First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int \frac{3x}{x+4}\,dx$.

Answer: $I = 3x - 12\ln|x+4| + C$

Solution: We first write

$$I = 3\int \frac{x}{x+4}\,dx = 3\int \left[1 - \frac{4}{x+4}\right]\,dx = 3x - 12\ln|x+4| + C\,.$$

(b) Calculate the indefinite integral $\int \arctan(x)\,dx$.

Answer: $I = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$

Solution: Let $u = \arctan(x)$ and $dv/dx = 1$. We calculate $du/dx = 1/(1+x^2)$ and $v = x$, so that one step of integration by parts gives

$$I = uv - \int v\frac{du}{dx}\,dx = x\arctan(x) - \int \frac{x}{(1+x^2)}\,dx\,.$$

In the integral, we let $u = 1 + x^2$ so that $x\,dx = du/2$. We integrate to get

$$I = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C\,.$$

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{1}{x\sqrt{x^2-1}}\,dx$ for $x > 1$.

Answer: $I = \text{arcsec}(x) + C = \arctan(\sqrt{x^2-1}) + C$.

Solution: Let $x = \sec\theta$ so that $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \tan\theta$ if $0 < \theta < \pi/2$. We calculate

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta = \int (1)d\theta = \theta + C$$

Now $\theta = \text{arcsec}(x)$ or $\theta = \arctan(\sqrt{x^2-1})$.

Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{\pi/4} \tan^2(x)\,dx$

Answer: $1 - \frac{\pi}{4}$

Solution: We use $\tan^2(x) = \sec^2(x) - 1$ to get

$$\int_0^{\pi/4} \tan^2(x)\ dx = \int_0^{\pi/4} \sec^2(x)\ dx - \int_0^{\pi/4} 1\,dx = \tan(x)\ |_0^{\pi/4} - x\ |_0^{\pi/4}\ .$$

Since $\tan(\pi/4) = 1$, this yields that $\int_0^{\pi/4} \tan^2(x)\,dx = 1 - \frac{\pi}{4}$.

(b) Calculate $\int_{-\pi}^{\pi} (1 + x^3) \cos^2(x)\,dx$.

Answer: π

Solution: $\int_{-\pi}^{\pi} (1 + x^3) \cos^2(x)\,dx = \int_{-\pi}^{\pi} \cos^2(x)\,dx + \int_{-\pi}^{\pi} x^3 \cos^2(x)\,dx$
Since $x^3 \cos^2(x)$ is an odd function on a symmetric interval the second term evaluates to zero. Then, by using $\cos^2(x) = 1/2 + \cos(2x)/2$ we get

$$\begin{aligned} \int_{-\pi}^{\pi} \cos^2(x)\,dx &= \int_{-\pi}^{\pi} \frac{1}{2}\,dx + \int_{-\pi}^{\pi} \frac{\cos(2x)}{2}\,dx \\ &= \pi + \frac{\sin(2x)}{4}\ |_{-\pi}^{\pi} \\ &= \pi + 0 = \pi\ . \end{aligned}$$

(c) (A Little Harder): Calculate $\int_0^\infty e^{-x}\cos(x)\,dx$.

Answer: $\frac{1}{2}$

Solution: Define $I = \int e^{-x}\cos(x)\,dx$.
We use integration by parts: We let $u = e^{-x}$ and $dv/dx = \cos(x)$ so that $v = \sin(x)$ and $u = -e^{-x}$. This gives

$$I = e^{-x}\sin(x) - \int -e^{-x}\sin(x)\,dx = e^{-x}\sin(x) + \int e^{-x}\sin(x)\,dx\,.$$

In the second integral substitute $u = e^{-x}$ and $dv/dx = \sin(x)$ so that $v = -\cos(x)$ and $du/dx = -e^{-x}$. Then,

$$\begin{aligned}
I &= e^{-x}\sin(x) + \left[e^{-x}(-\cos(x)) - \int (-1)e^{-x}(-\cos(x))\,dx\right] \\
&= e^{-x}\sin(x) + \left[-e^{-x}\cos(x) - \int e^{-x}\cos(x)\,dx\right] \\
&= e^{-x}\sin(x) + \left[-e^{-x}\cos(x) - I\right] \\
&= e^{-x}\sin(x) - e^{-x}\cos(x) - I \\
2I &= e^{-x}(\sin(x) - \cos(x)) \\
I &= \frac{1}{2}e^{-x}\left(\sin(x) - \cos(x)\right)
\end{aligned}$$

Since $\lim_{n\to\infty} e^{-n} = 0$ and $\sin(n) - \cos(n)$ is bounded (is at most 2) we have $\lim_{n\to\infty} I(n) = 0$. This gives,

$$\int_0^\infty e^{-x}\cos(x) = \lim_{n\to\infty}\int_0^n e^{-x}\cos(x) = \lim_{n\to\infty}(I(n) - I(0)) = -I(0) = 1/2\,.$$

Riemann Sum, FTC, and Volumes

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{8i}{n^2}\sqrt{1+\frac{4i^2}{n^2}}$$

by first writing it as a definite integral. Then, **evaluate this integral**.

Answer: $\frac{2}{3}(5\sqrt{5}-1)$

Solution: We let $\Delta x = 1/n$ and $x_i = i/n$ so that $a = 0$ and $b = 1$. Then, $f(x_i) = 8x_i\sqrt{1+4x_i^2}$. This yields that the Riemann is $\int_0^1 8x\sqrt{1+4x^2}\,dx$. Let $u = 1+4x^2$ so that $du = 8x\,dx$. When $x = 0$ then $u = 1$ and when $x = 1$ then $u = 5$. This gives,

$$I = \int_1^5 u^{1/2}\,du = \frac{2}{3}u^{3/2}|_1^5 = \frac{2}{3}\left(5\sqrt{5}-1\right).$$

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_0^x \cos^2(t)\,dt$ and $g(x) = x\,F(x^2)$. Calculate $g'(\sqrt{\pi})$.

Answer: $5\pi/2$

Solution: $g'(x) = xF'(x^2)(2x)+F(x^2) = 2x^2\cos^2(x^2)+F(x^2)$. We get $g'(\sqrt{\pi}) = 2\pi\cos^2(\pi)+F(\pi)$, and then calculate $F(\pi)$ as

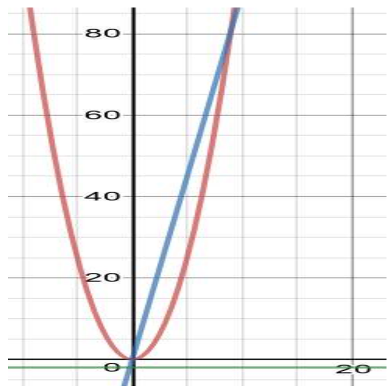
$$F(\pi) = \int_0^\pi \cos^2(t)\,dt = \int_0^\pi \frac{1}{2}\,dt + \int_0^\pi \frac{\cos(2t)}{2}\,dt = \frac{\pi}{2} + \frac{\sin(2t)}{4}|_0^\pi = \frac{\pi}{2}.$$

Since $\cos^2(\pi) = 1$, this yields that $g'(\sqrt{\pi}) = 5\pi/2$.

(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = x^2$ and $y = 9x$ about the horizontal line $y = -2$. **Do not evaluate the integral**.

Answer: $\pi \int_0^9 \left[(9x+2)^2 - (x^2+2)^2\right]\, dx$

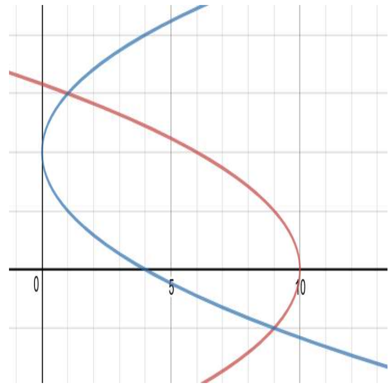
Solution:



The two curves intersect when $x^2 = 9x$, which yields $x = 0$ and $x = 9$. Define $y_T = 9x$ (top blue curve) and $y_B = x^2$ (bottom red curve), so that $y_T > y_B$ on $[0, 9]$. Then, at each x in $[0, 9]$, we have that $(y_T + 2)$ and $(y_B + 2)$ are the distances of the two curves from the axis of rotation $y = -2$ shown by the orange curve. This yields $V = \pi \int_1^5 \left[(y_T+2)^2 - (y_B+2)^2\right]\, dx = \pi \int_0^9 \left[(9x+2)^2 - (x^2+2)^2\right]\, dx$.

4. (a) 2 marks Plot the finite area enclosed by $y^2 = 10 - x$ and $x = (y-2)^2$.

Solution:
The area is the enclosed region between the blue and red curves:



The curves (as a function of y) are $x = 10 - y^2$ (red curve) and $x = (y-2)^2$ (blue curve), and they intersect when

$$10 - y^2 = (y-2)^2 \quad \rightarrow \quad 0 = 2y^2 - 4y - 6 \quad \rightarrow \quad 0 = (y-3)(y+1)\,.$$

This gives $y = 1$ and $y = 3$, corresponding to $x = 9$ and $x = 1$.

(b) 4 marks Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral**.

Answer: $\int_{-1}^{3} \left[(10 - y^2) - (y-2)^2\right]\, dy$

Solution: We label $x_T = 10 - y^2$ (red curve) and $x_B = (y-2)^2$ (blue curve), and observe that $x_T > x_B$ on $-1 < y < 3$. The area is best represented as an integral in y: we get $\int_{-1}^{3} \left[(10 - y^2) - (y-2)^2\right]\, dy$.

5. A solid has as its base the region in the xy-plane between $y = 1 - x^2/16$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are semi-circles with the diameter of the semi-circle in the base.

(a) 4 marks Write a definite integral that determines the volume of the solid.

Answer: $\frac{\pi}{8}\int_{-4}^{4}\left(1-\frac{x^2}{16}\right)^2\,dx$

Solution: For a cross-section along the $y-z$ plane we obtain a semi-circle with diameter $1-x^2/16$ which means the area $A(x)$ of the semi-circle is $\frac{\pi}{2}\left(\frac{1}{2}(1-\frac{x^2}{16})\right)^2$. Thus, the volume of the solid is $V = \int_{-4}^{4} A(x)\,dx$. This yields that

$$V = \frac{\pi}{8}\int_{-4}^{4}\left(1-\frac{x^2}{16}\right)^2\,dx\,.$$

(b) 2 marks **Evaluate the integral** to find the volume of the solid.

Answer: $\frac{13}{15}\pi$

Solution: Let $x = 4u$. Then, $dx = 4du$, so that using symmetry

$$V = \frac{\pi}{8}\int_{-1}^{1}(1-u^2)^2\,(4du) = \pi\int_{0}^{1}(1-u^2)^2\,du = \pi\int_{0}^{1}\left(1-2u^2+u^4\right)\,du\,.$$

This yields

$$V = \pi\left(1-\frac{1}{3}+\frac{1}{5}\right) = \frac{\pi}{15}(15-5+3) = \frac{13\pi}{15}\,.$$