First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int \sin^3(x)\,dx$.

Answer: $\frac{1}{3}\cos^3(x) - \cos(x) + C$

Solution: Use $\sin^2(x) = 1 - \cos^2(x)$ to convert the integral to $I = \int \big(1 - \cos^2(x)\big)\sin(x)\,dx$. Then let $u = \cos(x)$, so that $du = -\sin(x)dx$, and the integral is

$$I = \int (1-u^2)(-du) = -u + \frac{1}{3}u^3 + C\,.$$

Substituting $\cos(x)$ back for u gets the answer $I = \frac{1}{3}\cos^3(x) - \cos(x) + C$.

(b) Calculate the indefinite integral $\int \frac{1}{x(\ln x)^2}\,dx$ for $x > 0$.

Answer: $-\frac{1}{\ln x} + C$

Solution: Let $u = \ln x$ so $du = \frac{1}{x}dx$. The integral becomes

$$I = \int \frac{1}{u^2}du = -\frac{1}{u} + C\,.$$

Set $u = \ln(x)$ to get that $I = -\frac{1}{\ln x} + C$.

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{\sqrt{x^2-25}}{x}\, dx$ for $x > 5$.

Answer: $\sqrt{x^2-25} - 5\text{arcsec}(x/5) + C$

Solution: Use the trig substitution $x = 5\sec(\theta)$, so that $dx = 5\sec\theta\tan\theta d\theta$. The integral becomes

$$I = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta} 5\sec\theta\tan\theta d\theta = \int 5\tan^2\theta\, d\theta\,.$$

With the help of the identity $\tan^2\theta + 1 = \sec^2\theta$ this becomes

$$5\int(\sec^2\theta - 1)\, d\theta = 5\tan\theta - 5\theta + C.$$

When substituting back $x = 5\sec(\theta)$ we can make a triangle with hypotenuse x and adjacent side length 5, so that $\tan\theta = \frac{\sqrt{x^2-5^2}}{5}$. The final answer is $I = \sqrt{x^2-25} - 5\text{arcsec}(x/5) + C$.

Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{\pi/8} \tan^5(2x)\sec^2(2x)\,dx$.

Answer: $\frac{1}{12}$

Solution: Let $u = 2x$, $du = 2dx$, and the endpoints to the integral are now $u = 0$ and $u = \pi/4$. Then, we calculate

$$\int_0^{\pi/4} \tan^5(u)\sec^2(u)\frac{du}{2}\,.$$

With the substitution $w = \tan u$, we get $dw = \sec^2(u)\,du$, and the endpoints are now $w = 0$ and $w = 1$. The integral is

$$\frac{1}{2}\int_0^1 w^5 dw = \frac{1}{2}\Big[\frac{1}{6}w^6\Big]_0^1 = \frac{1}{12}\,.$$

(b) Calculate $\int_{-2}^{-1} \frac{1}{(x+2)^2+1}\,dx$.

Answer: $\frac{\pi}{4}$

Solution: Let $x+2 = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. The end-points are $-2+2 = \tan\theta \Rightarrow \theta = 0$ and $-1+2 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$. The integral is

$$\int_0^{\frac{\pi}{4}} \frac{1}{\tan^2\theta + 1}\sec^2 d\theta = \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}\,.$$

(c) (A Little Harder): Calculate $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

Answer: $\frac{2}{15}$

Solution: Method 1: Let $x = \sin(\theta)$, so that $dx = \cos(\theta)\,d\theta$. The end-points $x = 0$ and $x = 1$ become $\theta = 0$ and $\theta = \pi/2$. The integral becomes

$$\int_0^{\frac{\pi}{2}} \sin^3\theta\sqrt{1-\sin^2\theta}\cos(\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,d\theta\,.$$

Since the sin has an odd power we use the identity $\sin^2\theta = 1 - \cos^2\theta$ to get

$$\int_0^{\frac{\pi}{2}} \sin(\theta)(1-\cos^2\theta)\cos^2\theta\,d\theta\,.$$

Anti-differentiate with the substitution $w = \cos\theta$, $dw = -\sin(\theta)d\theta$ to get

$$\left[\frac{-1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}\,.$$

Method 2: Write the integral as

$$I \equiv \int_0^1 x^2\sqrt{1-x^2}\,(xdx)$$

Set $u = 1 - x^2$, so that $xdx = -du/2$. Since $x = 0$ and $x = 1$ map to $u = 1$ and $u = 0$, we use $x^2 = 1 - u$ and get

$$I = -\frac{1}{2}\int_1^0 (1-u)u^{1/2}\,du = \frac{1}{2}\int_0^1 \left(u^{1/2} - u^{3/2}\right)\,du = \frac{1}{2}\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{2}{15}\,.$$

Riemann Sum, FTC, and Volumes

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n^2}e^{-i^2/n^2}$$

by first writing it as a definite integral. Then, **evaluate this integral.**

Answer: $\int_0^1 2xe^{-x^2}\,dx = -\frac{1}{e}+1 = 1-e^{-1}$.

Solution: We identify $a=0$, $b=1$, $\Delta x=\frac{1}{n}$, $x_i=\frac{i}{n}$, and $f(x_i)=2x_ie^{-x_i^2}$. This yields

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n^2}e^{-i^2/n^2}=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x=\int_0^1 2xe^{-x^2}\,dx\,.$$

To calculate the integral we let $u=x^2$, so that $du=2dx$. The end-points in terms of u are 0 and 1. Then

$$S=\int_0^1 e^{-u}\,du=\left[-e^{-u}\right]_0^1=-\frac{1}{e}+1=1-e^{-1}\,.$$

(b) Define $F(x)$ and $g(x)$ by $F(x)=\int_0^x e^{-t}\,dt$ and $g(x)=\sqrt{F(x^2)}$. Calculate $g'(2)$.

Answer: $\frac{2e^{-4}}{\sqrt{1-e^{-4}}}$

Solution: The chain rule implies that

$$g'(x)=\frac{1}{2\sqrt{F(x^2)}}F'(x^2)(2x)\,.$$

By the fundamental theorem of calculus, $F'(x^2)=e^{-x^2}$. We can calculate

$$F(x)=\left[-e^{-t}\right]_0^x=-e^{-x}+1\,.$$

Together we have

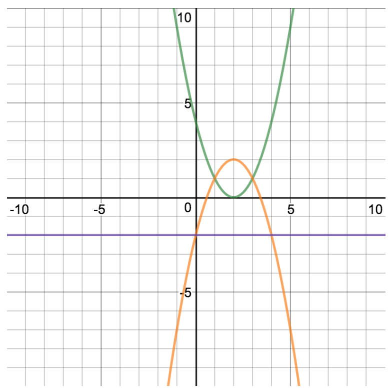
$$g'(x)=\frac{xe^{-x^2}}{\sqrt{-e^{-x^2}+1}}\,.$$

Evaluating at $x=2$ we get $g'(2)=\frac{2e^{-4}}{\sqrt{1-e^{-4}}}$. Alternatively, we could first compute $g(x)=\sqrt{-e^{-x^2}+1}$ and use the chain rule to differentiate.

(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = (x-2)^2$ and $y = 2-(x-2)^2$ about the horizontal line $y = -2$. **Do not evaluate the integral**.

Answer: $\pi \int_1^3 \left[\left(4-(x-2)^2\right)^2 - \left((x-2)^2+2\right)^2\right] dx$

Solution:

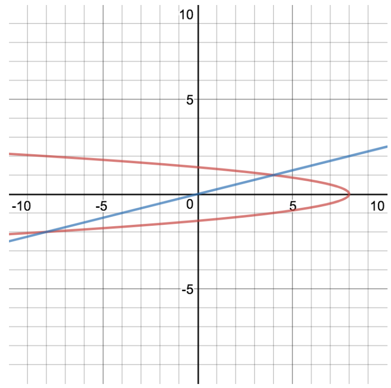


The curves intersect when $(x-2)^2 = 2-(x-2)^2$ which occurs at $(x-2)^2 = 1$ so that $x-2 = \pm 1$. This gives $x = 1$ and $x = 3$. The outer radius is $r_2(x) = 2-(x-2)^2-(-2) = 4-(x-2)^2$ and the inner radius is $r_1(x) = (x-2)^2-(-2)$. The volume of revolution is given by the formula

$$V = \pi \int_1^3 \left(r_2^2(x) - r_1^2(x)\right) dx\,.$$

4. (a) [2 marks] Plot the finite area enclosed by $4y^2 = 8 - x$ and $y = x/4$.

Solution: The area is the enclosed region between the blue and red curves:



(b) [4 marks] Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral**.

Answer: $\int_{-2}^{1} (8 - 4y^2 - 4y)\ dy$.

Solution: To find the intersection points we set $x = 4y$ and $4y^2 = 8 - 4y$. This yields

$$0 = 4(y^2 + y - 2) = 4(y + 2)(y - 1)\,,$$

so that $y = 1$ and $y = -2$. We label $x_T = 8 - 4y^2$ (red curve) and $x_B = 4y$ (blue curve), and observe that $x_T > x_B$ on $-2 < y < 1$. The area is best calculated as an integral in y, so that $A = \int_{-2}^{1} (x_T - x_B)\ dy = \int_{-2}^{1} (8 - 4y^2 - 4y)\ dy$.

5. A solid has as its base the region in the xy-plane between $y = 1 - x^2/16$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are isosceles right triangles (i.e. $45 - 45 - 90$ triangles) with the longest side (i.e. the hypoteneuse) in the base.

(a) 4 marks Write a definite integral that determines the volume of the solid.

Answer: $\frac{1}{4}\int_{-4}^{4}\left(1 - \frac{x^2}{16}\right)^2 dx$.

Solution: The intersection points with the x-axis are $x = \pm 4$. This gives, $V = \int_{-3}^{3} A(x)\,dx$ as the volume, where $A(x)$ is the cross-sectional area of the solid at position x. This cross-section is a $45 - 45 - 90$ triangle that has area $A(x) = [y(x)]([y(x)]/2)/2 = [y(x)]^2/4$. Here we have used the fact that the area of a $45-45-90$ triangle with baselength b is $bh/2$ where $h = b/2$ is the altitude of the triangle. This gives, $V = \frac{1}{4}\int_{-4}^{4}\left(1 - \frac{x^2}{16}\right)^2 dx$.

(b) 2 marks **Evaluate the integral** to find the volume of the solid.

Answer: $\frac{16}{15}$.

Solution: By symmetry we compute twice the volume between 0 and 4,

$$V = \frac{1}{2}\int_0^4 \left(1 - \frac{x^2}{16}\right)^2 dx\,.$$

We use $x = 4u$ so that $dx = 4\,du$ while $x = 0$ and $x = 4$ map to $u = 0$ and $u = 1$, respectively. This yields $V = 2\int_0^1 (1-u^2)^2\,du$, so that

$$V = 2\int_0^1 \left(1 - 2u^2 + u^4\right)\,du = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\frac{(15 - 10 + 3)}{15} = \frac{16}{15}\,.$$