION

SO $I = \lim_{L\to\infty}\left(L - (L^2+1)^{1/2}\right) - (0-1) = 1 + \lim_{L\to\infty}\left(L - (L^2+1)^{1/2}\right)$. (*)

BUT $L - (L^2+1)^{1/2} = L - L\left(1+\frac{1}{L^2}\right)^{1/2} = L\left[1 - \left(1+\frac{1}{L^2}\right)^{1/2}\right] \simeq L\left[1 - \left(1 + \frac{1}{2L^2} + \dots\right)\right]$ For large L.

WE GET $\lim_{L\to\infty} \left(L - (L^2+1)^{1/2} \right) = \lim_{L\to\infty} L \left(-\frac{1}{2L^2} + \dots \right) = 0.$

THUS FROM (*), $I = 1$.

EXAMPLE 13 FIND THE VALUE OF C FOR WHICH THE FOLLOWING INTEGRAL EXISTS:

$$I = \int_1^\infty \left(\frac{Cx}{\sqrt{4x^4+1}} - \frac{1}{x} \right) dx.$$

SOLUTION DEFINE $f(x) = \dfrac{Cx}{\sqrt{4x^4+1}} - \dfrac{1}{x}$.

FOR THE INTEGRAL TO EXIST $f(x)$ MUST DECAY <u>FASTER</u> THAN $1/x$ AS $x \to \infty$.

WE ESTIMATE USING $(1+y)^p \simeq 1 + py + \dots$ AS $y \to 0$ THAT

$$f(x) = \frac{Cx}{\sqrt{4x^4}\sqrt{1+\frac{1}{4x^4}}} - \frac{1}{x} = \frac{Cx}{2x^2} \frac{1}{\sqrt{1+\frac{1}{4x^4}}} - \frac{1}{x}$$

$$f(x) = \frac{C}{2x} \left(1 + \frac{1}{4x^4} \right)^{-1/2} - \frac{1}{x}$$

NOW FOR LARGE x, $\left(1 + \frac{1}{4x^4} \right)^{-1/2} \simeq 1 - \frac{1}{8x^4} \dots$ (PUT $p = -1/2$ AND $y = 1/4x^4$)

$$\text{SO} \quad f(x) \simeq \frac{C}{2x} - \frac{1}{x} + \frac{C}{2x}\left(-\frac{1}{8x^4} \right) + \dots$$

TO ELIMINATE THE $1/x$ BEHAVIOR AS $x \to +\infty$ SET $C = 2$.

THEN $f(x) \simeq -\dfrac{1}{8x^5}$ AS $x \to +\infty$ AND THE INTEGRAL WILL EXIST.

RECALL A SIMILAR, BUT SLIGHTLY EASIER WEBWORK PROBLEM WHEN WE WERE DOING IMPROPER INTEGRALS.

EXAMPLE 14 (IMPROPER INTEGRAL NEAR $X=0$).

(i) FIND MacLAURIN SERIES FOR $f(x)=\log(1+2X)$. WHAT IS THE RADIUS OF CONVERGENCE OF THE SERIES?

(ii) FOR WHAT VALUE OF "C" WILL THE FOLLOWING INTEGRAL CONVERGE AS AN IMPROPER INTEGRAL. $I=\int_0^1\left(\frac{CX-\log(1+2X)}{X^2}\right)dX$

(iii) FOR THE VALUE OF "C" IN (ii) CALCULATE I EXPLICITLY.

SOLUTION

(i) WE RECALL $\frac{1}{1-y}=1+y+y^2+\dots=\sum_{n=0}^{\infty}y^n$ FOR $|y|<1$.

INTEGRATE BOTH SIDES: $-\log(1-y)=\sum_{n=0}^{\infty}\frac{y^{n+1}}{n+1}$ FOR $|y|<1$.

NOW $\log(1-y)=-\sum_{n=0}^{\infty}\frac{y^{n+1}}{n+1}$ FOR $|y|<1$.

WE THEN REPLACE $y=-2X$.

$$\log(1+2X)=-\sum_{n=0}^{\infty}\frac{(-2x)^{n+1}}{n+1}=\sum_{n=0}^{\infty}\frac{-(-2)^{n+1}X^{n+1}}{n+1} \quad \text{FOR } |2X|<1 \longrightarrow |X|<\tfrac{1}{2}.$$

NOW NOTE: $-(-1)^{n+1}=(-1)^n$.

WRITING THIS COMPACTLY GIVES $\log(1+2x)=\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n X^{n+1}}{n+1}$ FOR $|X|<1/2$.

(ii) NOW CONSIDER $I=\int_0^1\frac{CX-\log(1+2X)}{X^2}\,dX$.

FOR $X\to 0$ WE USE OUR MacLaurin series to get $\log(1+2X)\simeq 2X-2X^2+\dots$

$$\frac{CX-\log(1+2X)}{X^2}\simeq\frac{CX-[2X-2X^2+\dots]}{X^2}=\frac{[(C-2)X+2X^2+\dots]}{X^2}.$$

TAKE $C=2$ SO THAT NEAR $X=0$ WE HAVE WITH $C=2$,

$\frac{2X-\log(1+2X)}{X^2}\longrightarrow 2$. HENCE THE INTEGRAL IS CONVERGENT WHEN $C=2$ SINCE $1/X$ SINGULARITY IS ELIMINATED

(iii) **HARD:** Now calculate the integral I for the value of c found in part (ii).

WE SET $c=2$ AND EVALUATE

$$I = \int_0^1 \frac{(2x - \log(1+2x))}{x^2}\,dx.$$

WE EVALUATE AS AN IMPROPER INTEGRAL

$$I = \lim_{b\to 0} \int_b^1 \left(\frac{2}{x} - \frac{\log(1+2x)}{x^2}\right) dx.$$

USE IBP ON SECOND INTEGRAL.

$u = \log(1+2x) \to du = \frac{2}{1+2x}\,dx$

$dv = \frac{1}{x^2}\,dx \to v = -\frac{1}{x}.$

$$I = \lim_{b\to 0}\left[\, 2\log x\Big|_b^1 - \left(-\frac{\log(1+2x)}{x}\Big|_b^1 + \int_b^1 \frac{2\,dx}{x(1+2x)}\right)\right]$$

$$= \lim_{b\to 0}\left[-2\log b + \log 3 - \frac{\log(1+2b)}{b} - \int_b^1 \frac{dx}{(x+\frac{1}{2})\,x}\right]$$

NOW USE PARTIAL FRACTIONS: $\frac{1}{x(x+\frac{1}{2})} = \frac{2}{x} - \frac{2}{x+\frac{1}{2}}$

$$= \lim_{b\to 0}\left[-2\log b + \log 3 - \frac{\log(1+2b)}{b} - \int_b^1 \frac{2}{x}\,dx + \int_b^1 \frac{2}{x+1/2}\,dx\right]$$

$$= \lim_{b\to 0}\left[-\cancel{2\log b} + \log 3 - \frac{\log(1+2b)}{b} - \cancel{2\log} x\Big|_b^1 + 2\log\left(x+\frac{1}{2}\right)\Big|_b^1\right]$$

$$= \lim_{b\to 0}\left[\log 3 - \frac{\log(1+2b)}{b} + 2\log\left(\frac{3}{2}\right) - 2\log\left(b+\frac{1}{2}\right)\right]$$

$$= \lim_{b\to 0}\left[\log 3 - \frac{\log(1+2b)}{b} + 2\log 3 - 2\log 2 - 2\log\left(b+\frac{1}{2}\right)\right].$$

$$= 3\log 3 - 2\log 2 - 2\lim_{b\to 0}\log\left(b+\frac{1}{2}\right) - \lim_{b\to 0}\frac{\log(1+2b)}{b}$$ USE MacLAURIN

$$= 3\log 3 - \cancel{2\log 2} + \cancel{2\log 2} - \lim_{b\to 0}\left[\frac{2b+\dots}{b}\right] = 3\log 3 - 2.$$

SO $I = 3\log 3 - 2 = \log(27) - 2.$