

ABSOLUTE AND CONDITIONAL CONVERGENCE: THE RATIO TEST

RECALL THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGES BY ALTERNATING SERIES TEST BUT $\sum_{n=1}^{\infty} \frac{1}{n}$

DIVERGES BY INTEGRAL TEST. YET $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ AND $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGE.

WE NOW INTRODUCE TWO DEFINITIONS.

DEFINITION (ABSOLUTE AND CONDITIONAL CONVERGENCE). CONSIDER THE SERIES $\sum_{n=1}^{\infty} a_n$.

(I) IF THE SERIES $\sum_{n=1}^{\infty} |a_n|$ CONVERGES, THEN WE SAY THAT

$\sum_{n=1}^{\infty} a_n$ IS ABSOLUTELY CONVERGENT.

(II) IF THE SERIES $\sum_{n=1}^{\infty} |a_n|$ DIVERGES, BUT $\sum_{n=1}^{\infty} a_n$ CONVERGES, THEN WE

SAY THAT $\sum_{n=1}^{\infty} a_n$ IS CONDITIONALLY CONVERGENT.

A KEY PROPERTY THAT WE ESTABLISHED EARLIER IS THE FOLLOWING.

THEOREM IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES, THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES. THUS, IF A SERIES

IS ABSOLUTELY CONVERGENT (IN THE SENSE OF THE DEFINITION), THEN THE SERIES CONVERGES.

PROOF NOTICE THAT $0 \leq a_n + |a_n| \leq 2|a_n|$. THUS BY COMPARISON TEST IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES, THEN SO DOES $\sum_{n=1}^{\infty} (a_n + |a_n|)$ CONVERGE. SUBTRACTING THE CONVERGENT SERIES $\sum_{n=1}^{\infty} |a_n|$ THEN IMPLIES THAT $\sum_{n=1}^{\infty} a_n$ CONVERGES. \square

A KEY TEST FOR ABSOLUTE CONVERGENCE OF A SERIES IS THE RATIO TEST, WHICH WE DESCRIBE NOW.

THE RATIO TEST IS REALLY USEFUL WHEN THE SERIES HAS EXPONENTS IN n , FACTORIAL OF n , ETC.. LIKE

$\sum_{n=1}^{\infty} \frac{n 2^n}{5^n}$, $\sum_{n=1}^{\infty} \frac{e^n}{n!}$, $\sum_{n=1}^{\infty} \frac{n^2 e^n}{4^n}$ ETC..

THEOREM (RATIO TEST) LET $N > 0$ BE AN INTEGER AND ASSUME THAT

(R2)

$a_n \neq 0$ FOR ALL $n \geq N$. THEN,

(I) IF $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, WE HAVE $\sum_{n=1}^{\infty} |a_n|$ CONVERGES AND SO $\sum_{n=1}^{\infty} a_n$ CONVERGES.

(II) IF $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ OR $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, THEN BOTH $\sum_{n=1}^{\infty} |a_n|$ AND $\sum_{n=1}^{\infty} a_n$ DIVERGE.

THERE IS NO CONCLUSION FROM THE RATIO TEST IF $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ (I.E. $L=1$), WHEN $L=1$, ANOTHER TEST IS NEEDED TO DETERMINE WHETHER $\sum |a_n|$ CONVERGES OR NOT.

REMARKS (i) IN (I), IF $L < 1$ THIS MEANS THAT $\sum a_n$ IS ABSOLUTELY CONVERGENT.

(ii) RATIO TEST IS NOT USEFUL FOR SERIES OF THE FORM $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ WHERE $P(n), Q(n)$ ARE POLYNOMIALS.

THIS IS BECAUSE WE WILL GET $L=1$ FOR SUCH SERIES. RATIO TEST IS USEFUL FOR SERIES WITH TERMS LIKE a^n, n^b, e^{-n} ETC.

(iii) WHEN $L > 1$ IN (II), THE DIVERGENCE OF BOTH $\sum |a_n|$ AND $\sum a_n$ RESULTS FROM THE FACT THAT $a_n \not\rightarrow 0$ AS $n \rightarrow \infty$.
(WE NOW SHOW THIS)

PROOF (OUTLINE OF RATIO TEST)

WE FIRST PROVE (I). SINCE $L < 1$ WE CAN PICK AN R WITH $0 < L < R < 1$. THEN SINCE $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow L < R$ AS $n \rightarrow \infty$, FOR SOME M SUFFICIENTLY LARGE WE MUST HAVE $\left| \frac{a_{n+1}}{a_n} \right| < R$ FOR ALL $n \geq M$.

THIS MEANS $|a_{n+1}| < R|a_n|$ FOR ALL $n \geq M$, SO THAT

$|a_{M+1}| < R|a_M|$, $|a_{M+2}| < R|a_{M+1}| < R^2|a_M|$ ETC... SO THAT

$$|a_{M+j}| < R^j |a_M|$$

THEN $\sum_{j=1}^{\infty} |a_{M+j}| < |a_M| \sum_{j=1}^{\infty} R^j = |a_M| \frac{R}{1-R} < \infty$ SINCE $0 < R < 1$.

WE CONCLUDE THAT $\sum_{k=1}^{\infty} |a_{M+k}| < \infty$.

(R3)

THIS MEANS THAT $\sum_{n=M+1}^{\infty} |a_n|$ CONVERGES AND SO $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^M |a_n| + \sum_{n=M+1}^{\infty} |a_n|$
— FINITE $< \infty$

IMPLIES THAT $\sum_{n=1}^{\infty} |a_n|$ CONVERGES. THIS PROVES (I).

NOW TO PROVE (II), PICK R IN $1 < R < L$. THEN FOR $n \geq M$, WE HAVE

$$\left| \frac{a_{n+1}}{a_n} \right| \geq R \rightarrow |a_{n+1}| \geq R |a_n| \text{ FOR ALL } n \geq M.$$

THIS $|a_{n+1}| \geq |a_n| > 0$. SINCE $|a_{n+1}| \not\rightarrow 0$ AS $n \rightarrow \infty$ WE HAVE

BY BASIC DIVERGENCE TEST THAT $\sum_{k=1}^{\infty} |a_{n+k}|$ IS DIVERGENT. THIS $\sum_{n=1}^{\infty} |a_n|$ DIVERGES.
THIS ALSO MEANS THAT $\sum_{n=1}^{\infty} a_n$ DIVERGES.

WE NOW DO MANY EXAMPLES TO ILLUSTRATE ABSOLUTE CONVERGENCE, CONDITIONAL CONVERGENCE, AND THE RATIO TEST.

EXAMPLE 1 CONSIDER $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ WHERE $a_n = (-1)^{n-1} \frac{1}{n}$.

BY ALTERNATING SERIES TEST $\sum_{n=1}^{\infty} a_n$ CONVERGES. BUT WE HAVE $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$

DIVERGES. THIS BY OUR DEFINITION IN (II) ON PAGE (R1), WE

CONCLUDE THAT $\sum_{n=1}^{\infty} a_n$ IS CONDITIONALLY CONVERGENT.

EXAMPLE 2 CONSIDER $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ WHERE $a_n = (-1)^{n-1} \frac{1}{n^2}$.

BY ALT. SERIES TEST, WE KNOW THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ CONVERGES. NEXT, WE TEST $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

WE OBTAIN THAT $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES BY INTEGRAL TEST. THIS $\sum_{n=1}^{\infty} |a_n|$ CONVERGES.

BY (I) OF THE DEFINITION ON PAGE (R1) WE SAY THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ IS

ABSOLUTELY CONVERGENT.

EXAMPLE 3 DISCUSS ABSOLUTE (CONDITIONAL) CONVERGENCE, OR DIVERGENCE OF

(R4)

EACH OF THE FOLLOWING:

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ (ii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5} \right)$ (iii) $\sum_{n=1}^{\infty} (-1)^n \frac{(n^2+1)}{n^4+5}$

(iv) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ (v) $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3+1}$ (vi) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

SOLUTION

(i) LET $a_n = (-1)^n b_n$ WITH $b_n = \frac{n}{n^2+1}$. SINCE b_n IS DECREASING FOR n LARGE

ENOUGH (EASILY SHOWN) AND $b_n > 0$ WITH $b_n \rightarrow 0$ AS $n \rightarrow \infty$, BY ALTERNATING SERIES TEST $\sum_{n=1}^{\infty} (-1)^n b_n$ CONVERGES. BUT $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$

DIVERGES BY USING LIMIT COMPARISON TEST WITH $1/n$. HENCE WE

CONCLUDE THAT $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ IS CONDITIONALLY CONVERGENT.

(ii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5} \right)$. LET $a_n = (-1)^n \left(\frac{2n+1}{6n+5} \right)$. FOR $n \rightarrow \infty$, $a_n \approx \frac{(-1)^n}{3}$.

SINCE $a_n \not\rightarrow 0$ AS $n \rightarrow \infty$, BY BASIC DIVERGENCE TEST $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5} \right)$ DIVERGES.

(iii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5} \right)$. LET $a_n = (-1)^n \left(\frac{n^2+1}{n^4+5} \right)$. LET'S FIRST TEST $\sum_{n=1}^{\infty} |a_n|$.

WE HAVE $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2+1}{n^4+5}$. SINCE $\frac{n^2+1}{n^4+5} \approx \frac{1}{n^2}$ FOR n LARGE

AND $\sum 1/n^2$ CONVERGES, WE EXPECT CONVERGENCE. WE LIMIT COMPARISON

TEST WITH $1/n^2$. WE CALCULATE $\lim_{n \rightarrow \infty} \frac{(n^2+1)/(n^4+5)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2(n^2+1)}{n^4+5} = 1$.

BY LIMIT COMPARISON TEST, $\sum |a_n|$ CONVERGES. THUS,

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5} \right)$ IS ABSOLUTELY CONVERGENT (AND $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5} \right)$ MUST CONVERGE).

(iv) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ WITH $a_n = (-1)^n n e^{-n^2}$.

LET'S FIRST TEST $\sum_{n=1}^{\infty} |a_n|$. WE CONSIDER THEN $\sum_{n=1}^{\infty} n e^{-n^2}$.

WE HAVE TWO APPROACHES:

RATIO TEST SINCE $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1) e^{-(n+1)^2}}{n e^{-n^2}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{n} \frac{e^{-(n^2+2n+1)}}{e^{-n^2}} \right|$
 $= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} e^{-2n-1} \right| = (1)(0) = 0$

WE HAVE $L = 0$ AND SO BY (I) OF RATIO TEST, $\sum_{n=0}^{\infty} |a_n|$ CONVERGES.

WE CONCLUDE THAT $\sum_{n=0}^{\infty} (-1)^n n e^{-n^2}$ IS ABSOLUTELY CONVERGENT.

INTEGRAL TEST CONSIDER $\sum_{n=1}^{\infty} n e^{-n^2}$ BY INTEGRAL TEST. DEFINE $b_n = f(n)$

WITH $f(x) = x e^{-x^2}$. CLEARLY $f(x) > 0$ ON $x > 1$ AND $f'(x) = e^{-x^2} (1 - 2x^2) < 0$ FOR $x \geq 1$.

SINCE $b_n > 0$ AND $b_{n+1} < b_n$ (DECREASING SEQUENCE) WE HAVE THAT $\sum_{n=1}^{\infty} n e^{-n^2}$ CONVERGES OR DIVERGES WHEN $\int_1^{\infty} f(x) dx$ CONVERGES OR DIVERGES, RESPECTIVELY.

WE CALCULATE $\int_1^{\infty} x e^{-x^2} dx = \lim_{L \rightarrow \infty} \int_1^L x e^{-x^2} dx = \lim_{L \rightarrow \infty} \left(-\frac{1}{2} e^{-x^2} \Big|_1^L \right) = \frac{1}{2} e^{-1} < \infty$.

THUS $\sum |a_n|$ CONVERGES AND SO $\sum a_n$ CONVERGES ABSOLUTELY.

(v) $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3 + 1}$. WE EXPECT CONVERGENCE SINCE $\frac{n \sin(n)}{n^3 + 1} \approx \frac{\sin(n)}{n^2}$ FOR LARGE n .

HOWEVER, WE CAN'T USE INTEGRAL TEST SINCE $f(x) = \frac{x \sin(x)}{x^3 + 1}$ IS NOT DECREASING FOR LARGE x . NOR CAN WE COMPARE WITH $\frac{1}{n^2}$ IN THE LIMIT COMPARISON

TEST SINCE $\lim_{n \rightarrow \infty} \left(\frac{(n \sin(n))}{n^3 + 1} \right) / \left(\frac{1}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{n^2 \sin(n)}{n^3 + 1}$ DOES NOT EXIST.

INSTEAD WE USE BASIC COMPARISON TEST WITH $|\sin(n)| < 1$.

WE HAVE $0 \leq \left| \frac{n \sin(n)}{n^3 + 1} \right| \leq \frac{n}{n^3 + 1}$

SINCE $n^3 + 1 \geq n^3$ WE HAVE $\left| \frac{n \sin(n)}{n^3 + 1} \right| \leq \frac{1}{n^2}$ AND $\sum \frac{1}{n^2}$ CONVERGES.

THUS, SINCE $\sum |a_n|$ CONVERGES, a_n IS SAID TO CONVERGE ABSOLUTELY.

(vi) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$. THE TERMS ARE POSITIVE AND WE CAN TEST FOR CONVERGENCE

WITH THE RATIO TEST. LET $a_n = \frac{2^n}{n!}$ AND CALCULATE $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n / n!}$

SINCE $\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} 2 = \left(\frac{n \cdot (n-1) \dots 1}{(n+1) \cdot n \cdot (n-1) \dots 1} \right) 2 = \frac{2}{n+1}$ WE HAVE $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0$.

WE CONCLUDE BY (I) OF THE RATIO TEST THAT $\sum_{n=0}^{\infty} a_n$ CONVERGES. (NO need to write absolute convergence since $a_n > 0$ FOR ALL n).

FURTHER REMARKS AND EXAMPLES:

REMARK THE RATIO TEST IS ALWAYS INCONCLUSIVE FOR RATIOS OF POLYNOMIALS SINCE

IF $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 3n}{n^3 + 4n^2}$ WE CALCULATE WITH $|a_n| = \frac{n^2 + 3n}{n^3 + 4n^2}$ THAT $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

SINCE $L=1$ THE RATIO TEST IS INCONCLUSIVE.

EXAMPLE $\sum_{n=1}^{\infty} \frac{n}{5^n}$. COULD WE USE INTEGRAL TEST WITH $f(x) = \frac{x}{5^x} = \frac{x}{e^{x \ln 5}} = x e^{-(\ln 5)x}$

OR WE RATIO TEST. LET $a_n = \frac{n}{5^n}$. WE WRITE $\frac{a_{n+1}}{a_n} = \frac{(n+1)/5^{n+1}}{n/5^n} = \left(\frac{n+1}{n} \right) \frac{1}{5}$.

SINCE $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{5} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) = \frac{1}{5} < 1$, THE SERIES CONVERGES.

EXAMPLE (HARDER) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$. WE HAVE $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} / (n+1)!}{n^n / n!} = \frac{n!}{(n+1)!} \frac{(n+1)^{n+1}}{n^n}$.

BUT $\frac{n!}{(n+1)!} = \frac{1}{n+1}$ SO $\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)} \frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n} \right)^n$.

NOW LET $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e = 2.7129... > 1$. BY (II)

OF RATIO TEST, $\sum n^n / n!$ DIVERGES. TO SEE WHY $\left(1 + \frac{1}{n} \right)^n \rightarrow e$ AS $n \rightarrow \infty$

TAKE LOG OF $\left(1 + \frac{1}{n} \right)^n$. CONSIDER $\lim_{n \rightarrow \infty} n \log \left(1 + \frac{1}{n} \right) = \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \lim_{x \rightarrow 0} \frac{(1/x)}{1} = 1$ BY

L'HOPITAL RULE. THUS $\left(1 + \frac{1}{n} \right)^n \rightarrow e^1$ AS $n \rightarrow \infty$.

EXAMPLE (HARDER)

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!} \quad \text{HERE} \quad a_n = \frac{n^n}{3^n n!}$$

WE CAN PROCEED AS IN PREVIOUS EXAMPLE TO GET

$$\frac{a_{n+1}}{a_n} = \frac{1}{3} \left(1 + \frac{1}{n}\right)^n$$

NOW $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} e = \frac{2.712...}{3} < 1$. THUS, THE SERIES NOW CONVERGES BY THE RATIO TEST.

EXAMPLE (HARDER)

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n^2+1)(n!)^2}$$

WE DEFINE $a_n = \frac{(2n)!}{(n^2+1)(n!)^2} = \frac{(2n)(2n-1)(2n-2)(2n-3)\dots 1}{(n^2+1)[n(n-1)\dots 1]^2}$

NOW $\frac{a_{n+1}}{a_n} = \frac{[2(n+1)]!}{[(n+1)^2+1][(n+1)!]^2} \cdot \frac{(n!)^2 (n^2+1)}{(2n)!} = \frac{(n^2+1)(2n+2)(2n+1)}{(n+1)^2+1 (n+1)^2}$

NOW LET $n \rightarrow \infty$. WE CALCULATE $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{4n^4 + \dots}{n^4 + \dots} = 4 > 1$.

BY RATIO TEST, SINCE $L = 4 > 1$, $\sum_{n=1}^{\infty} \frac{(2n)!}{(n^2+1)(n!)^2}$ DIVERGES.

EXAMPLE (HARDER)

SUPPOSE THAT $\sum_{n=1}^{\infty} \frac{n a_n - 2n + 1}{n+1}$ CONVERGES.

THEN CALCULATE $S = -\log a_1 + \sum_{n=1}^{\infty} \log \left(\frac{a_n}{a_{n+1}} \right)$.

SOLUTION

DEFINE $b_n = \frac{n a_n - 2n + 1}{n+1}$. IF THE SERIES CONVERGES WE MUST HAVE

THAT $b_n \rightarrow 0$ AS $n \rightarrow \infty$. WE WRITE $b_n = \frac{n(a_n - 2)}{n+1} + \frac{1}{n+1}$. THIS MEANS WE MUST

HAVE $a_n \rightarrow 2$ AS $n \rightarrow \infty$. NOW S IS A TELESCOPING SUM. THE N^{th} PARTIAL SUM IS $S_N = -\log a_1 + \sum_{n=1}^N (\log a_n - \log a_{n+1}) = -\log a_1 + (\log a_1 - \log a_2) + (\log a_2 - \log a_3) \dots + (\log a_N - \log a_{N+1})$

WE GET $S_N = -\log a_{N+1}$ WITH THIS CANCELLATION. THUS $S = -\log \left(\lim_{N \rightarrow \infty} a_{N+1} \right) = -\log 2$.