SEQUENCES

A SEQUENCE IS A LIST OF INFINITELY MANY NUMBERS WITH A SPECIFIED ORDER. WE WRITE

$$\{a_1, a_2, \ldots, a_n, \ldots\} \quad \text{OR} \quad \{a_n\}_{n=1}^{\infty}.$$

TYPICALLY IF $n =$ positive integer, we will have $\{a_n = f(n)\}_{n=1}^{\infty}$.

EXAMPLES

$$\left\{a_n = \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

$$\left\{a_n = \frac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots\right\}$$

$$\left\{a_n = e^{-n}\right\}_{n=1}^{\infty} = \left\{e^{-1}, e^{-2}, \ldots\right\}$$

DEFINITION A SEQUENCE IS SAID TO CONVERGE TO THE LIMIT A IF a_n APPROACHES A AS $n \to \infty$. WE WRITE

$$\lim_{n\to\infty} a_n = A \quad \text{OR} \quad a_n \to A \text{ AS } n \to \infty.$$

A SEQUENCE IS SAID TO CONVERGE IF IT CONVERGES TO SOME LIMIT. OTHERWISE IT IS SAID TO DIVERGE.

EXAMPLE

- $\left\{a_n = \frac{1}{n}\right\}_{n=1}^{\infty}$ CONVERGES TO ZERO SINCE $\lim_{n\to\infty} \frac{1}{n} = 0$.
- $\left\{a_n = (-1)^{n-1}\right\}_{n=1}^{\infty}$ diverges since a_n oscillates between -1 AND 1 FOR all n.

THEOREM IF $\lim_{x\to\infty} f(x) = L$ AND IF $a_n = f(n)$ FOR all $n = 1, 2, \ldots$ THEN $\lim_{n\to\infty} a_n = L$.

EXAMPLE DISCUSS CONVERGENCE OF $\left\{\frac{n}{4n+5}\right\}_{n=1}^{\infty}$ AND OF $\left\{\frac{2n^2+3}{n^2+n}\right\}_{n=1}^{\infty}$.

- DEFINE $f(x) = \dfrac{x}{4x+5}$. NOTICE THAT $\lim\limits_{x\to\infty} f(x) = \dfrac{1}{4}$.

  THUS $\lim\limits_{n\to\infty} \dfrac{n}{4n+5} = \dfrac{1}{4}$.

- NOW DEFINE $f(x) = \dfrac{2x^2+3}{x^2+x} \Longrightarrow f(x) = \dfrac{2x^2+3}{x^2\left(1+\frac{1}{x}\right)}$

  CLEARLY $\lim\limits_{x\to\infty} f(x) = 2$. THUS $\lim\limits_{n\to\infty} \dfrac{2n^2+3}{n^2+n} = 2$.

THE RULES YOU LEARNED ABOUT HOW TO WORK WITH $\lim\limits_{x\to\infty} f(x)$ ALSO APPLIES TO LIMITS LIKE $\lim\limits_{n\to\infty} a_n$.

<u>THEOREM</u> SUPPOSE THAT $\lim\limits_{n\to\infty} a_n = A$ AND $\lim\limits_{n\to\infty} b_n = B$.

THEN, FOR ANY CONSTANT C,

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$

(ii) $\lim\limits_{n\to\infty} (a_n - b_n) = A - B$.

(iii) $\lim\limits_{n\to\infty} C a_n = C A$

(iv) $\lim\limits_{n\to\infty} (a_n b_n) = A B$

(v) IF $B \neq 0$ THEN $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$.

<u>EXAMPLES</u> (i) $\lim\limits_{n\to\infty} \left(\dfrac{n^2}{2n^2+n} + 7e^{-n} \right) = \lim\limits_{n\to\infty} \dfrac{n^2}{n^2\left(2+\frac{1}{n}\right)} + 7 \lim\limits_{n\to\infty} e^{-n} = \dfrac{1}{2} + 0$.

(ii) $\lim\limits_{n\to\infty} \left(\dfrac{4n+5}{n+2} + ne^{-n} \right) = \lim\limits_{n\to\infty} \dfrac{4n+5}{n+2} + \lim\limits_{n\to\infty} ne^{-n}$

$= 4 + 0$ SINCE $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$.

SQUEEZE THEOREM

SUPPOSE $a_n \le c_n \le b_n$ FOR ALL $n = 1, 2, \ldots$ AND

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L.$$

THEN $$\lim_{n\to\infty} c_n = L.$$

EXAMPLE LET π_n BE THE n^{th} decimal digit of $\pi = 3.14159265359$

SO $\pi_1 = 1$, $\pi_2 = 4$, $\pi_3 = 1, \ldots$ CALCULATE $\displaystyle\lim_{n\to\infty}\left(1 + \frac{\pi_n}{n}\right)$

NOW $$\underset{\uparrow \atop a_n}{1} \le \underset{\uparrow \atop c_n}{1 + \frac{\pi_n}{n}} \le \underbrace{1 + \frac{9}{n}}_{b_n} \quad \text{FOR ALL } n = 1, 2, \ldots$$

WE HAVE $\displaystyle\lim_{n\to\infty} a_n = 1$ AND $\displaystyle\lim_{n\to\infty} 1 + \frac{9}{n} = 1.$

SO BY SQUEEZE THEOREM $\displaystyle\lim_{n\to\infty}\left(1 + \frac{\pi_n}{n}\right) = 1.$

THEOREM IF $\displaystyle\lim_{n\to\infty} a_n = L$ AND IF $g(x)$ IS CONTINUOUS AT $x = L$

THEN $$\lim_{n\to\infty} g(a_n) = g(L).$$

EXAMPLE (i) CALCULATE $\displaystyle\lim_{n\to\infty} \sin\left(\frac{\pi n}{n+1}\right)$

NOTICE $a_n = \dfrac{\pi n}{2n+1}$ AND LET $g(x) = \sin(x)$.

THEN $\displaystyle\lim_{n\to\infty} a_n = \frac{\pi}{2}$ AND $g(x)$ IS CONTINUOUS AT $x = \pi/2$

SO BY THEOREM $\displaystyle\lim_{n\to\infty} g(a_n) = g(\pi/2) = \sin(\pi/2) = 1.$

(ii) CALCULATE $\lim_{n\to\infty} e^{-(n^2+2/n^2+1)}$

LET $a_n = \dfrac{n^2+2}{n^2+1}$, AND $g(x) = e^{-x}$.

NOW $\lim_{n\to\infty} \dfrac{n^2+2}{n^2+1} = 1$ AND $g(x)$ IS CONTINUOUS AT $x=1$.

SO $\lim_{n\to\infty} g(a_n) = g(1) = e^{-1}$.

<u>EXAMPLE</u> DOES $\lim_{n\to\infty} \dfrac{2^n}{5^{n+1}}$ EXIST?

LET $a_n = \dfrac{2^n}{5^{n+1}} = \dfrac{1}{5}\left(\dfrac{2}{5}\right)^n$.

SINCE $2/5 < 1$ WE HAVE $\lim_{x\to\infty} \left(\dfrac{2}{5}\right)^x = 0$.

THUS $\lim_{n\to\infty} a_n = 0$.

<u>EXAMPLE</u> LET $a_n = \log(n^2+4) - \log(2n^2)$.

CALCULATE $\lim_{n\to\infty} a_n$.

WE USE PROPERTIES OF log: $a_n = \log\left(\dfrac{n^2+4}{2n^2}\right)$.

DEFINE $a_n = \dfrac{n^2+4}{2n^2}$. NOW CLEARLY $a_n \to \frac{1}{2}$ AS $n\to\infty$.

LET $g(x) = \log x$. THEN BY OUR THEOREM $\lim_{n\to\infty} g(a_n) = \log\left(\frac{1}{2}\right)$

SINCE $g(x)$ IS CONTINUOUS AT $x = 1/2$.

INFINITE SEQUENCES AND SERIES

CONSIDER THE SEQUENCE $\{a_n\}$. WE WANT TO SEE WHETHER THE INFINITE SERIES $\sum_{n=1}^{\infty} a_n$ IS FINITE (CONVERGES) OR IS INFINITE (DIVERGES).

DEFINITION DEFINE $S_N = \sum_{n=1}^{N} a_n$. S_N IS CALLED THE NTH PARTIAL SUM OF THE SERIES. IF $\lim_{N\to\infty} S_N$ EXISTS, WE DENOTE

$$S = \lim_{N\to\infty} S_N$$ AND WE SAY THAT THE SERIES CONVERGES WITH VALUE $\sum_{n=1}^{\infty} a_n = S$.

THEOREM (CLP THEOREM 3.2.9). SUPPOSE THAT $\sum_{n=1}^{\infty} a_n$ AND $\sum_{n=1}^{\infty} b_n$ ARE CONVERGENT SERIES WITH $\sum_{n=1}^{\infty} a_n = A$ AND $\sum_{n=1}^{\infty} b_n = B$. THEN FOR ANY REAL NUMBER c (INDEPENDENT OF n) WE HAVE

(i) $$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n = cA$$

(ii) $$\sum_{n=1}^{\infty} (a_n + b_n) = A + B \quad ; \quad \sum_{n=1}^{\infty} (a_n - b_n) = A - B.$$

REMARK THE PRODUCT AND RATIOS $\sum a_n b_n$ AND $\sum a_n/b_n$ ARE NOT SO SIMPLE TO CALCULATE. (THEY ARE <u>NOT</u> AB AND A/B).

THE SIMPLEST TYPE OF INFINITE SERIES THAT CAN BE ANALYZED ARE **GEOMETRIC SERIES**, WHICH HAVE THE GENERAL FORM

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \dots = a(1 + r + r^2 + r^3 + \dots).$$

THE MAIN RESULT FOR <u>GEOMETRIC SERIES</u> IS AS FOLLOWS:

<u>THEOREM</u> CONSIDER GEOMETRIC SERIES $\sum_{n=1}^{\infty} a r^{n}$.

THEN, IF $|r| < 1$ IT CONVERGES AND

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}.$$

IF $|r| \geq 1$ AND $a \neq 0$ THEN THE SERIES DIVERGES.

<u>PROOF</u> DEFINE $S_N = \sum_{n=1}^{N} a r^{n-1} = a\left(1 + r + \dots + r^{N-1}\right)$ AS NTH PARTIAL SUM.

WE CALCULATE

$$\begin{aligned}(1-r) S_N &= a\left(1 + r + r^2 + \dots + r^{N-1}\right)(1-r) \\ &= a\left(1 + \cancel{r} + \cancel{r^2} + \dots + r^{N-1} - \cancel{r} - \cancel{r^2} \dots - r^N\right) \\ &= a\left(1 - r^N\right).\end{aligned}$$

- THUS FOR $r \neq 1$, WE HAVE $S_N = \dfrac{a\left(1 - r^N\right)}{1-r}$.

NOW LET $N \to \infty$. IF $|r| < 1$ WE HAVE $\lim_{N\to\infty} S_N = \dfrac{a}{1-r} \to \sum_{n=1}^{\infty} a r^{n-1}$ CONVERGES

IF $|r| > 1$, $\lim_{N\to\infty} S_N$ DOES NOT EXIST $\to \sum_{n=1}^{\infty} a r^{n-1}$ DIVERGES

- NOTICE THAT IF $r = 1$, THEN $S_N = Na$ WHICH DIVERGES AS $N \to \infty$.

HENCE, GEOMETRIC SERIES CONVERGES IF AND ONLY IF $|r| < 1$. $\square$

<u>EXAMPLE 1</u> CALCULATE THE FOLLOWING INFINITE SUM (IN FORM OF A GEOMETRIC SEQUENCE)

$$9 - \frac{27}{5} + \frac{81}{25} - \frac{243}{125} + \dots$$

<u>SOLUTION</u> DEFINE $S \equiv 9 - \dfrac{27}{5} + \dfrac{81}{25} - \dfrac{243}{125} + \dots$

WE WRITE

$$S = 9\left(1 - \frac{3}{5} + \frac{9}{25} - \frac{27}{125} + \dots\right) = 9\left(1 + \left(-\frac{3}{5}\right) + \left(-\frac{3}{5}\right)^2 + \left(-\frac{3}{5}\right)^3 + \dots\right)$$

THUS SINCE $r = -3/5$ WE HAVE $S = 9\dfrac{1}{(1-r)} = \dfrac{9}{(1 + 3/5)} = \dfrac{45}{8}$.

EXAMPLE 2 CALCULATE $\sum_{n=1}^{\infty} \frac{4^n + 5^n}{9^n}$.

SOLUTION WE WRITE $S = \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^n = \frac{4}{9}\left(1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots\right) + \frac{5}{9}\left(1 + \left(\frac{5}{9}\right) + \left(\frac{5}{9}\right)^2 + \cdots\right)$

THUS USING geometric series result, $S = \frac{4}{9}\,\frac{1}{(1-4/9)} + \frac{5}{9}\,\frac{1}{(1-5/9)} = \frac{4}{9}\left(\frac{9}{5}\right) + \frac{5}{9}\left(\frac{9}{4}\right)$.

THIS GIVES $S = \frac{4}{5} + \frac{5}{4} = \frac{41}{20}$.

EXAMPLE 3 CALCULATE $S = \sum_{n=0}^{\infty} \frac{3^n}{8^{2n+1}}$.

SOLUTION WE WRITE $8^{2n} = (64)^n$ SO THAT $S = \frac{1}{8}\sum_{n=0}^{\infty} \frac{3^n}{8^{2n}} = \frac{1}{8}\sum_{n=0}^{\infty} \frac{3^n}{64^n} = \frac{1}{8}\sum_{n=0}^{\infty} \left(\frac{3}{64}\right)^n$.

THIS gives BY using geometric series result $S = \frac{1}{8}\,\frac{1}{(1-3/64)} = \frac{1}{8}\,\frac{64}{61} = \frac{8}{61}$.

EXAMPLE 4 CALCULATE $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}}$.

SOLUTION $2^{2n} = 4^n$ SO THAT $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} = \left(-\frac{1}{4}\right)\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = \left(-\frac{1}{4}\right)\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n$.

SINCE $r = -1/4$, WE GET $S = \left(-\frac{1}{4}\right)\frac{1}{(1-r)} = \left(-\frac{1}{4}\right)\frac{1}{(1+1/4)} = \left(-\frac{1}{4}\right)\left(\frac{4}{5}\right) = -\frac{1}{5}$.

EXAMPLE 5 NOTICE THAT $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^n$ AND $\sum_{n=1}^{\infty} \frac{9^n}{5 \cdot 3^{2n}}$ BOTH DIVERGE.

FOR FIRST SUM, $r = -4/3$ SO $|r| > 1 \rightarrow$ divergence. FOR SECOND SUM WE USE $3^{2n} = (3^2)^n = 9^n$ SO THAT $\sum_{n=1}^{\infty} \frac{9^n}{5 \cdot 3^{2n}} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{9^n}{9^n} = \frac{1}{5}\sum_{n=1}^{\infty} (1)^n \rightarrow$ SO $r = 1 \rightarrow$ divergence.

EXAMPLE 6 WRITE THE REPEATING DECIMAL $0.12\overline{34} = 0.12343434\ldots$ AS A RATIO OF INTEGERS.

SOLUTION $.12343434\ldots = \frac{12}{100} + \frac{34}{10{,}000} + \frac{34}{(10{,}000)(100)} + \frac{34}{10{,}000\,(100)^2} + \cdots$

$$= \frac{12}{100} + \frac{34}{10{,}000}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots\right)$$

$$= \frac{12}{100} + \frac{34}{10{,}000}\,\frac{1}{(1-1/100)} = \frac{12}{100} + \frac{34}{10{,}000}\left(\frac{100}{99}\right) = \frac{12}{100} + \frac{34}{9900}$$

THUS $.12343434\ldots = \frac{12(99) + 34}{9900} = \frac{1222}{9900}$.

A SECOND TYPE OF INFINITE SERIES THAT IS EASY TO ANALYZE IS A <u>TELESCOPING INFINITE SERIES</u> WHICH HAS THE GENERAL FORM

$$S = \sum_{n=1}^{\infty} (a_n - a_{n+1})$$

<u>THEOREM</u> SUPPOSE THAT $\lim_{n\to\infty} a_n = A$. THEN,

$$S = a_1 - A, \quad \text{AND SO } S \text{ IS A CONVERGENT SERIES.}$$

<u>PROOF</u> WE WRITE $S_N = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots + (a_N - a_{N+1})$,

WHERE $S_N \equiv \sum_{n=1}^{N} (a_n - a_{n+1})$ IS N^{TH} PARTIAL SUM. BY CANCELLATION $S_N = a_1 - a_{N+1}$.

NOW LETTING $N \to \infty$, $\lim_{N\to\infty} S_N = a_1 - \lim_{N\to\infty} a_{N+1} = a_1 - A$.

<u>EXAMPLE 1</u> DETERMINE WHETHER $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ CONVERGES. IF IT CONVERGES FIND THE SUM OF THE INFINITE SERIES.

<u>SOLUTION</u> DEFINE $b_n \equiv \frac{1}{n(n+1)}$. WE USE PARTIAL FRACTIONS TO WRITE b_n IN FORM OF A TELESCOPING SERIES. BY PARTIAL FRACTIONS

$$b_n \equiv \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

DEFINE $a_n = \frac{1}{n}$. THEN $b_n = a_n - a_{n+1}$. SINCE $\lim_{n\to\infty} a_n = 0$ WE HAVE BY THEOREM ABOVE THAT $S \equiv \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1}$.

THUS $S \equiv a_1 = 1$. WE CONCLUDE THAT

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

EXAMPLE 1 CALCULATE $\displaystyle\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$.

SOLUTION DEFINE $a_n = \cos\left(\frac{\pi}{n}\right)$ AND N^{TH} PARTIAL SUM, FOR $N>3$,

$$S_N \equiv \sum_{n=3}^{N}(a_n - a_{n+1}) = \sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right).$$

NOW $S_N = (a_3 - a_4) + (a_4 - a_5) + \cdots + (a_N - a_{N+1}) = a_3 - a_{N+1}$.

THUS $\displaystyle\lim_{N\to\infty} S_N = a_3 - \lim_{N\to\infty} a_{N+1} = a_3 - \lim_{N\to\infty}\cos\left(\frac{\pi}{N+1}\right) = a_3 - \cos(0)$.

SO $\displaystyle\lim_{N\to\infty} S_N = \cos\left(\frac{\pi}{3}\right) - \cos(0) = \frac{1}{2} - 1 = -\frac{1}{2}$.

EXAMPLE 2 CALCULATE $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1}\right)$.

SOLUTION WE BREAK INTO TWO PIECES.

DEFINE $\displaystyle S \equiv \sum_{n=2}^{\infty}\frac{2^{n+1}}{3^n}$ geometric series

AND $\displaystyle S' \equiv \sum_{n=2}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$ telescoping series.

- FOR S WE WRITE $\displaystyle S = 2\sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^n = 2\left(\frac{4}{9}\right)\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$

$$S = \frac{8}{9}\left(\frac{1}{1-2/3}\right) = \frac{8}{9}(3) = \frac{24}{9}.$$

- FOR S', DEFINE $a_n = \frac{1}{2n-1}$. THEN $\displaystyle a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+2-1} = \frac{1}{2n+1}$.

THUS LET $\displaystyle S'_N = \sum_{n=2}^{N}(a_n - a_{n+1}) = \sum_{n=2}^{N}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$, FOR $N>2$.

WE HAVE BY TELESCOPING SERIES $S'_N = a_2 - a_{N+1} = \frac{1}{3} - a_{N+1}$. LET $N\to\infty$

SO THAT $S'_N \to 1/3$ AS $N\to\infty$ SINCE $a_{N+1}\to 0$.

THEN $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{24}{9} + \frac{1}{3} = \frac{27}{9} = 3$.

EXAMPLE 3 DISCUSS CONVERGENCE OF INFINITE SERIES $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$.

SOLUTION DEFINE NTH PARTIAL SUM

$S_N \equiv \sum_{n=1}^{N} \log\left(1+\frac{1}{n}\right)$. WE WRITE $S_N = \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = \sum_{n=1}^{N}(\log(n+1) - \log n)$.

LOOKS LIKE A TELESCOPING SERIES, BUT MUST BE CAREFUL. WE CALCULATE

$$S_N = (\log(N+1) - \cancel{\log N}) + (\cancel{\log N} - \log(N-1)) + \dots + (\cancel{\log 2} - \log 1) = \log(N+1) - \log 1.$$

BUT $\log 1 = 0$ SO $S_N = \log(N+1)$. SINCE $\lim_{N\to\infty} S_N$ DOES NOT EXIST WE HAVE THAT $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$ diverges.

IN OUR EXAMPLE $a_n = \log n$ WHICH DOES NOT VANISH OR tend to a limit as $n \to \infty$, (IN $S_N = \sum_{n=1}^{N}(a_{n+1} - a_n)$).

- IN GENERAL IT IS VERY DIFFICULT TO CALCULATE A CONVERGENT INFINITE SUM.
- WE NEED CRITERIA FOR ESTABLISHING WHETHER AN INFINITE SERIES CONVERGES OR DIVERGES.

DIVERGENCE TEST

SUPPOSE THAT THE INFINITE SERIES $\sum_{n=1}^{\infty} a_n$ CONVERGES. THEN WE MUST HAVE THAT $a_n \to 0$ AS $n \to \infty$.

PROOF LET $S_N = \sum_{n=1}^{N} a_n$ BE NTH PARTIAL SUM.

THEN $S_N - S_{N-1} = a_N$

SINCE $\lim_{N\to\infty} S_N = S$ WE MUST ALSO HAVE $\lim_{N\to\infty} S_{N-1} = S$ AND SO

$$\lim_{N\to\infty}(S_N - S_{N-1}) = \lim_{N\to\infty} a_N \Longrightarrow \lim_{N\to\infty} a_N = S - S = 0.$$

SO $a_N \to 0$ AS $N \to \infty$.

SUMMARIZING WE HAVE: $\boxed{\text{IF } \sum_{n=1}^{\infty} a_n \text{ CONVERGES} \implies \lim_{n\to\infty} a_n = 0. \quad (*)}$

THE CONTRAPOSITIVE OF THIS STATEMENT IS A USEFUL TEST FOR DIVERGENCE.

THEOREM SUPPOSE THAT THE SEQUENCE $\{a_n\}$ DOES NOT CONVERGE TO ZERO AS $n \to \infty$ (i.e. $\lim_{n\to\infty} a_n \neq 0$), THEN THE INFINITE SERIES $\sum_{n=1}^{\infty} a_n$ DIVERGES.

REMARK (IMPORTANT) THIS RESULT IS THE CONTRAPOSITIVE OF THE BOXED STATEMENT (*). THE THEOREM HAS NOTHING TO SAY ABOUT THE CASE WHERE $a_n \to 0$ AS $n \to \infty$. THE INFINITE SERIES MAY OR MAY NOT CONVERGE, WE CAN'T TELL WITHOUT MORE DETAILED INFORMATION.

BELOW WE WILL SHOW THAT IF $a_n = 1/n^2$, THEN $\sum_{n=1}^{\infty} 1/n^2 < \infty$ BUT IF $a_n = 1/n$, THEN $\sum_{n=1}^{\infty} 1/n$ diverges.

BOTH HAVE PROPERTY THAT $a_n \to 0$ AS $n \to \infty$.

EXAMPLE 1 DISCUSS THE CONVERGENCE OF THE SERIES $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$.

SOLUTION DEFINE $a_n = \frac{5^n}{3^n + 4^n}$. IF WE CAN SHOW THAT $a_n \not\to 0$ AS $n \to \infty$ THE SERIES DIVERGES BY OUR THEOREM. WE WRITE $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\underbrace{\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n}_{\to 0}}$.

SINCE $r^n \to 0$ AS $n \to \infty$ ONLY WHEN $|r| < 1 \implies \lim_{n\to\infty} a_n \neq 0$.

EXAMPLE 2 EACH OF THE FOLLOWING SERIES DIVERGE SINCE $a_n \not\to 0$ AS $n \to \infty$

(i) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{4n+4}\right)$, (ii) $\sum_{n=1}^{\infty} \cos\left(\frac{\pi n}{n+1}\right)$, (iii) $\sum_{n=1}^{\infty} e^{-\left(\frac{n+1}{2n+1}\right)}$

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{\sqrt{n^4 + 4}}$.

HARMONIC SERIES (OPTIONAL)

CONSIDER THE HARMONIC SERIES $\sum_{n=1}^{\infty} \frac{1}{n}$.

DEFINE $S_N \equiv \sum_{n=1}^{N} \frac{1}{n}$. WE WILL SHOW (BY TRICKERY) THAT S_N DIVERGES (HAS NO LIMITING VALUE) AS $N \to \infty$. WE OBSERVE

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} \geq 1 + \frac{1}{2} + \frac{2}{4} = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{4}{8}} \geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} = 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{4}{8}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{\geq \frac{8}{16}}$$

SO $S_{16} \geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} \geq 1 + \frac{4}{2}$

THUS CONTINUING ON WE GET $S_{2^n} > 1 + \frac{n}{2}$.

LETTING $n \to \infty$ IT FOLLOWS THAT $S_{2^n} \to \infty$ AS $n \to \infty$.

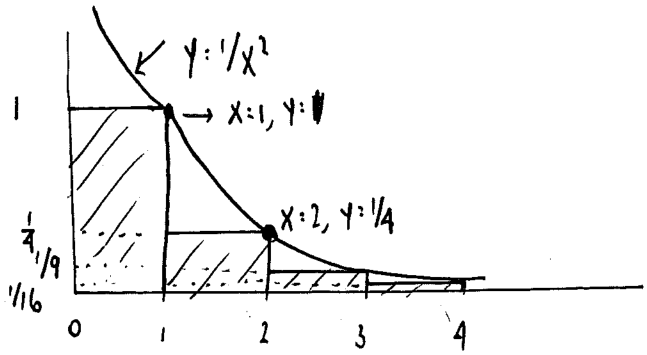
$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

WE NOW PROVIDE A MORE general test (based on integrals) to determine whether a series converges OR NOT.

INTEGRAL TEST

WE BEGIN WITH SEEKING WHETHER $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES OR DIVERGES.

WE FIRST DRAW A VISUAL PICTURE:



WE FIRST DRAW RECTANGLES OF HEIGHT $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25} \ldots$ RESPECTIVELY WITH WIDTH = 1 FOR EACH RECTANGLE.

THEN $\sum_{n=1}^{\infty} \frac{1}{n^2}$ = AREA OF ALL THE RECTANGLES.

NOW THE CURVED LINE IS $y = \frac{1}{x^2}$ WHICH GOES THROUGH TOP RIGHT CORNER OF EACH RECTANGLE.

FROM THE PICTURE $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + \lim_{b\to\infty} \int_1^b x^{-2}\,dx$

THUS, $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b = 2.$ WE CONCLUDE THAT $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$

$\Longrightarrow$ THE SERIES CONVERGES.

THIS METHOD ONLY PROVIDES A <u>BOUND</u> ON THE INFINITE SERIES AND NOT SUM SPECIFIC VALUE FOR INFINITE SERIES (IT IS <u>NOT</u> A RIEMANN SUM).

IN MATH 316 YOU LEARN THAT $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ SO $\frac{\pi^2}{6} < 2.$

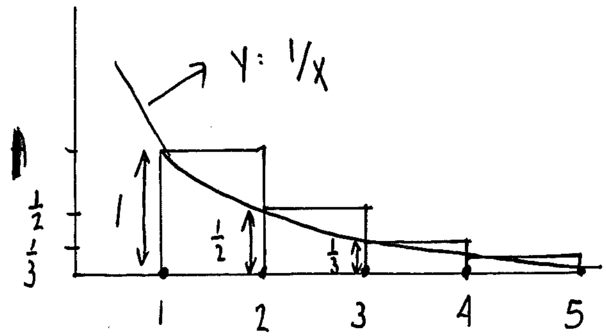
THE LITTLE IDEA HERE IS THE FOLLOWING:

IF WE ANTICIPATE THAT $\sum_{n=1}^{\infty} a_n$ CONVERGES AND $a_n > 0 \;\forall n$,

TRY TO WRITE $\sum_{n=1}^{\infty} a_n <$ INTEGRAL THAT CONVERGES.

NOW CONSIDER HARMONIC SERIES $\sum_{n=1}^{\infty} \frac{1}{n}$ WHICH WE HAVE SHOWN EARLIER DIVERGES. WE TRY TO WRITE $\sum_{n=1}^{\infty} \frac{1}{n} >$ divergent integral.

WE CONSIDER THE FOLLOWING PICTURE:



$$\sum_{n=1}^{\infty} \frac{1}{n} = \text{AREA OF RECTANGLES ABOVE} > \int_1^{\infty} \frac{1}{x}\, dx.$$

THUS $$\sum_{n=1}^{\infty} \frac{1}{n} > \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx = \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

THUS $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

WE NOW CAN GENERALIZE THESE OBSERVATIONS IN THE FORM OF AN "INTEGRAL TEST" FOR CONVERGENCE OR DIVERGENCE.

<u>THEOREM</u> (INTEGRAL TEST - CLP 3.3.5). LET N_0 BE A POSITIVE INTEGER AND LET $f(x)$ BE A CONTINUOUS FUNCTION FOR ALL $x \geq N_0$. FURTHERMORE, ASSUME THAT

(i) $f(x) \geq 0$ FOR ALL $x \geq N_0$, AND

(ii) $f(x)$ DECREASES AS x INCREASES, AND

(iii) $f(n) = a_n$ FOR ALL $n \geq N_0$.

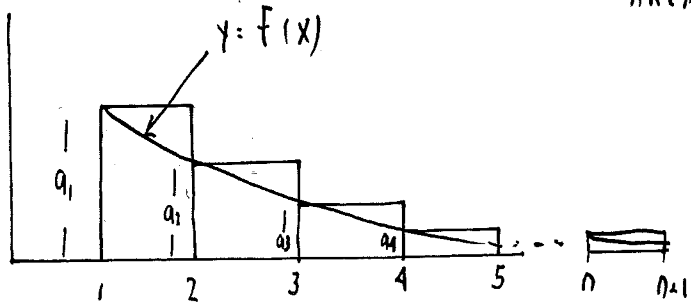
THEN,

(I) IF $\int_{N_0}^{\infty} f(x)\, dx$ IS CONVERGENT, THEN $\sum_{n=1}^{\infty} a_n$ IS CONVERGENT.

(II) IF $\int_{N_0}^{\infty} f(x)\, dx$ diverges, THEN $\sum_{n=1}^{\infty} a_n$ diverges.

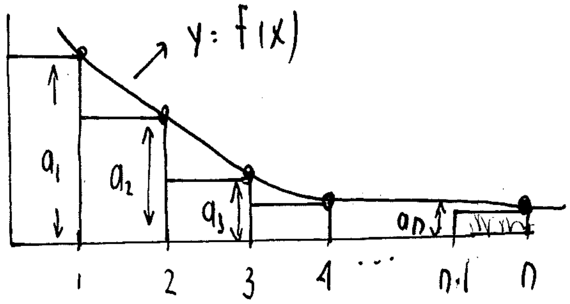
PROOF WE SUPPOSE THAT $N_0 = 1$ SINCE ALL WE NEED IS TO ESTIMATE THE TAIL BEHAVIOR OF THE SERIES $\sum_{n=N_0}^{\infty} a_n$ TO ESTABLISH CONVERGENCE OR DIVERGENCE.

WE DRAW TWO PICTURES:



- AREA OF RECTANGLES ARE
$$a_1 + a_2 + \dots + a_n \geq \int_1^{n+1} f(x)\,dx$$

NOW DRAW



. AREA OF RECTANGLES ARE
$$a_1 + a_2 + \dots + a_n \leq a_1 + \int_1^{n} f(x)\,dx.$$

NOW COMBINING THESE RESULTS WE GET
$$\int_1^{n+1} f(x)\,dx \leq a_1 + \dots + a_n \leq a_1 + \int_1^{n} f(x)\,dx \qquad (*)$$

THUS IF $\int_1^{\infty} f(x)\,dx$ IS FINITE $\longrightarrow \sum_{n=1}^{\infty} a_n < \infty$.

IF $\int_1^{\infty} f(x)\,dx$ IS INFINITE, THEN $\int_1^{n+1} f(x)\,dx < \sum_{n=1}^{\infty} a_n \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 1 DOES $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ diverge or converge?

SOLUTION DEFINE $f(x) = \frac{1}{x^{3/2}}$. THEN FOR $x > 1$, $f(x) > 0$ AND $f(x)$ IS DECREASING.

WE OBSERVE THAT $\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^{b} x^{-3/2}\,dx = \lim_{b\to\infty} -2x^{-1/2} \Big|_1^{b} = 2.$

SINCE $\int_1^{\infty} f(x)\,dx = 2 < \infty \Longrightarrow$ BY (I) OF THEOREM $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges

MOREOVER, $\sum_{n=1}^{\ell} \frac{1}{n^{3/2}} \leq a_1 + \int_1^\infty f(x)\,dx = 1 + 2 = 3.$

EXAMPLE FOR WHAT VALUES OF $p>0$ DOES $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGE OR DIVERGE. (P-TEST)

SOLUTION DEFINE $f(x) = \frac{1}{x^p}$ FOR $p>0$ AND $x \geq 1$.

THEN $\int_1^\infty x^{-p}\,dx$ CONVERGES IF $p>1$ AND DIVERGES IF $p \leq 1$

THUS, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGES IF $p>1$ AND DIVERGES IF $0 < p \leq 1$. (INTEGRAL TEST)

TRIVIALLY, THE SERIES DIVERGES IF $p \leq 0$ SINCE $a_n \not\to 0$ AS $n \to \infty$ FOR $p \leq 0$.

REMARK BY SERIES P-TEST, $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ CONVERGES BUT $\sum_{n=1}^{\infty} \frac{1}{n^{.99}}$ DIVERGES.

ANOTHER (CONVENIENT) WAY TO WRITE THE INTEGRAL TEST IS THE FOLLOWING:

THEOREM IF $f(x)$ IS CONTINUOUS, POSITIVE, AND DECREASING ON $[N_0, \infty)$ AND $f(n) = a_n$, THEN

(I)′ IF $\int_{N_0}^\infty f(x)\,dx$ IS CONVERGENT, SO IS $\sum_{n=N_0}^{\infty} a_n$

(II)′ IF $\int_{N_0}^\infty f(x)\,dx$ IS DIVERGENT, SO IS $\sum_{n=N_0}^{\infty} a_n$.

REMARK SINCE $\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{N_0-1} a_n}_{\text{FINITE}} + \sum_{n=N_0}^{\infty} a_n$, THIS RESULT APPLIES TO $\sum_{n=1}^{\infty} a_n$ AS WELL.

SO ALL WE NEED IS THAT $f(x)$ IS DECREASING FOR $x \geq N_0$; $f(x)$ COULD BE INCREASING ON $1 \leq x < N_0$.

EXAMPLE 1 DISCUSS CONVERGENCE /DIVERGENCE OF $\sum_{n=1}^{\infty} \frac{\log n}{n}$.

SOLUTION DEFINE $f(x) = x^{-1} \ln x$. THEN $f'(x) = -x^{-2} \ln x + x^{-2} = x^{-2}(1 - \ln x)$.

SO $f'(x) < 0$ FOR $x > e$ AND $f(x) > 0$ ON $x \geq 1$.

SINCE $2.71\ldots = e < 3$ WE TAKE $N_0 = 3$ IN OUR THEOREM AND WRITE
$$\sum_{n=1}^{\infty} \frac{\log n}{n} = \frac{\log(1)}{1} + \frac{\log(2)}{2} + \sum_{n=3}^{\infty} \frac{\log n}{n}.$$
WE THEN SEE IF $I = \int_3^{\infty} \frac{\log x}{x}\, dx$ converges or diverges.
$$I = \lim_{L\to\infty} \int_3^{L} \frac{\log x}{x}\, dx = \lim_{L\to\infty} \int_{\log 3}^{\log L} u\, du = \frac{1}{2} \lim_{L\to\infty} u^2 \Big|_{\log 3}^{\log L} = \infty.$$
$-u = \log x$

THUS $\int_3^{\infty} \frac{\log x}{x}\, dx$ diverges. BY $(\mathrm{II})'$ OF THEOREM $\Longrightarrow \sum_{n=3}^{\infty} \frac{\log n}{n}$ diverges.

HENCE $\sum_{n=1}^{\infty} \frac{\log n}{n}$ is divergent as well.

EXAMPLE 2 DISCUSS CONVERGENCE/DIVERGENCE OF $\sum_{n=1}^{\infty} n e^{-n^2}$.

SOLUTION DEFINE $f(x) = x e^{-x^2}$.

THEN $f'(x) = e^{-x^2}(1 - 2x^2) < 0$ IF $x > \frac{1}{\sqrt{2}}$. SINCE $\frac{1}{\sqrt{2}} < 1$

WE HAVE $f'(x) < 0$ ON $x \geq 1$ AND $f(x) > 0$ ON $x > 1$. WE USE INTEGRAL TEST WITH $N_0 = 1$. WE CALCULATE $\int_1^{\infty} x e^{-x^2} dx = \lim_{L\to\infty} \int_1^{L} x e^{-x^2} dx = \lim_{L\to\infty} \left(\frac{-1}{2}\right) e^{-x^2} \Big|_1^{L} = \frac{1}{2} e^{-1} < \infty$.

THUS $\int_1^{\infty} x e^{-x^2} dx < \infty \Longrightarrow \sum_{n=1}^{\infty} n e^{-n^2}$ IS FINITE (CONVERGES) BY PART $(\mathrm{I})'$ OF THEOREM.

EXAMPLE 3 FOR WHAT VALUES OF p WITH $p \geq 0$ DOES $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converge?

SOLUTION DEFINE $f(x) = x^{-1} [\log x]^{-p}$. THEN $f'(x) = -x^{-2} (\log x)^{-p} - p x^{-2} (\log x)^{-p-1}$.

NOW $f'(x) = x^{-2} (\log x)^{-p} \left[-1 - \frac{p}{\log x} \right] < 0$ IF $x \geq 2$. TAKE $N_0 = 2$ AND

CONSIDER $I = \int_2^{\infty} \frac{1}{x (\log x)^p}\, dx = \lim_{L\to\infty} \int_2^{L} \frac{1}{x(\log x)^p}\, dx$. let $u = \log x$.
$du = {}^1/_x\, dx$

SO $I = \lim_{L\to\infty} \int_{\log 2}^{\log L} \frac{1}{u^p}\, du$ BUT $\int_a^{\infty} \frac{1}{u^p}\, du$ IS FINITE ONLY IF $p > 1$.
(FROM IMPROPER integral Notes)

THUS, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges only if $p > 1$.

FINALLY, WE WOULD LIKE TO ESTIMATE THE REMAINDERS IN "CALCULATING" CONVERGENT SERIES BY TAKING N TERMS IN A PARTIAL SUM.

ESTIMATING REMAINDERS

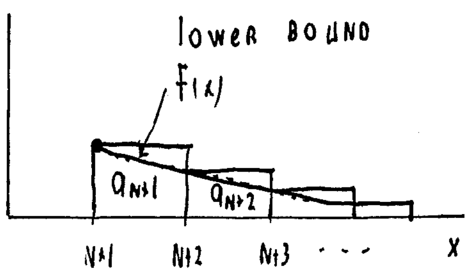
SUPPOSE $f(x)$ IS DECREASING ON $x \geq N$, $f > 0$ ON $x \geq N$ AND $f(n) = a_n$. SUPPOSE THAT $S = \sum_{n=1}^{\infty} a_n < \infty$ (CONVERGES). DEFINE R_N BY

$$R_N \equiv S - S_N \quad \text{WITH} \quad S_N = \sum_{n=1}^{N} a_n.$$

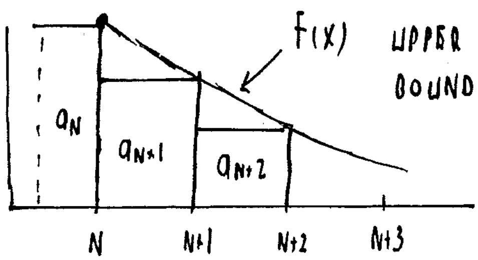
THEN WE HAVE $R_N = a_{N+1} + a_{N+2} + \cdots$ AND THE ESTIMATE

$$(**) \quad \int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$$

OUTLINE OF PROOF



$$\int_{N+1}^{\infty} f(x)\,dx < a_{N+1} + a_{N+2} + \cdots$$



$$a_{N+1} + a_{N+2} + \cdots < \int_{N}^{\infty} f(x)\,dx.$$

COMBINING LOWER + UPPER BOUNDS WE GET

$$\int_{N+1}^{\infty} f(x)\,dx < a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)\,dx$$

EXAMPLE USING A COMPUTER WE ESTIMATE $\sum_{n=1}^{100} \frac{1}{n^2} = 1.634984$.

DETERMINE A BOUND (lower AND upper) FOR THE REMAINDER $\sum_{n=101}^{\infty} \frac{1}{n^2}$.

SOLUTION HERE $a_n = \frac{1}{n^2}$ AND $N = 100$. WE WANT TO ESTIMATE $R_N \equiv a_{N+1} + a_{N+2} + \cdots$

FROM OUR FORMULA $(**)$ WE HAVE $\int_{101}^{\infty} \frac{1}{x^2}\,dx < R_{100} < \int_{100}^{\infty} \frac{1}{x^2}\,dx$.

CALCULATING THE INTEGRALS WE GET $\frac{1}{101} < R_{100} < \frac{1}{100}$ AND SO THE TAIL OF THE SERIES WILL ONLY influence second decimal point IN 1.634984..