ASYMPTOTIC STABILITY OF HARMONIC MAPS UNDER THE SCHRÖDINGER FLOW

STEPHEN GUSTAFSON, KYUNGKEUN KANG, and TAI-PENG TSAI

Abstract

For Schrödinger maps from $\mathbb{R}^2 \times \mathbb{R}^+$ *to the 2-sphere* $\mathbb{S}^2$*, it is not known if finite energy solutions can form singularities (blow up) in finite time. We consider equivariant solutions with energy near the energy of the two-parameter family of equivariant harmonic maps. We prove that if the topological degree of the map is at least four, blowup does not occur, and global solutions converge (in a dispersive sense, i.e., scatter) to a fixed harmonic map as time tends to infinity. The proof uses, among other things, a time-dependent splitting of the solution, the* generalized Hasimoto transform*, and Strichartz (dispersive) estimates for a certain two space–dimensional linear Schrödinger equation whose potential has critical power spatial singularity and decay. Along the way, we establish an energy-space local well-posedness result for which the existence time is determined by the length scale of a nearby harmonic map.*

Contents

DUKE MATHEMATICAL JOURNAL
Vol. 145, No. 3, © 2008 DOI 10.1215/00127094-2008-058
Received 14 May 2007. Revision received 26 December 2007.
2000 *Mathematics Subject Classification.* Primary 35Q55; Secondary 35B40.
Gustafson's work supported in part by a Natural Sciences and Engineering Research Council of Canada grant.
Kang's work supported in part by Korea Research Foundation grant KRF-2006-003-C00020 and by a Pacific Institute for the Mathematical Sciences postdoctoral fellowship at the University of British Columbia, Vancouver.
Tsai's work supported in part by a Natural Sciences and Engineering Research Council of Canada grant.

1. Introduction and main results

The *Schrödinger flow* for maps from $\mathbb{R}^n$ to $\mathbb{S}^2$ (also known as the *Schrödinger map*, and, in ferromagnetism, as the *Heisenberg model* or *Landau-Lifshitz equation*) is given by the equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \Delta \mathbf{u}, \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x). \tag{1.1}$$

Here, $\mathbf{u} = \mathbf{u}(x, t)$ is the unknown map from $\mathbb{R}^n \times \mathbb{R}^+$ to the 2-sphere

$$\mathbb{S}^2 := \left\{\mathbf{u} \in \mathbb{R}^3 \,\middle|\, |\mathbf{u}| = 1\right\} \subset \mathbb{R}^3;$$

Δ denotes the Laplacian in $\mathbb{R}^n$, and $\times$ denotes the cross product of vectors in $\mathbb{R}^3$. A somewhat more geometric way of writing equation (1.1) is

$$\frac{\partial \mathbf{u}}{\partial t} = J P \Delta \mathbf{u}, \tag{1.2}$$

where $P = P^{\mathbf{u}}$ denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_{\mathbf{u}}\mathbb{S}^2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 \mid \boldsymbol{\xi} \cdot \mathbf{u} = 0\}$$

to $\mathbb{S}^2$ at $\mathbf{u}$ (so that $P\Delta\mathbf{u} = \Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u}$) and

$$J = J^{\mathbf{u}} := \mathbf{u}\times$$

is a rotation through $\pi/2$ on the tangent plane $T_{\mathbf{u}}\mathbb{S}^2$.

On the one hand, equation (1.1) is a borderline case of the *Landau-Lifshitz-Gilbert* equations that model dynamics in isotropic ferromagnets (including dissipation):

$$\frac{\partial \mathbf{u}}{\partial t} = aP\Delta\mathbf{u} + bJP\Delta\mathbf{u}, \quad a \geq 0 \tag{1.3}$$

(see, e.g., [15]). The Schrödinger flow corresponds to the case of $a = 0$. The case of $b = 0$ is the well-studied harmonic map heat flow, for which some finite-energy solutions do blow up in finite time (see [4]).

On the other hand, equation (1.1) is a particular case of the Schrödinger flow for maps from a Riemannian manifold into a Kähler manifold (see, e.g., [8], [24], [10], [7]). We consider only the case of maps $\mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{S}^2$ in this article.

We refer the reader to our previous article [11] for more detailed background on (1.1) (and further references), limiting the discussion here to a list of a few basic facts we need in order to state our results.

Energy conservation

Equation (1.1) formally conserves the *energy*

$$\mathcal{E}(\mathbf{u}) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2\,dx = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{j=1}^{n}\sum_{k=1}^{3}\left|\frac{\partial u_k}{\partial x_j}\right|^2 dx. \tag{1.4}$$

The space dimension $n = 2$ is critical in the sense that $\mathcal{E}(\mathbf{u})$ is invariant under scaling. In general,

$$\mathcal{E}\big(\mathbf{u}(\cdot)\big) = s^{2-n}\mathcal{E}\left(\mathbf{u}\left(\frac{\cdot}{s}\right)\right) \tag{1.5}$$

for $s > 0$.

Equivariant maps

Fix $m \in \mathbb{Z}$ a nonzero integer. By an *m-equivariant map* $\mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ we mean a map of the form

$$\mathbf{u}(r,\theta) = e^{m\theta R}\,\mathbf{v}(r), \tag{1.6}$$

where (r,θ) are polar coordinates on $\mathbb{R}^2$, $\mathbf{v} : [0,\infty) \to \mathbb{S}^2$, and R is the matrix generating rotations around the u_3-axis:

$$R = \begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad e^{\alpha R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{1.7}$$

Radial maps arise as the case of $m = 0$. The class of m-equivariant maps is formally preserved by the Schrödinger flow.

Topological lower bound on energy

If $\mathbf{u}$ is m-equivariant, we have $|\nabla \mathbf{u}|^2 = \left|\frac{\partial \mathbf{u}}{\partial r}\right|^2 + r^{-2}\left|\frac{\partial \mathbf{u}}{\partial \theta}\right|^2 = \left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + (m^2/r^2)|R\mathbf{v}|^2$, and so

$$\mathcal{E}(\mathbf{u}) = \pi\int_0^\infty \left(\left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + \frac{m^2}{r^2}(v_1^2+v_2^2)\right) r\,dr. \tag{1.8}$$

If $\mathcal{E}(\mathbf{u}) < \infty$, then $\mathbf{v}(r)$ is continuous, and the limits $\lim_{r\to 0}\mathbf{v}(r)$ and $\lim_{r\to\infty}\mathbf{v}(r)$ exist (see [11]). So we must have $\mathbf{v}(0), \mathbf{v}(\infty) = \pm\widehat{\mathbf{k}}$, where $\widehat{\mathbf{k}} = (0,0,1)^T$. Without loss of generality, we fix $\mathbf{v}(0) = -\widehat{\mathbf{k}}$. The two cases $\mathbf{v}(\infty) = \pm\widehat{\mathbf{k}}$ then correspond to different topological classes of maps. We denote by Σ_m the class of m-equivariant maps with $\mathbf{v}(\infty) = \widehat{\mathbf{k}}$:

$$\begin{aligned}\Sigma_m = \Big\{\mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 \,\Big|\, \mathbf{u} = e^{m\theta R}\mathbf{v}(r),\ \mathcal{E}(\mathbf{u}) < \infty,\\ \mathbf{v}(0) = -\widehat{\mathbf{k}},\ \mathbf{v}(\infty) = \widehat{\mathbf{k}}\Big\}.\end{aligned} \tag{1.9}$$

For $\mathbf{u} \in \Sigma_m$, the energy $\mathcal{E}(\mathbf{u})$ can be rewritten

$$\begin{aligned}\mathcal{E}(\mathbf{u}) &= \pi \int_0^\infty \Big\{\Big|\frac{\partial \mathbf{v}}{\partial r}\Big|^2 + \frac{m^2}{r^2}|J^{\mathbf{v}}R\mathbf{v}|^2\Big\}\, r\,dr\\ &= \pi \int_0^\infty \Big|\frac{\partial \mathbf{v}}{\partial r} - \frac{|m|}{r}J^{\mathbf{v}}R\mathbf{v}\Big|^2 r\,dr + \mathcal{E}_{\min}\end{aligned} \tag{1.10}$$

(recall that $J^{\mathbf{v}} := \mathbf{v}\times$) with

$$\mathcal{E}_{\min} = 2\pi \int_0^\infty \mathbf{v}_r \cdot \frac{|m|}{r} J^{\mathbf{v}}R\mathbf{v}\, r\,dr = 2\pi|m| \int_0^\infty (v_3)_r\,dr = 4\pi|m|. \tag{1.11}$$

Thus for $\mathbf{u} \in \Sigma_m$, there is a nontrivial lower bound for the energy:

$$\mathbf{u} \in \Sigma_m \implies \mathcal{E}(\mathbf{u}) \geq 4\pi|m|. \tag{1.12}$$

In general, one has $\mathcal{E}(\mathbf{u}) \geq 4\pi|\deg|$, where deg is the degree of the map $\mathbf{u}$, considered as a map from $\mathbb{S}^2$ to itself (defined, e.g., by integrating the pullback by $\mathbf{u}$ of the volume form on $\mathbb{S}^2$).

Harmonic maps

For a map $\mathbf{u} \in \Sigma_m$, the topological lower bound (1.12) is saturated if and only if

$$\frac{\partial \mathbf{v}}{\partial r} = \frac{|m|}{r} J^{\mathbf{v}}R\mathbf{v}, \tag{1.13}$$

and the minimal energy is attained (i.e., (1.13) is satisfied) precisely at the two-parameter family of harmonic maps

$$\mathcal{O}_m := \Big\{e^{(m\theta+\alpha)R}\mathbf{h}\Big(\frac{r}{s}\Big) \,\Big|\, s > 0,\ \alpha \in \mathbb{R}\Big\}, \tag{1.14}$$

where

$$\mathbf{h}(r) = \begin{pmatrix} h_1(r) \\ 0 \\ h_3(r) \end{pmatrix}, \quad h_1(r) = \frac{2}{r^{|m|} + r^{-|m|}}, h_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}}. \tag{1.15}$$

The rotation parameter α is determined only up to shifts of 2π (i.e., really $\alpha \in \mathbb{S}^1$). The fact that $\mathbf{h}(r)$ satisfies (1.13) means

$$(h_1)_r = -\frac{m}{r} h_1 h_3, \qquad (h_3)_r = \frac{m}{r} h_1^2. \tag{1.16}$$

Note that $\mathcal{O}_m$ is just the orbit of the harmonic map $e^{m\theta R}\mathbf{h}(r)$ under the symmetries of the energy $\mathcal{E}$ which preserve equivariance: scaling and rotation. Explicitly, the maps in $\mathcal{O}_m$ are of the form

$$u(r,\theta) = \begin{pmatrix} \cos(m\theta + \alpha) h_1\left(\frac{r}{s}\right) \\ \\ \sin(m\theta + \alpha) h_1\left(\frac{r}{s}\right) \\ \\ h_3\left(\frac{r}{s}\right) \end{pmatrix}. \tag{1.17}$$

Of course, these harmonic maps are each static solutions of the Schrödinger flow (1.1). In fact, it is not hard to show that they are the only m-equivariant static solutions (though this fact plays no role in our analysis).

The orbital stability of $\mathcal{O}_m$

We recall the main result of [11].

THEOREM 1.1 ([11, Th. 1.1])
There exist $\delta > 0$ and $C_1, C_2 > 0$ such that if $\mathbf{u} \in C([0,T); \dot{H}^2 \cap \Sigma_m)$ is a solution of the Schrödinger flow (1.1) conserving energy and satisfying

$$\delta_1^2 := \mathcal{E}(\mathbf{u}_0) - 4\pi|m| < \delta^2,$$

then there exist $s(t) \in \mathscr{C}([0,T);(0,\infty))$ and $\alpha(t) \in \mathscr{C}([0,T);\mathbb{R})$ such that

$$\left\| \mathbf{u}(x,t) - e^{(m\theta + \alpha(t))R}\mathbf{h}\left(\frac{r}{s(t)}\right) \right\|_{\dot{H}_1(\mathbb{R}^2)} \le C_1 \delta_1, \quad \forall t \in [0,T). \tag{1.18}$$

Moreover, $s(t) > C_2/\|\mathbf{u}(t)\|_{\dot{H}^2(\mathbb{R}^2)}$. Furthermore, if $T < \infty$ is the maximal time of existence for $\mathbf{u}$ in $\dot{H}^2$ (i.e., if $\lim_{t\to T^-}\|\mathbf{u}(t)\|_{\dot{H}^2(\mathbb{R}^2)} = \infty$), then

$$\liminf_{t\to T^-} s(t) = 0. \tag{1.19}$$

Theorem 1.1 can be viewed, on the one hand, as an *orbital stability* result for the family $\mathcal{O}_m$ of harmonic maps (at least up to the possible blowup time) and, on the other hand, as a characterization of blowup for energy near $\mathcal{E}_{\min}$: solutions blow up if and only if the *length scale* $s(t)$ goes to zero. Here, $s(t)$ (and the rotation angle $\alpha(t)$) is determined simply by finding, at each time t, the harmonic map that is $\dot{H}^1$-closest to $\mathbf{u}(t)$. More precisely, a continuous map

$$\begin{aligned} \left\{\mathbf{u} \in \Sigma_m \,\middle|\, \mathcal{E}(\mathbf{u}) < 4\pi|m| + \delta^2\right\} &\to \mathbb{R}^+ \times (\mathbb{R} \bmod 2\pi), \\ \mathbf{u} &\mapsto \big(s(\mathbf{u}),\ \alpha(\mathbf{u})\big) \end{aligned} \tag{1.20}$$

is constructed in [11] which, for m-equivariant maps with energy close to $4\pi|m|$, identifies the unique $\dot{H}^1$-closest harmonic map:

$$\left\| \mathbf{u} - e^{[m\theta+\alpha(\mathbf{u})]R} h\Big(\frac{r}{s(\mathbf{u})}\Big)\right\|_{\dot{H}^1} = \min_{s\in\mathbb{R}^+,\alpha\in\mathbb{R}} \left\|\mathbf{u} - e^{[m\theta+\alpha]R} h\Big(\frac{r}{s}\Big)\right\|_{\dot{H}^1}. \tag{1.21}$$

Then we set $s(t) := s(\mathbf{u}(t))$.

In this article, we continue our study of the Schrödinger flow for equivariant maps with energy close to the harmonic map energy. We begin with an energy-space local well-posedness theorem for such maps. It is worth noting that despite a great deal of recent work on the local well-posedness problem in two space dimensions (see [22], [9], [16], [1], [12]; see also [17], [13] for the *modified Schrödinger map* case), there is no general result for energy-space initial data. For our special class of data, however, we do have such a result. Before stating it, let us first make precise the sense in which our energy-space solution solves the Schrödinger map problem.

Definition 1.2 (Weak solutions)
Let $Z := \{\mathbf{u} : \mathbb{R}^n \to \mathbb{S}^2,\ D\mathbf{u} \in L^2\}$ be the energy space. We say that $\mathbf{u}(x,t)$ is a *weak solution* of the Schrödinger flow (1.1) on the time interval $I = [0,T]$, with initial data $\mathbf{u}_0 \in Z$, if

(1) $\mathbf{u} \in L^\infty(I;Z) \cap C_{\text{weak}}([0,T];Z)$,

(2) $\mathbf{u}(0) = \mathbf{u}_0$,

(3) $\iint_{\mathbb{R}^n\times I} \{\mathbf{u}\cdot\phi_t - \mathbf{u}\times\partial_j\mathbf{u}\cdot\partial_j\phi\}\,dx\,dt = 0, \forall\phi \in C_c^1(I\times\mathbb{R}^n;\mathbb{R}^3)$.

Remark 1.3
It is not strictly necessary to require that $D\mathbf{u}$ be weakly continuous in t (in property (1)). The weak form of the equation (property (3)) implies that $\mathbf{u}_t \in L^\infty([0,T];H^{-1})$, and so, after redefinition on a set of time measure zero, $\mathbf{u} \in \mathrm{Lip}([0,T];H^{-1})$ and $D\mathbf{u} \in \mathrm{Lip}([0,T];H^{-2})$. Since we also have $D\mathbf{u} \in L^\infty([0,T];L^2)$, we can prove $D\mathbf{u} \in C_{\text{weak}}([0,T];L^2)$.

We have the following.

THEOREM 1.4 (Local well-posedness)
Let $|m| \geq 1$. There exist $\delta > 0$ and $\sigma > 0$ such that the following hold. Suppose that $\mathbf{u}_0 \in \Sigma_m$ and $\mathcal{E}(\mathbf{u}_0) = 4\pi m + \delta_0^2$, $\delta_0 \in (0, \delta]$. Let $s_0 := s(\mathbf{u}_0)$, as defined in (1.20) and (1.21). Then there is a unique weak solution $\mathbf{u}(t)$ of (1.1),

$$\mathbf{u}(t) \in C(I; \Sigma_m), \quad I = [0, \sigma s_0^2].$$

Moreover, $\mathcal{E}(\mathbf{u}(t)) = \mathcal{E}(\mathbf{u}_0)$ for $t \in I$. If, furthermore, $\mathbf{u}_0 \in \dot{H}^2$, then $\mathbf{u}(t) \in C(I; \Sigma_m \cap \dot{H}^2)$. Suppose that $\mathbf{u}_0^n \to \mathbf{u}_0$ in Σ_m, and let $\mathbf{u}^n$ denote the corresponding solutions of (1.1); then $\mathbf{u}^n \to \mathbf{u}$ in $C(I, \Sigma_m)$.

It is worth emphasizing that the existence time furnished by Theorem 1.4 depends not on the energy $\|\mathbf{u}_0\|_{\dot{H}^1}^2$ of the initial data (reflecting the energy-space critical nature of the equation in dimension $n = 2$) but rather on $s(\mathbf{u}_0)$, the length scale of the $\dot{H}^1$-nearest harmonic map.

There are at least two ways to define blowup for these solutions. Suppose $\mathbf{u}(t) \in C([0, T), \Sigma_m \cap \dot{H}^k)$, $0 < T < \infty$ with $k = 1$ or 2. If $k = 1$, we say that $\mathbf{u}(t)$ blows up at $t = T$ if $\lim_{t \to T^-} \mathbf{u}(t)$ does not exist in $\dot{H}^1$. If $k = 2$, we say that $\mathbf{u}(t)$ blows up at $t = T$ if $\limsup_{t \to T-} \|\mathbf{u}(t)\|_{\dot{H}^2} = \infty$.

For $\mathbf{u}_0 \in \Sigma_m \cap \dot{H}^k$, $k = 1, 2$, denote by $T_{\max}^k$ the maximal time such that there is a unique solution $\mathbf{u}(t) \in C([0, T_{\max}^k); \Sigma_m \cap \dot{H}^k)$.

COROLLARY 1.5
Under the same assumptions as in Theorem 1.4, suppose that the solution $\mathbf{u}(t) \in C([0, T), \Sigma_m \cap \dot{H}^k)$, $k = 1$ or 2, and $T < \infty$.

(i) (Blowup alternative) *The solution $\mathbf{u}(t)$ blows up at time T (i.e., $T = T_{\max}^1$) if and only if $\liminf_{t \to T-} s(\mathbf{u}(t)) = 0$. In this case, $s(\mathbf{u}(t)) \leq C\sqrt{T - t}$, and if $k = 2$, $T = T_{\max}^1 = T_{\max}^2$ with $\|\mathbf{u}(t)\|_{\dot{H}^2} \geq C(T - t)^{-1/2}$.*

(ii) (Lower bound for $T_{\max} := T_{\max}^1$) *We have $T_{\max} \geq \sigma[s(\mathbf{u}_0)]^2$. (Here, σ is the constant from Theorem 1.4.)*

Corollary 1.5(i) improves Theorem 1.1 by giving explicit bounds.

We also have $\dot{H}^1$ local well-posedness for the small-energy equivariant case considered in [5]. Since the energy is conserved, local well-posedness implies global well-posedness.

THEOREM 1.6 (Small-energy local well-posedness)
Let $|m| \geq 1$. There exist $\delta > 0$ and $\sigma > 0$ such that the following hold. Suppose that $\mathbf{u}_0 = e^{m\theta R} v_0(r)$ and $\mathcal{E}(\mathbf{u}_0) \leq \delta^2$; then there is a unique weak solution $\mathbf{u}(t, r, \theta) = e^{m\theta R}\mathbf{v}(t, r)$ of (1.1) such that $\mathbf{u}(t) \in C([0, \sigma]; \dot{H}^1)$. Moreover, $\mathcal{E}(\mathbf{u}(t)) = \mathcal{E}(\mathbf{u}_0)$ for

$t \in [0, \sigma]$. *Suppose that* $\mathbf{u}_0^n$ *are equivariant,* $\mathbf{u}_0^n \to \mathbf{u}_0$ *in* $\dot{H}^1$*, and let* $\mathbf{u}^n$ *denote the corresponding solutions of (1.1); then* $\mathbf{u}^n \to \mathbf{u}$ *in* $C([0, \sigma], \dot{H}^1)$.

Note that this result does *not* cover the radial case ($m = 0$).

The question of whether singularities can form in the Schrödinger flow is open. So far, it has only been shown that they cannot form for *small-energy* radial or equivariant solutions (see [5]). Our Theorem 1.1 leaves open the question of whether finite-time blowup can occur for maps in Σ_m with energies near $\mathcal{E}_{\min} = 4\pi|m|$. The main result of this article shows that when $|m| \geq 4$, it does *not*. Moreover, we show that these solutions converge (in a dispersive sense) to specific harmonic maps as $t \to \infty$. Here is the main result.

THEOREM 1.7 (Main result)
Let $|m| \geq 4$. *Let* (r, p) *satisfy* $2 < r \leq \infty$, $2 \leq p < \infty$, *with* $1/r + 1/p = 1/2$. *There exist positive constants* δ, C, *and* C_p *such that if* $\mathbf{u}_0 \in \Sigma_m$ *satisfies*

$$\delta_1^2 := \mathcal{E}(\mathbf{u}_0) - 4\pi|m| < \delta^2,$$

then for the corresponding solution $\mathbf{u}(t)$ *of the Schrödinger flow (guaranteed by Theorem 1.4),*

(1) *there is no finite-time blowup:* $T_{\max} = \infty$*;*

(2) *there exist* $s(t) \in \mathscr{C}([0, \infty); (0, \infty))$ *and* $\alpha(t) \in \mathscr{C}([0, \infty); \mathbb{R})$ *such that*

$$\left\| \nabla \left[\mathbf{u}(x, t) - e^{(m\theta + \alpha(t))R} \mathbf{h}\left(\frac{r}{s(t)}\right) \right] \right\|_{(L_t^\infty L_x^2 \cap L_t^r L_x^p)(\mathbb{R}^2 \times [0,\infty))} \leq C_p \delta_1; \qquad (1.22)$$

(3) *furthermore,*

$$\left| \frac{s(t)}{s(\mathbf{u}_0)} - 1 \right| + |\alpha(t) - \alpha(\mathbf{u}_0)| \leq C\delta_1^2, \quad \forall t > 0,$$

and there exist $s_+ > 0$ *and* α_+ *with*

$$s(t) \to s_+, \qquad \alpha(t) \to \alpha_+, \qquad \text{as } t \to \infty. \qquad (1.23)$$

Remark 1.8

(1) The $L_t^\infty L_x^2$ (energy-space) estimate in (1.22) already follows from Theorem 1.1. The other space-time estimates in (1.22) further imply asymptotic *convergence* to the family of harmonic maps (at least, in a time-averaged sense—the best we can expect without further assumptions on the initial data). The convergence results (1.22) and (1.23) are precisely what we mean when we say the harmonic maps are *asymptotically stable* under the Schrödinger flow for $|m| \geq 4$.

(2) Note that for $|m| = 1, 2, 3$, the fate of solutions with energy near $\mathcal{E}_{\min}$ is still an open question. Our restriction $|m| > 3$ is connected with the slow spatial decay of the harmonic map component $h_1(r) \sim (\text{const})r^{-|m|}$ as $r \to \infty$. For a somewhat technical reason, we need $r^2 h_1(r) \in L^2(r\,dr)$ (see Lem. 2.4), which requires $|m| > 3$. For seemingly more fundamental reasons, we need $r h_1(r) \in L^2(r\,dr)$ (see (2.17)), which holds if $|m| > 2$.

(3) The recent work [21] on the analogous *wave map* problem imposes the same restriction of $|m| \geq 4$, but it proves that *blowup* is possible in this class, suggesting that singularity formation is a more delicate question for Schrödinger maps than for wave maps.

We end the introduction with a few words about our approach. One key observation, already used in [11], is that the tangent vector field

$$\mathbf{W} := \frac{\partial \mathbf{v}}{\partial r} - \frac{|m|}{r} J^{\mathbf{v}} R \mathbf{v}$$

measures the deviation of the map $\mathbf{u}$ *from harmonicity*. (This is indicated, e.g., by (1.13).) Furthermore, when expressed in an appropriate orthonormal frame, the coordinates of $\mathbf{W}$ satisfy a nonlinear Schrödinger-type equation that is suitable for obtaining estimates—this is the *generalized Hasimoto transform* introduced in [5] to study the small-energy problem.

In the present article, this nonlinear Schrödinger-type partial differential equation is coupled to a 2-dimensional dynamical system describing the dynamics of the scaling and rotation parameters $s(t)$ and $\alpha(t)$, a careful choice of which must be made at each time in order to allow estimation. This is all done in Section 2.

The key to proving convergence of the solution to a harmonic map is then to obtain dispersive estimates—in this case, Strichartz-type estimates—for the linear part of our nonlinear Schrödinger equation. The potential appearing in the corresponding Schrödinger operator turns out to have $(\text{const}/|x|^2)$-behavior both at the origin and as $|x| \to \infty$, which is a *borderline* case not treatable by purely perturbative methods. Fortunately, a recent series of articles by Burq, Planchon, Stalker, and Tahvildar-Zadeh [2], [3] addresses the problem of obtaining dispersive estimates when the potential has just this *critical* decay rate, provided the potential satisfies a *repulsivity* condition (which, in particular, rules out bound states). Though their relevant results are for dimension $n \geq 3$, we are able to adapt their approach to prove the estimates that we need in our 2-dimensional setting. This is done in Section 3.

Finally, in Section 4, we prove Theorem 1.7 by applying the linear estimates of Section 3 to the coupled nonlinear system of Section 2.

Since the proofs of Theorems 1.4 and 1.6 and Corollary 1.5 are independent of the rest of the article, they are postponed until Appendix A. Some lemmas are proved in Appendix B.

Remark 1.9
From here on, we assume that $m > 0$. For $m < 0$, simply make the change of variable $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$.

Notation. Throughout the article, the letter C is used to denote a generic constant, the value of which may change from line to line. Vectors in $\mathbb{R}^3$ appear in boldface, while their components appear in regular type: for example, $\mathbf{u} = (u_1, u_2, u_3)$.

2. The dynamics near the harmonic maps

2.1. Splitting the solution

Let $\mathbf{u}(x, t) = e^{m\theta R}\mathbf{v}(r, t) \in \Sigma_m$ be a solution of the Schrödinger map equation (1.1). We write our solution as a harmonic map with time-varying parameters, plus a perturbation:

$$\mathbf{v}(r, t) = e^{\alpha(t)R}\left[\mathbf{h}(\rho) + \boldsymbol{\xi}(\rho, t)\right], \quad \rho := \frac{r}{s(t)}. \tag{2.1}$$

In Section 2.3, we take up the central question of precisely how to do this splitting (i.e., the choice of $s(t)$ and $\alpha(t)$).

It is convenient and natural to single out the component of the perturbation $\boldsymbol{\xi}$ which is tangent to $\mathbb{S}^2$ at $\mathbf{h}$:

$$\boldsymbol{\xi}(\rho, t) = \boldsymbol{\eta}(\rho, t) + \gamma(\rho, t)\mathbf{h}(\rho), \quad \boldsymbol{\eta}(\rho, t) \in T_{\mathbf{h}(\rho)}\mathbb{S}^2,$$

so that $\boldsymbol{\eta} \cdot \mathbf{h} \equiv 0$. Thus the original map $\mathbf{u}$ is written

$$\mathbf{u}(x, t) = e^{[m\theta+\alpha(t)]R}\Big[\big(1 + \gamma(\rho, t)\big)\mathbf{h}(\rho) + \boldsymbol{\eta}(\rho, t)\Big];$$
$$\rho = \frac{r}{s(t)}, \ \boldsymbol{\eta}(\rho, t) \in T_{\mathbf{h}(\rho)}\mathbb{S}^2.$$

The pointwise constraint $|\mathbf{v}| \equiv 1$ forces

$$1 \equiv |\mathbf{h} + \boldsymbol{\xi}|^2 = |(1+\gamma)\mathbf{h} + \boldsymbol{\eta}|^2 = (1+\gamma)^2 + |\boldsymbol{\eta}|^2,$$

so $\gamma(\rho, t) \le 0$ and $|\boldsymbol{\eta}(\rho, t)| \le 1$. If $|\boldsymbol{\xi}| \le 1$, then

$$\gamma(\rho, t) = +\big(1 - |\boldsymbol{\eta}(\rho, t)|^2\big)^{1/2} - 1 \in [-1, 0]. \tag{2.2}$$

A convenient orthonormal basis of $T_{\mathbf{h}(\rho)}\mathbb{S}^2$ is given by

$$\widehat{\boldsymbol{\jmath}} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad J^{\mathbf{h}(\rho)}\widehat{\boldsymbol{\jmath}} = \begin{pmatrix} -h_3(\rho) \\ 0 \\ h_1(\rho) \end{pmatrix},$$

and we express tangent vectors like $\boldsymbol{\eta} \in T_{\mathbf{h}}\mathbb{S}^2$ in this basis via the invertible linear map

$$\mathbf{V}^{\rho} : \mathbb{C} \to T_{\mathbf{h}(\rho)}\mathbb{S}^2,$$
$$z = z_1 + i z_2 \mapsto z_1\widehat{\boldsymbol{\jmath}} + z_2 J^{\mathbf{h}(\rho)}\widehat{\boldsymbol{\jmath}}.$$

So we write

$$\boldsymbol{\eta}(\rho,t) = \mathbf{V}^{\rho}\big(z(\rho,t)\big),$$

and in this way, the complex function $z(\rho,t)$, together with a choice of the parameters $s(t)$ and $\alpha(t)$, gives a full description of the original solution $\mathbf{u}(x,t)$, provided $|\boldsymbol{\xi}| \leq 1$.

From (2.2), we find

$$|z| = |\boldsymbol{\eta}| \leq \frac{1}{2} \implies |\gamma| \lesssim |z|^2, \qquad |\gamma_\rho| \lesssim |z||z_\rho|. \tag{2.3}$$

These estimates, together with results in [11], show that if s and α are chosen appropriately, then for $\mathcal{E}(\mathbf{u}) - 4\pi m$ small,

$$\|z\|_X^2 \lesssim \mathcal{E}(\mathbf{u}) - 4\pi m \lesssim \|z\|_X^2,$$

where $X := \{z : [0,\infty) \to \mathbb{C} \mid z_\rho \in L^2(\rho\, d\rho),\ z/\rho \in L^2(\rho\, d\rho)\}$ with

$$\|z\|_X^2 := \int_0^\infty \left\{ |z_\rho(\rho)|^2 + \frac{|z(\rho)|^2}{\rho^2} \right\} \rho\, d\rho. \tag{2.4}$$

The space X is therefore the natural space for z, corresponding to the energy space for the original map $\mathbf{u}$. The fact that

$$z \in X \implies z \text{ continuous in } (0,\infty),\ z(0+) = z(\infty-) = 0, \text{ and } \|z\|_{L^\infty} \lesssim \|z\|_X \tag{2.5}$$

follows easily from the change of variable $\rho^m = e^y$ and Sobolev imbedding on $\mathbb{R}$ (see [11]).

Remark 2.1
Unless explicitly stated otherwise, Lebesgue norms of radial functions such as $z(\rho)$ are always with respect to the $\mathbb{R}^2$ Lebesgue measure $\rho\, d\rho$.

2.2. Equation for the perturbation

The next step is to derive an equation for $z(\rho, t)$. In terms of $\mathbf{v}(r, t)$, the Schrödinger map equation can be written as

$$\mathbf{v}_t = \mathbf{v} \times \Big(\mathbf{v}_{rr} + \frac{1}{r}\mathbf{v}_r + \frac{m^2}{r^2} R^2 \mathbf{v}\Big). \tag{2.6}$$

Using (2.1), we find

$$e^{-\alpha R}\mathbf{v}_t = [\dot{\alpha} R - s^{-1}\dot{s}\rho\partial_\rho](\mathbf{h} + \boldsymbol{\xi}) + \boldsymbol{\xi}_t, \tag{2.7}$$

$$s^2 e^{-\alpha R}(\mathbf{v} \times M_r \mathbf{v}) = (\mathbf{h} + \boldsymbol{\xi}) \times (M_\rho \mathbf{h} + M_\rho \boldsymbol{\xi}), \tag{2.8}$$

where

$$M_\rho := \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2} R^2$$

(and the right-hand sides are evaluated at $(\rho = r/s(t), t)$).

Consider first (2.8). Since $\Delta \mathbf{H} + |\nabla \mathbf{H}|^2 \mathbf{H} = 0$ for $\mathbf{H} = e^{m\theta R}\mathbf{h}$, we have

$$M\mathbf{h} = -2\frac{m^2}{\rho^2} h_1^2 \mathbf{h}, \tag{2.9}$$

where $M = M_\rho$. Thus

$$\begin{aligned}\text{RHS of (2.8)} &= \mathbf{h} \times M\boldsymbol{\xi} + \boldsymbol{\xi} \times \Big(-2\frac{m^2}{\rho^2} h_1^2 \mathbf{h}\Big) + \boldsymbol{\xi} \times M\boldsymbol{\xi} \\ &= \mathbf{h} \times \Big(M + 2\frac{m^2}{\rho^2} h_1^2\Big)\boldsymbol{\xi} + \boldsymbol{\xi} \times M\boldsymbol{\xi}.\end{aligned}$$

Keeping in mind (2.3), we write

$$\text{RHS of (2.8)} = \mathbf{h} \times \Big(M + 2\frac{m^2}{\rho^2} h_1^2\Big)\big(\mathbf{V}^\rho(z)\big) + \mathbf{F_1},$$

where $\mathbf{F_1} = \mathbf{h} \times (M + 2(m^2/\rho^2)h_1^2)\gamma\mathbf{h} + \boldsymbol{\xi} \times M\boldsymbol{\xi}$ is the nonlinear part. By (2.9), we have $M\gamma\mathbf{h} = 2\gamma_\rho \mathbf{h}_\rho + (\cdots)\mathbf{h} = 2\gamma_\rho (m/\rho)\widehat{\mathbf{k}} + (\cdots)\mathbf{h}$, and hence

$$\mathbf{F_1} = -2\gamma_\rho \frac{m}{\rho} h_1 \widehat{\boldsymbol{\jmath}} + \boldsymbol{\xi} \times M\boldsymbol{\xi}. \tag{2.10}$$

Using $R^2\widehat{\boldsymbol{\jmath}} = -\widehat{\boldsymbol{\jmath}}$, $R^2 J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}} = h_1 h_3 \mathbf{h} - h_3^2 J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}}$, $(J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}})_\rho = -(m/\rho)h_1\mathbf{h}$, and $(J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}})_{\rho\rho} = -(m^2/\rho^2)h_1^2 J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}} - ((m/\rho)h_1)_\rho\mathbf{h}$ (all easy computations), we find that the linear part can be rewritten as

$$\mathbf{h} \times \Big(M + 2\frac{m^2}{\rho^2} h_1^2\Big)\big(\mathbf{V}^\rho(z)\big) = -\mathbf{h} \times [\mathbf{V}^\rho(Nz)] = \mathbf{V}^\rho(-iNz),$$

where N denotes the differential operator $N := -\partial_\rho^2 - (1/\rho)\partial_\rho + (m^2/\rho^2)(1 - 2h_1^2)$.

Because $\boldsymbol{\xi}_t = \mathbf{V}^\rho(z_t) + \gamma_t \mathbf{h}$, (2.7) and (2.8) give

$$s^2[\mathbf{V}^\rho(z_t) + \gamma_t \mathbf{h}] + [s^2 \dot{\alpha} R - s\dot{s}\rho\partial_\rho](\mathbf{h} + \boldsymbol{\xi}) = \mathbf{V}^\rho(-iNz) + \mathbf{F_1}$$

or

$$\mathbf{V}^\rho(s^2 z_t + iNz) = \mathbf{F}, \tag{2.11}$$

where

$$\mathbf{F} := \mathbf{F_1} + [-s^2\dot{\alpha}R + s\dot{s}\rho\partial_\rho](\mathbf{h} + \boldsymbol{\xi}) - s^2\gamma_t \mathbf{h}.$$

Because the left-hand side of (2.11) lies in $T_{\mathbf{h}}\mathbb{S}^2$, the right-hand side does also, and hence $\mathbf{F} \cdot \mathbf{h} \equiv 0$. We can rewrite (2.11) on the complex side by applying $(V^\rho)^{-1}$:

$$is^2 \frac{\partial z}{\partial t} = Nz + i(\mathbf{V}^\rho)^{-1}\mathbf{F}, \quad N = -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2}(1 - 2h_1^2). \tag{2.12}$$

This is the equation that we sought for $z(\rho, t)$.

In order to see the form of the *nonlinear* terms $(\mathbf{V}^\rho)^{-1}(\mathbf{F})$ more clearly, we compute

$$(\mathbf{V}^\rho)^{-1}\big(R\mathbf{h}(\rho)\big) = h_1(\rho), \qquad (\mathbf{V}^\rho)^{-1}\big(\rho\partial_\rho \mathbf{h}(\rho)\big) = imh_1,$$

$$(\mathbf{V}^\rho)^{-1}\big(P^{\mathbf{h}(\rho)} R \mathbf{V}^\rho(z)\big) = izh_3, \qquad (\mathbf{V}^\rho)^{-1}\big(P^{\mathbf{h}(\rho)} \rho\partial_\rho \mathbf{V}^\rho(z)\big) = \rho z_\rho,$$

where $P^{\mathbf{h}(\rho)}$ denotes the orthogonal vector projection onto $T_{\mathbf{h}(\rho)}\mathbb{S}^2$. Thus, using $\mathbf{h}+\boldsymbol{\xi} = (1+\gamma)\mathbf{h} + \mathbf{V}^\rho(z)$,

$$(\mathbf{V}^\rho)^{-1}(\mathbf{F}) = [-s^2\dot{\alpha} + ims\dot{s}](1+\gamma)h_1 - s^2\dot{\alpha}izh_3 + s\dot{s}\rho z_\rho + (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)}\mathbf{F_1}). \tag{2.13}$$

2.3. Orthogonality condition and parameter equations

We have not yet specified $s(t)$ and $\alpha(t)$. The main result of [11] says that if the energy is close to $\mathcal{E}_{\min}$ (i.e., $\delta_1^2 := \mathcal{E}(\mathbf{u}) - \mathcal{E}_{\min} \ll 1$), then there exist continuous $s(t) > 0$ and $\alpha(t) \in \mathbb{R}$ such that $\|e^{m\theta R}\boldsymbol{\xi}\|_{\dot{H}^1} \lesssim \delta_1$ as long as $s(t)$ stays away from zero. The choice of the parameters was simple and natural: at each time t, $s(t)$ and $\alpha(t)$ were chosen so as to minimize $\|e^{m\theta R}\boldsymbol{\xi}\|_{\dot{H}^1}$. In this article, we are forced into a different choice of $s(t)$ and $\alpha(t)$, as we now explain.

Supposing for a moment that $s(t) \equiv 1$, the linearized equation for $z(\rho, t)$ can be read from (2.12):

$$i\partial_t z = Nz. \tag{2.14}$$

The factorization

$$N = L_0^* L_0, \quad L_0 := \partial_\rho + \frac{m}{\rho} h_3 = h_1 \partial_\rho \frac{1}{h_1} \tag{2.15}$$

(where the adjoint L_0^* is taken in the $L^2(\rho d\rho)$ inner product) shows that $\ker N = \operatorname{span}\{h_1\}$. In particular, (2.14) admits the constant (in time) solution $z(\rho, t) \equiv h_1(\rho)$. Since we would like $z(\rho, t)$ to have some decay in time, we must choose $s(t)$ and $\alpha(t)$ in such a way as to avoid such constant solutions. Since N is self-adjoint in L^2, the natural choice is to work in the subspace of functions z satisfying

$$(z,\ h_1)_{L^2} = \int_0^\infty z(\rho) h_1(\rho) \rho\, d\rho \equiv 0, \tag{2.16}$$

which is invariant under the linear flow (2.14).

Recall, however, that the *energy space* for z is the space X (defined in (2.4)). Certainly, the linear flow (2.14) does *not* preserve the subspace $\{f \in X, \langle f, h_1 \rangle_X = 0\}$ (since N is not self-adjoint in X). In fact, neither z nor h_1 lies in L^2 in general. The best we can do is

$$|(z, h_1)_{L^2}| = \left| \left(\frac{z}{\rho}, \rho h_1 \right)_{L^2} \right| \le \|z\|_X \|\rho h_1\|_{L^2}.$$

So to make sense of (2.16), we require

$$\rho h_1(\rho) = \frac{2\rho}{\rho^m + \rho^{-m}} \in L^2(\rho\, d\rho), \tag{2.17}$$

which holds only if $m \ge 3$. This is one of the reasons that we cannot handle the small $|m|$ cases in Theorem 1.7. The further restriction $m > 3$ is needed in Proposition 2.3 in Section 2.4.

In order to ensure that condition (2.16) holds for all times t, it suffices to impose it initially and then ensure that the time derivative of the inner product vanishes for all t. Differentiating (2.16) with respect to t, and using equations (2.12), (2.13), and (2.16), yields a system of ordinary differential equations (ODEs) for $s(t)$ and $\alpha(t)$:

$$[s^2\dot\alpha - ims\dot s]\big(h_1,\ (1+\gamma)h_1\big)_{L^2} = \big(h_1,\ (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)}\mathbf{F_1}) - s^2\dot\alpha i h_3 z + s\dot s \rho z_\rho\big)_{L^2}. \tag{2.18}$$

The orthogonality condition (2.16) is precisely the one that ensures the terms linear in z disappear from (2.18) and hence the key property that $\dot{s}$ and $\dot{\alpha}$ be at least *quadratic* in z. More precisely, the system (2.18) leads to the following estimate.

LEMMA 2.2
If $\|z\|_X \ll 1$, *then*

$$|s\dot{s}| + |s^2\dot{\alpha}| \lesssim \left\|\frac{z}{\rho^2}\right\|_{L^2}^2 + \left\|\frac{z_\rho}{\rho}\right\|_{L^2}^2.$$

Proof
Using

$$|(h_1, h_3 z)| \lesssim \|\rho h_1\|_{L^2}\left\|\frac{z}{\rho}\right\|_{L^2} \lesssim \|z\|_X \ll 1,$$

$$|(h_1, \rho z_\rho)| \lesssim \|\rho h_1\|_{L^2}\|z_\rho\|_{L^2} \lesssim \|z\|_X \ll 1,$$

$$|(h_1, \gamma h_1)| \lesssim \|\rho^2 h_1^2\|_{L^\infty}\left\|\frac{z}{\rho}\right\|_{L^2}^2 \lesssim \|z\|_X^2 \ll 1,$$

in (2.18), we arrive at

$$|s\dot{s}| + |s^2\dot{\alpha}| \lesssim \left|\left(h_1, (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)}\mathbf{F_1})\right)\right|. \tag{2.19}$$

To finish the proof of the lemma, we need to find $(\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}}\mathbf{F_1})$ explicitly. Using the calculation of Lemma B.1 in Appendix B, we have

$$\begin{aligned}&\left(h_1, (\mathbf{V}^\rho)^{-1}P^{\mathbf{h}}\mathbf{F_1}\right)_{L^2}\\ &\quad= \int_0^\infty \Big(i(h_1)_\rho(-\gamma z_\rho + z\gamma_\rho) + \frac{m}{\rho}h_1^2\big(-2\gamma_\rho - iz_2(z_1)_\rho + iz_1(z_2)_\rho\big)\\ &\qquad\qquad + \frac{m}{\rho}(h_1^2)_\rho(\gamma^2 - iz_2 z) + i\frac{m^2}{\rho^2}(2h_1^2 - 1)h_1\gamma z\Big)\rho\, d\rho.\end{aligned}$$

Now, using inequality (2.5), together with $(h_1)_\rho = -(m/\rho)h_1h_3$, and the fact that $\rho^2 h_1(\rho)$ is bounded for $m \geq 2$, the estimat

$$\left|\left(h_1, (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}}\mathbf{F_1})_{L^2}\right)\right| \lesssim \left\|\frac{z}{\rho^2}\right\|_{L^2}^2 + \left\|\frac{z_\rho}{\rho}\right\|_{L^2}^2$$

follows. Together with (2.19), this completes the proof of Lemma 2.2. □

2.4. A nonlinear Schrödinger equation suited to estimates

We need to prove that $z(\rho, t)$ has some decay in time, but the nonlinear Schrödinger-type equation (2.12) is not suitable for obtaining such estimates, for at least two reasons. First, as noted previously, the linearized equation has constant solutions, and so the orthogonality condition (2.16) has to be explicitly used in order to get any decay whatsoever. Second, and maybe more seriously, some of the nonlinear terms contain derivatives (even two derivatives) of z, leading to a loss of regularity. Fortunately, there is a neat way around these problems: the *generalized Hasimoto transform* of [5] yields an equation without these difficulties, as we now explain.

Let $\mathbf{u} = e^{m\theta R}\mathbf{v}(r) \in \Sigma_m$. From (1.10), it is clear that the tangent vector

$$\mathbf{W}(r) := \mathbf{v}_r(r) - \frac{m}{r} J^{\mathbf{v}} R\mathbf{v}(r) \in T_{\mathbf{v}(r)}\mathbb{S}^2$$

plays a distinguished role. In particular, $\mathbf{u}$ is a harmonic map if and only if $\mathbf{W} \equiv 0$. Indeed, the Schrödinger map equation (1.1), written in terms of $\mathbf{v}(r, t)$, can be factored as

$$\frac{\partial \mathbf{v}}{\partial t} = J^{\mathbf{v}}\Big[D_r^{\mathbf{v}} + \frac{1}{r} - \frac{m}{r}v_3\Big]\mathbf{W}, \tag{2.20}$$

where

$$D_r^{\mathbf{v}} := P^{\mathbf{v}(\mathbf{r})}\partial_r$$

denotes the *covariant derivative* (with respect to r, along $\mathbf{v}$). The idea is to write an equation for $\mathbf{W}$ in an appropriately intrinsic way.

Following [5], let $\mathbf{e}(r) \in T_{\mathbf{v}(r)}\mathbb{S}^2$ be a unit-length tangent field satisfying the *gauge condition*

$$D_r^{\mathbf{v}}\mathbf{e} \equiv 0, \quad e|_{r=\infty} = \hat{\jmath}. \tag{2.21}$$

Expressing $\mathbf{W}$ in the orthonormal frame $\{\mathbf{e}, \ J^{\mathbf{v}}\mathbf{e}\}$,

$$\mathbf{W} = q_1\mathbf{e} + q_2 J^{\mathbf{v}}\mathbf{e},$$

and using (2.20) and (2.21), it is not difficult to arrive at the following equation for the complex function $q(r, t) := q_1(r, t) + iq_2(r, t)$:

$$\begin{aligned} iq_t &= -\Big(\partial_r + \frac{m}{r}v_3\Big)\Big(\partial_r + \frac{1}{r} - \frac{m}{r}v_3\Big)q + Sq \\ &= \Big(- \Delta_r + \frac{1}{r^2}\big((1 - mv_3)^2 + mr(v_3)_r\big)\Big)q + Sq, \end{aligned} \tag{2.22}$$

where the function $S(r, t)$ arises as $D_t^{\mathbf{v}}\mathbf{e} = SJ^{\mathbf{v}}\mathbf{e}$. From the curvature relation

$$[D_r, D_t]\mathbf{e} = -\mathrm{Re}\Big[\Big(\partial_r + \frac{1}{r} - \frac{m}{r}v_3\Big)q\ \overline{\Big(q + \frac{m}{r}\nu\Big)}\Big]J^{\mathbf{v}}\mathbf{e},$$

where $P^{\mathbf{v(r)}}\widehat{\mathbf{k}} = \widehat{\mathbf{k}} - v_3\mathbf{v} = \nu_1\mathbf{e} + \nu_2 J^{\mathbf{v}}\mathbf{e}$, we find

$$S = \mathrm{Re}\int_r^\infty \Big(\partial_\tau + \frac{1}{\tau} - \frac{m}{\tau}v_3(\tau,t)\Big)q(\tau,t)\,\overline{\Big(q(\tau,t) + \frac{m}{\tau}\nu(\tau,t)\Big)}\,d\tau. \tag{2.23}$$

Thus the term in (2.22) involving S is nonlocal and nonlinear. We can simplify the expression for S by integrating by parts in the term involving $\partial_\tau q$, and using the relation $\nu_r = -v_3(q + (m/r)\nu)$, to arrive at

$$S(r,t) = -\frac{1}{2}Q(r,t) + \int_r^\infty \frac{1}{\tau}Q(\tau,t)\,d\tau, \quad Q := |q|^2 + \frac{2m}{r}\mathrm{Re}(\bar{\nu}q). \tag{2.24}$$

Thus equation (2.22) resembles a cubic nonlinear Schrödinger equation, keeping in mind that

(a) there are nonlocal nonlinear terms;

(b) it is not self-contained: the unknown map $\mathbf{v}(r,t)$ itself appears in several places (including through ν).

Furthermore, since

$$\delta_1^2 = \mathcal{E}(\mathbf{u}) - 4\pi m = \frac{1}{2}\|\mathbf{W}\|_{L^2}^2 = \pi\|q\|_{L^2(r\,dr)}^2,$$

we are dealing with a *small L^2 data problem* for equation (2.22) (even though the map $\mathbf{u}$ is not a small-energy map). This is what allows us the estimates we need.

Because of the fact (b) mentioned above, and in order to close the estimate of Lemma 2.2, we need to be able to control z (and hence $\mathbf{v}$) in terms of q. This is possible only if we have a supplementary condition such as (2.16) (since $q = 0$ just means $\mathbf{v}(r) = e^{\alpha R}h(r/s)$ for some s, α). Parts of the proof of the following estimates are a simple adaptation of the corresponding argument in [11], where the orthogonality condition was somewhat different.

PROPOSITION 2.3

If $m \geq 3$ and (2.16) holds, and if $\|z\|_X \ll 1$, then for $2 \leq p < \infty$,

(1) $\|z_\rho\|_{L^p} + \|z/\rho\|_{L^p} \lesssim s^{1-2/p}\|q\|_{L^p}$;

(2) *if* $m > 3$, $\|z_\rho/\rho\|_{L^2} + \|z/\rho^2\|_{L^2} \lesssim s\|q/r\|_{L^2}$.

Proof

The first observation is that, modulo nonlinear terms, $q(r)$ is equivalent to $(1/s)(L_0 z)(r/s)$, where $L_0 = \partial_\rho + (m/\rho)h_3(\rho)$. Precisely,

$$\begin{aligned}
s\mathbf{W}(s\rho) = \mathbf{V}^{\rho}(L_0 z) &+ \frac{m}{\rho} z_1 (h_1 z_2 + h_3 \gamma) \widehat{\boldsymbol{\jmath}} \\
&+ \frac{m}{\rho}\big(- h_1 z_1^2 + [h_3 z_2 - h_1(1+\gamma)]\gamma \big) J^{\mathbf{h}} \widehat{\boldsymbol{\jmath}} \\
&+ \Big(\gamma_\rho + \frac{m}{\rho}[h_1 z_2 \gamma - h_3 |z|^2] \Big) \mathbf{h}.
\end{aligned}$$

Using (2.5), it follows easily that for $2 \le p \le \infty$,

$$\|L_0 z\|_{L^p} \lesssim s^{1-2/p} \|q\|_{L^p} + (\|z\|_X + \|z\|_X^3) \Big\| |z_\rho| + \frac{|z|}{\rho} \Big\|_{L^p},$$

$$\Big\| \frac{1}{\rho} L_0 z \Big\|_{L^2} \lesssim s \Big\| \frac{1}{r} q \Big\|_{L^2} + (\|z\|_X + \|z\|_X^3) \Big\| \frac{|z_\rho|}{\rho} + \frac{|z|}{\rho^2} \Big\|_{L^2}.$$

In light of these estimates, and $\|z\|_X \ll 1$, Proposition 2.3 follows from the next lemma.

LEMMA 2.4
For $m \ge 3$ and $z(\rho)$ satisfying (2.16),
(1) $\|z\|_X \lesssim \|L_0 z\|_{L^2}$;
(2) $\| |z_\rho| + |z|/\rho \|_{L^p} \lesssim \|L_0 z\|_{L^p}$ *for* $2 \le p < \infty$;
(3) *if* $m > 3$, $\| |z_\rho|/\rho + |z|/\rho^2 \|_{L^2} \lesssim \|L_0 z/\rho\|_{L^2}$.

Proof
An estimate very similar to the first one here is proved in [11]. (Only the orthogonality condition is different.) Here, we prove the first and third statements together, by showing

$$\Big\| \frac{|z_\rho|}{\rho^b} + \frac{|z|}{\rho^{1+b}} \Big\|_{L^2} \lesssim \Big\| \frac{L_0 z}{\rho^b} \Big\|_{L^2}$$

for $-1 \le b \le 1$. If this is false, we have a sequence $\{z_j\}$ with

$$\begin{aligned}
&\Big\| \frac{(z_j)_\rho}{\rho^b} \Big\|_{L^2}^2 + \Big\| \frac{z_j}{\rho^{1+b}} \Big\|_{L^2}^2 = 1, \\
&\int z_j(\rho) h_1(\rho) \rho \, d\rho = 0, \qquad (2.25) \\
&\Big\| \frac{L_0 z_j}{\rho^b} \Big\|_{L^2} \to 0.
\end{aligned}$$

It follows that up to subsequence, $z_j \to z^*$ weakly in H^1 and strongly in L^2 on compact subsets of $(0, \infty)$ and that $L_0 z^* = 0$. Hence $z^*(\rho) = C h_1(\rho)$ for some

$C \in \mathbb{C}$. Integration by parts gives

$$\left\| \frac{L_0 z_j}{\rho^b} \right\|_{L^2}^2 = \left\| \frac{(z_j)_\rho}{\rho^b} \right\|_{L^2}^2 + m \int_0^\infty \frac{|z_j|^2}{\rho^{2b+2}} \big(m + 2bh_3(\rho) - 2mh_1^2(\rho)\big) \rho \, d\rho,$$

and so, defining $V(\rho) := m + 2bh_3(\rho) - 2mh_1^2(\rho)$, we see that for any $\epsilon < 1/m$,

$$\limsup_{j \to \infty} m \int_0^\infty \frac{|z_j|^2}{\rho^{2b+2}} [V(\rho) - \epsilon] \rho \, d\rho \le -m\epsilon.$$

If $2|b| + \epsilon < m$ (which certainly holds under our assumptions that $|b| \le 1$ and $m > 3$), then $\{\rho \mid V(\rho) - \epsilon \le 0\}$ is a compact subset of $(0, \infty)$, and so

$$m \int_{V-\epsilon \le 0} \frac{|C|^2 h_1^2(\rho)}{\rho^{2b+1}} [V(\rho) - \epsilon] \rho \, d\rho = \lim_{j \to \infty} m \int_{V-\epsilon \le 0} \frac{|z_j|^2}{\rho^{2b+2}} [V(\rho) - \epsilon] \rho \, d\rho \le -m\epsilon,$$

which implies that $C \neq 0$. Finally, for any $\epsilon' > 0$,

$$\begin{aligned} 0 &= \lim_{j \to \infty} \int_0^\infty z_j(\rho) h_1(\rho) \rho \, d\rho \\ &= \int_{\epsilon'}^{1/\epsilon'} C h_1^2(\rho) \rho \, d\rho + \lim_{j \to \infty} \Big(\int_0^{\epsilon'} + \int_{1/\epsilon'}^\infty \Big) z_j(\rho) h_1(\rho) \rho \, d\rho. \end{aligned}$$

Since $\|z_j/\rho^{1+b}\|_{L^2} \le 1$, and $\rho^{1+b} h_1 \in L^2$ (this is precisely where we need $m > 3$ for $b = 1$), the last integrals are uniformly small in ϵ', and we arrive at

$$0 = \int_0^\infty C h_1^2(\rho) \rho \, d\rho,$$

contradicting $C \neq 0$.

We now prove the second statement. First, note that following [11, proof of Lem. 4.4], the estimate

$$\left\| |z_\rho| + \frac{|z|}{\rho} \right\|_{L^p} \lesssim \|L_0 z\|_{L^p} + \|L_0 z\|_{L^2} \tag{2.26}$$

can be deduced from the X-estimate above (the case of $b = 0$). Now, fix a smooth cutoff function $\Phi(t)$ with $\Phi(t) = 1$ for $t \in [0, 1]$, $\Phi(t) = 0$ for $t \in [2, \infty)$, and $\Phi_t(t) < 0$ for $t \in (1, 2)$. Let $\phi(\rho) := \Phi(t)$ with $t = (\rho/s)^\beta$, where $s \gg 1$ and $0 < \beta \ll 1$ are such that

$$\varepsilon_1 = \|\rho \phi_\rho(\rho)\|_{L^\infty} \lesssim \beta$$

and

$$\varepsilon_2 := \big\| \rho[1-\phi(\rho)]h_1(\rho) \big\|_{L^2(\rho\, d\rho)} \le \| \rho h_1(\rho) \|_{L^2((s,\infty),\rho\, d\rho)}$$

are sufficiently small. Now, using (2.16),

$$\Big| \int h_1 z\phi\rho\, d\rho \Big| = \Big| \int h_1 z(1-\phi)\rho\, d\rho \Big| \le \varepsilon_2 \Big\| \frac{z}{\rho} \Big\|_{L^2(\rho\, d\rho)}.$$

Observe that the proof of the X-estimate above (and hence also of (2.26)) works even if $\big| \int h_1 z\rho\, d\rho \big| = o(1)\|z/\rho\|_{L^2}$, and so provided ε_2 is sufficiently small, we can apply (2.26) to obtain

$$\begin{aligned} \Big\| \frac{z}{\rho} \Big\|_p &\le \Big\| \frac{z\phi}{\rho} \Big\|_p + \Big\| \frac{z(1-\phi)}{\rho} \Big\|_p \\ &\lesssim \|L_0(z\phi)\|_p + \|L_0(z\phi)\|_2 + \Big\| \frac{z(1-\phi)}{\rho} \Big\|_p \\ &\lesssim \|L_0(z\phi)\|_p + \Big\| \frac{z(1-\phi)}{\rho} \Big\|_p. \end{aligned}$$

Now, $1-\phi$ is supported for $\rho \ge s \gg 1$, and on this set, $h_3(\rho) \ge 1/2$. Then an easy adaptation, [5, Lem. 3.6] (using $m > 1$), yields

$$\Big\| \frac{z(1-\phi)}{\rho} \Big\|_p \lesssim \big\| L_0\big(z(1-\phi)\big) \big\|_p,$$

and hence

$$\begin{aligned} \Big\| \frac{z}{\rho} \Big\|_p &\lesssim \|L_0(z\phi)\|_p + \big\| L_0\big(z(1-\phi)\big) \big\|_p \\ &\lesssim \|L_0(z)\phi\|_p + \|L_0(z)(1-\phi)\|_p + \|z\phi_\rho\|_p. \end{aligned}$$

Since $\|z\phi_\rho\|_p \le \varepsilon_1 \|z/\rho\|_p$, we conclude that

$$\|z_\rho\|_p + \Big\| \frac{z}{\rho} \Big\|_p \le C\|L_0(z)\|_p + C\varepsilon_1 \Big\| \frac{z}{\rho} \Big\|_p.$$

If ε_1 is small enough, the last term can be absorbed to the left-hand side.

That completes the proof of the lemma. □

End of proof of Proposition 2.3

Hence the proposition is proved. □

Combining Proposition 2.3 with Lemma 2.2 leads to the following.

COROLLARY 2.5

Under the conditions of Proposition 2.3, if $m > 3$,

$$|s^{-1}\dot{s}| + |\dot{\alpha}| \lesssim \left\| \frac{q}{r} \right\|_{L^2}^2. \tag{2.27}$$

This is our main estimate of the harmonic map parameters $s(t)$ and $\alpha(t)$.

2.5. Nonlinear estimates

We can now use Proposition 2.3 to estimate the nonlinear terms in (2.22). The idea is that from the splitting of Section 2.1, we expect $v_3(r,t) = h_3(r/s(t)) +$ "small." We *freeze* the scaling factor $s(t)$ at, say, $s_0 := s(0)$ (and, without loss of generality, we rescale the solution so that $s_0 = 1$) and treat the corresponding correction as a nonlinear term:

$$iq_t + \Delta_r q - \frac{1 + m^2 - 2mh_3(r)}{r^2} q = Uq + Sq, \tag{2.28}$$

where

$$U := \frac{1}{r^2}\Big[m(v_3 - h_3)\big(m(v_3 + h_3) - 2\big) + mr\big((v_3)_r - (h_3)_r\big)\Big].$$

(Here, we have used $r(h_3)_r = mh_1^2$ and $h_1^2 + h_3^2 = 1$.) Recall, from (2.24),

$$S(r,t) = -\frac{1}{2}Q(r,t) + \int_r^\infty \frac{1}{\tau} Q(\tau,t)\,d\tau, \qquad Q := |q|^2 + \frac{2m}{r}\operatorname{Re}(\bar{v}q).$$

The next lemma estimates the right-hand side of (2.28) in various space-time norms.

LEMMA 2.6

Provided that (2.16) holds and that $\|z\|_X \ll 1$, *we have*

$$\|rUq\|_{L_t^2 L_x^2} \lesssim \big((1 + \|s^{-1}\|_{L_t^\infty})\|s-1\|_{L_t^\infty} + \|q\|_{L_t^\infty L_x^2}\big)\left\|\frac{q}{r}\right\|_{L_t^2 L_x^2} + \|s^{-1}\|_{L_t^\infty}^{1/2}\|q\|_{L_t^4 L_x^4}^2 \tag{2.29}$$

and

$$\|Sq\|_{L_t^{4/3} L_x^{4/3}} \lesssim \|q\|_{L_t^4 L_x^4}\Big(\|q\|_{L_t^4 L_x^4}^2 + \left\|\frac{q}{r}\right\|_{L_t^2 L_x^2}\Big). \tag{2.30}$$

Proof

Recall

$$v_3(r) = h_3\Big(\frac{r}{s}\Big) + \xi_3\Big(\frac{r}{s}\Big) = \left(1 + \gamma\Big(\frac{r}{s}\Big)\right)h_3\Big(\frac{r}{s}\Big) + h_1\Big(\frac{r}{s}\Big)z_2\Big(\frac{r}{s}\Big),$$

and set, as usual, $\rho = r/s$. Estimate (2.29) follows from $\|z\|_{L^\infty} \lesssim \|z\|_X$, the estimates in Proposition 2.3, and

- $|h_3(r/s) - h_3(r)| = \left|\int_1^s \frac{d}{d\tau} h_3(r/\tau)\,d\tau\right| = m\left|\int_1^s (1/\tau) h_1^2(r/\tau)\,d\tau\right| \lesssim [\min(1,s)]^{-1}|s-1|$,
- $r|[h_3(r/s)]_r - [h_3(r)]_r| = m|h_1^2(r/s) - h_1^2(r)| \lesssim [\min(1,s)]^{-1}|s-1|$.

For estimate (2.30), begin with

$$\|Sq\|_{L_x^{4/3}L_t^{4/3}} \le \|q\|_{L_t^4 L_x^4}\|S\|_{L_t^2 L_x^2}.$$

Using the Hardy-type inequality $\|\cdot\|_{L_x^2} \lesssim \|r\partial_r \cdot\|_{L_x^2}$ yields

$$\|S\|_{L_t^2 L_x^2} \lesssim \|Q\|_{L_t^2 L_x^2} \lesssim \|q\|^2_{L_t^4 L_x^4} + \|\nu\|_{L_t^\infty L_x^\infty}\left\|\frac{q}{r}\right\|_{L_t^2 L_x^2}.$$

And since $|\nu| = |\widehat{\mathbf{k}} - v_3\mathbf{v}| \lesssim 1$, we arrive at (2.30). □

3. Dispersive estimates for critical-decay potentials in two dimensions

In order to establish any decay (dispersion) of solutions of (2.28), we need good dispersive estimates for the linear part

$$iq_t = -q_{rr} - \frac{1}{r}q_r + \frac{1}{r^2}(1 + m^2 - 2mh_3)q. \tag{3.1}$$

This turns out to be a little tricky since it is a *borderline* case in two senses: the space dimension is 2, and the potential has $(1/r^2)$-behavior both at the origin and at infinity; that is,

$$\frac{1}{r^2}\big(1 + m^2 - 2mh_3(r)\big) \sim \begin{cases} \dfrac{(1+m)^2}{r^2}, & r \to 0, \\ \dfrac{(1-m)^2}{r^2}, & r \to \infty. \end{cases} \tag{3.2}$$

In this section, we consider linear Schrödinger operators like the one appearing on the right-hand side of (3.1). More precisely, let

$$H = -\Delta + \frac{1}{r^2} + V(r), \quad V \in C^\infty(0,\infty),\ 0 \le r^2 V(r) \le \text{const}. \tag{3.3}$$

Such an operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2\backslash\{0\})$, extends to a self-adjoint operator on a domain $D(H)$ with $C_0^\infty(\mathbb{R}^2\backslash\{0\}) \subset D(H) \subset L^2(\mathbb{R}^2)$, and generates a one-parameter unitary group e^{-itH} such that for $\phi \in L^2$, $\psi = e^{-itH}\phi$ is the solution of the linear Schrödinger equation $i\psi_t = H\psi$ with initial data $\psi|_{t=0} = \phi$ (see, e.g., [18]).

Our goal is to obtain dispersive space-time (*Strichartz*) estimates for e^{-itH} of the sort which hold for the *free* ($H = -\Delta$) evolution:

$$\|e^{it\Delta}\phi\|_{L_t^r L_x^p} + \left\| \int_0^t e^{i(t-s)\Delta} f(s)\, ds \right\|_{L_t^r L_x^p} \lesssim \|\phi\|_{L^2} + \|f\|_{L_t^{\tilde r'} L_x^{\tilde p'}}, \tag{3.4}$$

where (r, p) and $(\tilde r, \tilde p)$ are *admissible* pairs of exponents:

$$(r, p) \text{ admissible} \longleftrightarrow \frac{1}{r} + \frac{1}{p} = \frac{1}{2}, \quad 2 < r \le \infty,$$

and $p' = p/(p-1)$ denotes the Hölder dual exponent. The *endpoint* case of (3.4), $(r, p) = (2, \infty)$, is known to be false in general but true for radial ϕ and f, save for the *double-endpoint* case of $r = \tilde r = 2$ (see [23]).

Perturbative arguments to extend estimates like (3.4) to Schrödinger operators with potentials (in general, one has to include a projection onto the continuous spectral subspace in order to avoid bound states, which do not disperse) cannot work for borderline behavior like (3.2). Fortunately, the problem of obtaining dispersive estimates when the potential has this critical falloff (and singularity) has been taken up in recent articles by Burq, Planchon, Stalker, and Tahvildar-Zadeh [2], [3]. In place of a perturbative argument, the authors make a repulsivity assumption on the potential (which, in particular, rules out bound states), and they prove more or less directly—by identities—that solutions have some time decay, in a spatially weighted space-time sense (a *Kato smoothing*–type estimate). This approach is ideally suited to our present problem: the operator appearing in (3.1) satisfies the following repulsivity property. When written in the form (3.3),

$$-r^2 \big(rV(r)\big)_r + 1 \ge \nu \quad \text{for some } \nu > 0. \tag{3.5}$$

We cannot rely directly on the results of [2] and [3] here. The article [2] considers only potentials (const)$/r^2$, while the results of [3] hold in dimension at least 3 only and do not immediately extend to dimension 2 for two reasons: one is the failure of the Hardy inequality, and the other is the failure of the double-endpoint Strichartz estimate (even for radial functions). However, we can recover the argument from [3] by exploiting the radial symmetry of our functions to avoid the Hardy inequality, and we can avoid the use of the double-endpoint Strichartz estimate by following the approach of [2], which in turn follows [20].

THEOREM 3.1
Suppose that the Schrödinger operator H satisfies conditions (3.3) and (3.5). Let $\phi = \phi(r)$ be radially symmetric. Then for any admissible pair (r, p), we have

$$\|e^{-itH}\phi\|_{L_t^r L_x^p} + \left\| \frac{1}{|x|} e^{-itH}\phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}. \tag{3.6}$$

If $f = f(r,t)$ is radially symmetric and $(\tilde{r}, \tilde{p})$ is another admissible pair, then

$$\Big\| \int_0^t e^{-i(t-s)H} f(x,s)\, ds \Big\|_{L_t^r L_x^p} + \Big\| \frac{1}{|x|} \int_0^t e^{-i(t-s)H} f(x,s)\, ds \Big\|_{L_t^2 L_x^2} \lesssim \min\big(\|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}, \| |x| f \|_{L_t^2 L_x^2} \big). \tag{3.7}$$

Remark 3.2

In [3], the single endpoint Strichartz estimate ((3.6) with $r = 2$) is also obtained for dimensions at least 3. In two dimensions, though it holds in the *free*, radial case, we do not know if it holds for our operators. However, it is *essential* to the present article to have an estimate with L_t^2-decay (L_t^r with $r > 2$ is simply not enough; see Sec. 4). Our way around this problem is to use the above *weighted* $L_t^2 L_x^2$-estimate that arises naturally in the approach of [3].

Proof of Theorem 3.1

Parts of the proof are perturbative, so we identify a reference operator:

$$H = -\Delta + \frac{1}{r^2} + V =: H_0 + V.$$

Note that $H_0 = -\Delta + 1/r^2$ satisfies the "usual" Strichartz estimates (those satisfied by $-\Delta$, as in (3.4)) on radial functions since H_0 is simply $-\Delta$ conjugated by $e^{i\theta}$ when acting on such functions.

Step 1. Following [3], we begin with weighted resolvent estimates.

LEMMA 3.3

For $f = f(r)$ radial,

$$\sup_{\mu \notin \mathbb{R}} \Big\| \frac{1}{|x|} (H - \mu)^{-1} f \Big\|_{L^2(\mathbb{R}^2)} \lesssim \| |x| f \|_{L^2(\mathbb{R}^2)}. \tag{3.8}$$

Proof

We can assume that $f \in C_0^\infty(0, \infty)$, with the lemma then following from a standard density argument. Set $u := (H - \mu)^{-1} f$ so that $(H - \mu) u = f$, and note that $u = u(r)$ is radial since f is. To avoid the use of the Hardy inequality in [3], we change variables from $u(r)$ to

$$v(x) := e^{i\theta} u(r)$$

and use $|\nabla v|^2 = |u_r|^2 + (1/r^2)|u|^2$, so

$$\left\| \frac{v}{|x|} \right\|_{L^2} \lesssim \|v\|_{H^1}. \tag{3.9}$$

In terms of v, the equation for u becomes

$$(-\Delta + V - \mu)v = \tilde{f}, \tag{3.10}$$

where $\tilde{f}(x) := e^{i\theta} f(r) \in L^2$, and so $v \in D(-\Delta + V) \subset H^2$. The proof of the lemma now follows the corresponding proof in [3] precisely, using $\frac{-d^2}{d\theta^2} \geq 1$ on functions of our form $e^{i\theta} f(r)$, and with (3.9) (rather than Hardy) providing $v/|x| \in L^2$ where needed. □

Step 2. As in [3], the next step is to invoke [14] to conclude that the resolvent estimate (3.8) implies the following *Kato smoothing* weighted L^2-estimate for the propagator: for $\phi = \phi(r)$,

$$\left\| \frac{1}{|x|} e^{-itH} \phi \right\|_{L^2_t L^2_x} \lesssim \|\phi\|_{L^2}. \tag{3.11}$$

This is one part of (3.6). Note that the reference operator H_0 also satisfies the weighted estimate (3.11) (a fact that follows from the same argument). Another direct consequence of the resolvent estimate (3.8) is the inhomogeneous version of (3.11),

$$\left\| \frac{1}{|x|} \int_0^t e^{-i(t-s)H} f(\cdot, s)\, ds \right\|_{L^2_t L^2_x} \lesssim \| |x| f \|_{L^2_t L^2_x}, \tag{3.12}$$

which is one part of (3.7). The estimate (3.12) is probably standard, but we did not see a proof, and so we supply one in Section B.2.

Step 3. Next, we establish more of the inhomogeneous estimates in (3.7) but first for the reference operator H_0. Since we do not have the double-endpoint Strichartz estimate available, we now depart from [3] and henceforth follow [2] (which in turn relies partly on [20]). Note that by (3.11) for H_0, for any $\psi \in L^2_x$,

$$\begin{aligned}\left(\psi, \int_0^\infty e^{isH_0} f(\cdot, s)\, ds\right)_{L^2_x} &= \int_0^\infty ds \left(e^{-isH_0}\psi, f(\cdot, s)\right)_{L^2_x} \\ &\leq \left\| \frac{1}{|x|} e^{-isH_0} \psi \right\|_{L^2_t L^2_x} \| |x| f \|_{L^2_t L^2_x} \lesssim \|\psi\|_{L^2} \| |x| f \|_{L^2_t L^2_x},\end{aligned}$$

yielding

$$\left\| \int_0^\infty e^{isH_0} f(\cdot, s)\, ds \right\|_{L_x^2} \lesssim \| |x| f \|_{L_t^2 L_x^2}.$$

Hence by the Strichartz estimates for H_0, for (r, p) admissible,

$$\begin{aligned} \left\| \int_0^\infty e^{-i(t-s)H_0} f(\cdot, s)\, ds \right\|_{L_t^r L_x^p} &= \left\| e^{-itH_0} \int_0^\infty e^{isH_0} f(\cdot, s)\, ds \right\|_{L_t^r L_x^p} \\ &\lesssim \left\| \int_0^\infty e^{isH_0} f(\cdot, s)\, ds \right\|_{L_x^2} \lesssim \| |x| f \|_{L_t^2 L_x^2}. \end{aligned}$$

Finally, the required estimate

$$\left\| \int_0^t e^{-i(t-s)H_0} f(\cdot, s)\, ds \right\|_{L_t^r L_x^p} \lesssim \| |x| f \|_{L_t^2 L_x^2} \tag{3.13}$$

follows from a general argument of Christ and Kiselev [6] (see also [2]).

Step 4. To obtain the remaining part of (3.6) (the Strichartz estimate), we use (3.11) and (3.13) in a perturbative argument. We have

$$e^{-itH}\phi = e^{-itH_0}\phi + i \int_0^t e^{-i(t-s)H_0} V e^{-isH} \phi\, ds,$$

and so for (r, p) admissible,

$$\begin{aligned} \|e^{-itH}\phi\|_{L_t^r L_x^p} &\lesssim \|\phi\|_{L^2} + \| |x| V e^{-isH} \phi \|_{L_t^2 L_x^2} \\ &\le \|\phi\|_{L^2} + \| |x|^2 V \|_{L^\infty} \left\| \frac{1}{|x|} e^{-isH} \phi \right\|_{L_t^2 L_x^2} \\ &\lesssim \|\phi\|_{L^2}. \end{aligned}$$

This finishes the proof of (3.6).

Step 5. It remains to prove the rest of the inhomogeneous estimates in (3.7). But given (3.6), these follow again from the argument used in step 3.

That completes the proof of Theorem 3.1. □

COROLLARY 3.4
If $m \ge 2$, estimates (3.6) and (3.7) hold for the operator

$$H := -\Delta + \frac{1}{r^2}(1 + m^2 - 2mh_3)$$

coming from the Schrödinger map problem.

Proof
We have

$$\frac{1}{r^2}(1+m^2-2mh_3)=\frac{1}{r^2}+V(r), \quad V(r)=\frac{m}{r^2}\big(m-2h_3(r)\big).$$

So for $m \geq 2$,

$$(m+1)^2 \geq 1+r^2V(r) \geq (m-1)^2 \geq 1,$$

and

$$1-r^2(rV)_r = 1+m\big(m-2h_3(r)+2mh_1^2(r)\big) \geq 1+m(m-2) \geq 1.$$

Thus conditions (3.3) and (3.5) both hold with $\nu = 1$. □

4. Proof of the main theorem

Let $\mathbf{u} \in C([0, T_{\max}); \Sigma_m)$ be the solution of the Schrödinger map equation (1.1) with initial data $\mathbf{u}_0$ (given by Th. 1.4). Energy is conserved:

$$\mathcal{E}\big(\mathbf{u}(t)\big)=\mathcal{E}(\mathbf{u}_0)=4\pi m+\delta_1^2.$$

We begin by splitting the initial data $\mathbf{u}(0)$, using the following lemma, which is proved in Section B.3.

LEMMA 4.1
If $m \geq 3$ and δ is sufficiently small, then for any map $\mathbf{u} \in \Sigma_m$ with $\mathcal{E}(\mathbf{u}) \leq 4\pi m+\delta^2$, there exist $s > 0$, $\alpha \in \mathbb{R}$, and a complex function $z(\rho)$ such that

$$\mathbf{u}(r,\theta)=e^{[m\theta+\alpha]R}\left[\left(1+\gamma\left(\frac{r}{s}\right)\right)\mathbf{h}\left(\frac{r}{s}\right)+\mathbf{V}^{r/s}\left(z\left(\frac{r}{s}\right)\right)\right] \tag{4.1}$$

with z satisfying (2.16); that is,

$$\int_0^\infty z(\rho)h_1(\rho)\rho\,d\rho=0, \tag{4.2}$$

and $\|z\|_X^2 \lesssim \mathcal{E}(\mathbf{u})-4\pi m$.

Invoking Lemma 4.1, we have

$$\mathbf{u}_0=e^{[m\theta+\alpha_0]R}\left[\left(1+\gamma\left(\frac{r}{s_0}\right)\right)\mathbf{h}\left(\frac{r}{s_0}\right)+\mathbf{V}^{r/s_0}\left(z\left(\frac{r}{s_0}\right)\right)\right]$$

with z_0 satisfying the orthogonality condition (2.16), and

$$\|z_0\|_X \lesssim \delta_1 \ll 1.$$

Now, rescale by setting

$$\hat{\mathbf{u}}(x,t) := \mathbf{u}(s_0 x, s_0^2 t).$$

Then $\hat{\mathbf{u}}$ is another solution of the Schrödinger map equation (1.1), and

$$\hat{\mathbf{u}}(x,0) = e^{[m\theta+\alpha_0]R}\big[\big(1+\gamma_0(r)\big)\mathbf{h}(r) + \mathbf{V}^r\big(z_0(r)\big)\big].$$

Let $q(r,t)$ be the complex function derived from the Schrödinger map $\hat{\mathbf{u}}$, as in Section 2.4.

Suppose that (r, p) is an admissible pair of exponents. Define a space-time norm Y by

$$\|q\|_Y := \|q\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^4 \cap L_t^r L_x^p} + \Big\|\frac{q}{r}\Big\|_{L_t^2 L_x^2}.$$

As long as $\|z\|_X \lesssim \|q\|_{L_x^2}$ remains sufficiently small, Corollary 3.4, together with estimates (2.29) and (2.30), yields

$$\|q\|_Y \lesssim \|q(0)\|_{L^2} + \big[(1+\|s^{-1}\|_{L_t^\infty})\|s-1\|_{L_t^\infty} + (1+\|s^{-1}\|_{L_t^\infty})\|q\|_Y + \|q\|_Y^2\big]\|q\|_Y. \tag{4.3}$$

We also have

$$\hat{\mathbf{u}} = e^{[m\theta+\alpha(t)]R}\Big[\Big(1+\gamma\Big(\frac{r}{s(t)},t\Big)\Big)\mathbf{h}\Big(\frac{r}{s(t)}\Big) + \mathbf{V}^{r/s(t)}\Big(z\Big(\frac{r}{s(t)},t\Big)\Big)\Big],$$

with $z(\rho,t)$ satisfying (2.16), $s(0)=1$, $\alpha(0)=\alpha_0$, and, by Corollary 2.5, $s(t) \in C([0,T];\mathbb{R}^+)$ and $\alpha(t) \in C([0,T];\mathbb{R})$, with

$$\|s^{-1}\dot{s}\|_{L_t^1} + \|\dot{\alpha}\|_{L_t^1} \lesssim \|q\|_Y^2. \tag{4.4}$$

Taking $\|q(0)\|_{L^2} \lesssim \delta_1$ sufficiently small, estimates (4.3) and (4.4) yield

$$\|q\|_Y \lesssim \delta_1, \qquad \|s^{-1}\dot{s}\|_{L_t^1} + \|\dot{\alpha}\|_{L_t^1} \lesssim \delta_1^2. \tag{4.5}$$

(And, in particular, $\|z\|_X \ll 1$ continues to hold.) Since

$$\Big|\nabla\Big[\hat{\mathbf{u}} - e^{[m\theta+\alpha(t)]R}\mathbf{h}\Big(\frac{r}{s(t)}\Big)\Big]\Big| \lesssim \frac{1}{s}\Big(|z_\rho| + \Big|\frac{z}{\rho}\Big|\Big)(1+|z|),$$

the estimates of Proposition 2.3 give

$$\Big\|\nabla\Big[\hat{\mathbf{u}} - e^{[m\theta+\alpha(t)]R}\mathbf{h}\Big(\frac{r}{s(t)}\Big)\Big]\Big\|_Y \lesssim \|q\|_Y \lesssim \delta_1. \tag{4.6}$$

Estimate (4.5) shows

(a) that $s(t) \geq \text{const} > 0$, and hence, by Corollary 1.5, we must have $T_{\max} = \infty$;

(b) that

$$s(t) \to s_\infty \in (1 - c\delta_1^2, 1 + c\delta_1^2), \qquad \alpha(t) \to \alpha_\infty \in (\alpha_0 - c\delta_1^2, \alpha_0 + c\delta_1^2)$$

as $t \to \infty$.

Finally, undoing the rescaling, $\mathbf{u}(r, t) = \hat{\mathbf{u}}(r/s_0, t/s_0^2)$, yields the estimates of Theorem 1.7. □

Appendices

A. Local well-posedness

In this appendix, we prove Theorem 1.4 and Corollary 1.5 on the local well-posedness of the Schrödinger flow (1.1) when the data $\mathbf{u}_0 \in \Sigma_m$ has energy $\mathcal{E}(\mathbf{u}_0) = 4\pi m + \delta_0^2$ close to the harmonic map energy, $0 < \delta_0 \leq \delta \ll 1$. In Section A.1, we show that z (and hence $\mathbf{u}$) can be reconstructed from q, s, and α; this section is time-independent. In Section A.2, we set up the equations for the existence proof. In Section A.3, we show that we have a contraction mapping, and we complete the proofs of Theorem 1.4 and Corollary 1.5. In Section A.4, we discuss the small-energy case.

Recall the decomposition $\mathbf{u}(r, \theta) = e^{m\theta R}\mathbf{v}(r)$, and recall

$$\mathbf{v}(r) = e^{\alpha R}\,[\mathbf{h}(\rho) + \boldsymbol{\xi}(\rho)] = e^{\alpha R}\begin{bmatrix}(1+\gamma)h_1 - h_3 z_2\\ z_1\\ (1+\gamma)h_3 + h_1 z_2\end{bmatrix}(\rho), \tag{A.1}$$

where $\rho = r/s$, $\boldsymbol{\xi} = z_1\widehat{\jmath} + z_2 h \times \widehat{\jmath} + \gamma b h$, and $\gamma = \sqrt{1 - |z|^2} - 1$. The time-dependence of $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\xi}$, α, s, and γ has been dropped from (A.1). The equation $D_r\mathbf{e} = 0$ is equivalent to

$$\mathbf{e}_r = -(\mathbf{v}_r \cdot \mathbf{e})\mathbf{v}. \tag{A.2}$$

Recall that $q\mathbf{e} = \mathbf{v}_r - (m/r)J^{\mathrm{v}}R\mathbf{v}$ with $\nu\mathbf{e} = J^{\mathrm{v}}R\mathbf{v} = \widehat{\mathbf{k}} - v_3\mathbf{v}$. By substituting in (A.1) and using $L_0\mathbf{h} = (m/r)\,\widehat{\mathbf{k}}$, $q\mathbf{e}$ should satisfy

$$se^{-\alpha R}q\mathbf{e}(r) = (L_0 z)(\rho)\,\widehat{\jmath} + \mathbf{G_0}(z)(\rho), \quad \rho = \frac{r}{s}, \tag{A.3}$$

where

$$\mathbf{G_0}(z)(\rho) := se^{-\alpha R}\Big[\mathbf{v}_r - \frac{m}{r}(\widehat{\mathbf{k}} - v_3\mathbf{v})\Big] - (L_0 z)\,\widehat{\jmath} = \gamma_\rho\mathbf{h} + \frac{m}{\rho}(\gamma\widehat{\mathbf{k}} + \gamma h_3\mathbf{h} + \xi_3\boldsymbol{\xi}) \tag{A.4}$$

and $\|\mathbf{G_0}(z)\|_{L^2} \lesssim \|z\|_X^2$ when $\|z\|_X \ll 1$. In other words, q is rescaled $L_0 z$, plus error.

In this appendix, we choose a different orthogonality condition for z, instead of (2.16). Specifically, we choose the unique s and α so that

$$\langle h_1, z\rangle_X = 0. \tag{A.5}$$

(Recall that $\langle f, g\rangle_X = \int_0^\infty (\bar{f}_r g_r + (m^2/r^2)\bar{f} g) r\,dr$.) Condition (A.5) makes sense for all $m \neq 0$ and suffices for the proof of local well-posedness. In contrast, (2.16) makes sense only if $|m| \geq 3$, but it is necessary for the study of the time-asymptotic behavior. In [11, Sec. 2], we chose s and α to minimize $\|\mathbf{u} - e^{(m\theta+\alpha)R}\mathbf{h}(\cdot/s)\|_{\dot{H}^1}$. The resulting equations in [11, Lem. 2.6] are $\langle h_1, z_1\rangle_X = 0$ and $\langle h_1, z_2\rangle_X = \int_0^\infty (4m^2/\rho^2) h_1^2 h_3 \gamma(\rho)\rho\,d\rho$. Condition (A.5) is similar but has no error term. The unique choice of s and α can be proved by the implicit function theorem, similar to the proof of Lemma 4.1, and it is skipped. It is important to point out, however, that the parameter s used here, though not the same as $s(\mathbf{u})$ defined in (1.20) and (1.21), is nonetheless comparable: $s = s(\mathbf{u})(1 + O(\delta_0^2))$. (This comes immediately from the implicit-function-theorem argument.) Thus we can state the local well-posedness result (Th. 1.4) in terms of $s(\mathbf{u}_0)$.

A.1. Reconstruction of z and $\mathbf{u}$ from q, s, and α

In this section, all maps are time-independent. For a given map $\mathbf{u} = e^{m\theta R}\mathbf{v}(r) \in \Sigma_m$ with energy close to $4\pi m$, we can define s, α, z, and q. The three quantities s, α, and z determine $\mathbf{u}$ and hence q. Conversely, as is done in Lemma A.2 of this section, we can recover z and $\mathbf{u}$ if s, α, and q are given, assuming that $\|q\|_{L^2} \leq \delta$. Before that, we first prove difference estimates for $\delta\mathbf{e}$ in Lemma A.1.

For given $s > 0$, $\alpha \in \mathbb{R}$, and $z \in X$ small, we define $\mathbf{v}(r) = \mathbf{V}(z, s, \alpha)(r)$ by (A.1), and we define $\mathbf{e}(r) = \hat{\mathbf{E}}(z, s, \alpha)(r)$ by the ODE

$$\mathbf{e}(z)(\infty) = e^{\alpha R}\hat{\jmath}, \qquad \mathbf{e}_r = -(\mathbf{v}_r \cdot \mathbf{e})\mathbf{v}, \quad \text{where } \mathbf{v} = \mathbf{V}(z, s, \alpha). \tag{A.6}$$

Note that the boundary condition is different from that in (2.21). By setting $z_b = 0$, the proof of Lemma A.1 shows that oscillation $\mathbf{e} \lesssim \|z\|_X$ and $\mathbf{e}$ converges as $r \to 0, \infty$, which makes sense of this boundary condition. Also, denote $\hat{\mathbf{E}}(z) = \hat{\mathbf{E}}(z, 1, 0)$. Simple comparison shows

$$\hat{\mathbf{E}}(z, s, \alpha) = e^{\alpha R}\hat{\mathbf{E}}(z^s), \quad z^s(r) := z\Big(\frac{r}{s}\Big). \tag{A.7}$$

LEMMA A.1

Suppose that $z_l \in X$, $l = a, b$, are given with $\|z_l\|_X$ sufficiently small. Let $\delta z := z_a - z_b$, let $\delta\mathbf{v} := \mathbf{V}(z_a, 1, 0) - \mathbf{V}(z_b, 1, 0)$, and let $\delta\mathbf{e} := \hat{\mathbf{E}}(z_a) - \hat{\mathbf{E}}(z_b)$. Then

$$\|\delta\mathbf{v}\|_X + \|\delta\mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X.$$

Proof
Note that

$$\|\mathbf{h}_r\|_{L^2(rdr)} \le C; \qquad \|\boldsymbol{\xi}^l\|_X \lesssim \|z^l\|_X + \|z^l\|_X^2, \quad l = a, b. \tag{A.8}$$

Since $\delta\mathbf{v} = \delta\boldsymbol{\xi} = (\delta z)\widehat{\boldsymbol{\jmath}} + (\delta\gamma)\mathbf{h}$,

$$\|\delta\mathbf{v}\|_X + \|\delta\boldsymbol{\xi}\|_X \lesssim (1 + \|z_a\|_X + \|z_b\|_X)\|\delta z\|_X \lesssim \|\delta z\|_X. \tag{A.9}$$

For $\delta\mathbf{e}$, write $\delta\mathbf{e} = (\delta e_1, \delta e_2, \delta e_3)$, and write

$$\delta e_{j,r} = -(\delta\boldsymbol{\xi}_r \cdot e_a)v_{a,j} - (\mathbf{v}_{b,r} \cdot \delta\mathbf{e})v_{a,j} - (\mathbf{v}_{b,r} \cdot \mathbf{e}_b)\delta\boldsymbol{\xi}_j, \quad j = 1, 2, 3. \tag{A.10}$$

First, consider δe_2. Integrate in r. Using (A.8), (A.9), $v_{a,2} = z_{a,1}$, and $v_{l,r} \in L^2(r\,dr)$,

$$\begin{aligned} |\delta e_2(\tau)| &\lesssim \int_\tau^\infty \left(\left|(\delta\boldsymbol{\xi}_r \cdot \mathbf{e}_a)\frac{z_a}{r}\right| + \left|(\mathbf{v}_{b,r} \cdot \delta\mathbf{e})\frac{z_a}{r}\right| + \left|(\mathbf{v}_{b,r} \cdot \mathbf{e}_b)\frac{\delta z_1}{r}\right|\right) r\,dr \\ &\lesssim (1 + \max_{l=a,b}\|z_l\|_X)\|\delta z\|_X + \max_{l=a,b}\|z_l\|_X\|\delta\mathbf{e}\|_{L^\infty}. \end{aligned} \tag{A.11}$$

Next, we consider δe_1 and δe_3. Equation (A.10) for $j = 1, 3$ can be written as a vector equation for $x = (\delta e_1, \delta e_3)^T$:

$$x_r = A(r)x + F, \tag{A.12}$$

where

$$A(r) = -\begin{bmatrix} h_1 \\ h_3 \end{bmatrix}[h_{1,r}, h_{3,r}] = \frac{m}{r}h_1\begin{bmatrix} h_1h_3, & -h_1^2 \\ h_3^2, & -h_1h_3 \end{bmatrix}$$

and

$$F = \begin{bmatrix} F_1 \\ F_3 \end{bmatrix}, \qquad F_j = -(\delta\boldsymbol{\xi}_r \cdot \mathbf{e}_a)v_{a,j} - (\boldsymbol{\xi}_{b,r} \cdot \delta\mathbf{e})h_j - (\mathbf{v}_{b,r} \cdot \delta\mathbf{e})\xi_{a,j} - (\mathbf{v}_{b,r} \cdot \mathbf{e}_b)\delta\xi_j,$$
$$j = 1, 3.$$

To simplify the linear part $\tilde{x}_r = A(r)\tilde{x}$, let $y = U^{-1}\tilde{x}$, where

$$U(r) = \begin{bmatrix} h_1, & -h_3 \\ h_3, & h_1 \end{bmatrix}, \qquad U^{-1} = \begin{bmatrix} h_1, & h_3 \\ -h_3, & h_1 \end{bmatrix}.$$

Then y satisfies

$$y_r = (U^{-1})_r Uy + U^{-1}AUy = \frac{m}{r}h_1\begin{bmatrix} 0, & 0 \\ -1, & 0 \end{bmatrix}y.$$

This linear system can be solved explicitly,

$$y(r)=\begin{bmatrix}1, & 0\\ p(\rho,r), & 1\end{bmatrix}y(\rho), \qquad p(\rho,r)=-\Big(\int_\rho^r \frac{m}{r}h_1(\tau)\,d\tau\Big)=-[2\arctan\tau^m]_\rho^r.$$

Thus the linear system $\tilde{x}_r=A(r)\tilde{x}$ has the solution $\tilde{x}(r)=P(\rho,r)\tilde{x}(\rho)$ with the propagator

$$P(\rho,r)=U(r)\begin{bmatrix}1, & 0\\ p(\rho,r), & 1\end{bmatrix}U^{-1}(\rho).$$

The original system (A.12) with $x(0)=0$ has the solution

$$x(r)=\int_0^r P(\rho,r)F(\rho)\,d\rho.$$

To estimate $x(r)$, the two terms of F_3 with h_3 as the last factor,

$$\tilde{F}_3=-(\delta\boldsymbol{\xi}_r\cdot\mathbf{e}_a)h_3-(\boldsymbol{\xi}_{b,r}\cdot\delta\mathbf{e})h_3,$$

require special care since $\tilde{F}_3$ may not be in $L^1(dr)$. Other terms can be estimated as follows:

$$\begin{aligned}\left|\int_\infty^r P(\rho,r)\begin{bmatrix}F_1\\ F_3-\tilde{F}_3\end{bmatrix}d\rho\right| &\lesssim \int_0^\infty |F_1|+|F_3-\tilde{F}_3|\,dr\\ &\lesssim \|\delta z\|_X+(\|z_a\|_X+\|z_b\|_X)\,\|\delta\mathbf{e}\|_{L^\infty}.\end{aligned}$$

We treat $\tilde{F}_3$ by integration by parts:

$$\begin{aligned}&\int_\infty^r P(\rho,r)\begin{bmatrix}0\\ \tilde{F}_3\end{bmatrix}d\rho=\int_\infty^r P(\rho,r)\begin{bmatrix}0\\ -(\delta\boldsymbol{\xi}_\rho\cdot\mathbf{e}_a+\boldsymbol{\xi}_{b,\rho}\cdot\delta\mathbf{e})h_3\end{bmatrix}d\rho\\ &=-\begin{bmatrix}0\\ (\delta\boldsymbol{\xi}\cdot\mathbf{e}_a+\boldsymbol{\xi}_b\cdot\delta\mathbf{e})h_3\end{bmatrix}(r)\\ &\quad+\int_\infty^r P(\rho,r)\begin{bmatrix}0\\ (\delta\boldsymbol{\xi}\cdot\mathbf{e}_{a,\rho}+\boldsymbol{\xi}_b\cdot\delta e_\rho)h_3+(\delta\boldsymbol{\xi}\cdot\mathbf{e}_a+\boldsymbol{\xi}_b\cdot\delta\mathbf{e})h_{3,\rho}\end{bmatrix}d\rho\\ &\quad+\int_\infty^r P_\rho(\rho,r)\begin{bmatrix}0\\ (\delta\boldsymbol{\xi}\cdot\mathbf{e}_a)h_3+(\boldsymbol{\xi}_b\cdot\delta\mathbf{e})h_3\end{bmatrix}d\rho=\sum_{j=1}^3 I_j.\end{aligned}$$

Now, we estimate the right-hand side one by one. For I_1,

$$|I_1|\lesssim\|\delta\boldsymbol{\xi}\|_{L^\infty}+\|\boldsymbol{\xi}_b\|_{L^\infty}\|\delta\mathbf{e}\|_{L^\infty}\lesssim\|\delta z\|_X+\|z_b\|_X\|\delta\mathbf{e}\|_{L^\infty}.$$

For I_2, observe that

$$\|\mathbf{e}_{a,r}\|_{L^2} \le C, \qquad \|\delta \mathbf{e}_r\|_{L^2} \lesssim \|\delta z\|_X + \|\delta \mathbf{e}\|_{L^\infty},$$

due to the fact that $\mathbf{e}_{a,r} = -(\mathbf{v}_{a,r} \cdot \mathbf{e}_a)\mathbf{v}_a$ and $\delta \mathbf{e}_r = -(\mathbf{v}_{a,r} \cdot \mathbf{e}_a)\mathbf{v}_a + (\mathbf{v}_{b,r} \cdot \mathbf{e}_b)\mathbf{v}_b$. Thus

$$|I_2| \lesssim \|\delta z\|_X + \|z_b\|_X \left(\|\delta z\|_X + \|\delta \mathbf{e}\|_{L^\infty}\right).$$

To estimate the last term I_3, note that

$$\begin{aligned} &P_\rho(\rho, r) \\ &= \frac{m}{\rho} h_1(\rho) U(r) \cdot \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot U^{-1}(\rho) + \begin{bmatrix} 1 & 0 \\ p(\rho, r) & 1 \end{bmatrix} \cdot \begin{bmatrix} -h_3, & h_1 \\ -h_1, & -h_3 \end{bmatrix} (\rho) \right\}, \end{aligned}$$

and hence $|P_\rho(\rho, r)| \lesssim h_1(\rho)/\rho$. We get

$$|I_3| \lesssim \int_r^\infty \left|\frac{h_1}{\rho}\right| \big(|(\delta \boldsymbol{\xi} \cdot \mathbf{e}_a) h_3| + |(\boldsymbol{\xi}_b \cdot \delta \mathbf{e}) h_3|\big)\, d\rho \lesssim \left\|\frac{h_1}{\rho}\right\|_{L^2} (\|\delta z\|_X + \|z_b\|_X \|\delta \mathbf{e}\|_{L^\infty}).$$

Summing up, we have shown

$$\|\delta \mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X + (\|z_a\|_X + \|z_b\|_X)\, \|\delta \mathbf{e}\|_{L^\infty}.$$

Since $\|z_a\|_X + \|z_b\|_X \ll 1$, we can absorb the last term to the left-hand side. The lemma is proved. □

LEMMA A.2

For given $s > 0$, $\alpha \in \mathbb{R}$, and $q \in L^2_{\mathrm{rad}}$ with $\|q\|_{L^2} \le \delta$, there is a unique function $z = Z(q, s, \alpha) \in X$ such that $\langle h_1, z\rangle_X = 0$, $\|z\|_X \lesssim \delta$, and the functions $\mathbf{v} = \mathbf{V}(Z, s, \alpha)$ $\mathbf{e} = \hat{\mathbf{E}}(Z, s, \alpha)$ satisfy (5.3). Moreover, $Z(q, s, \alpha)$ is independent of α and continuous in q and s.

Proof

Simple comparison shows

$$Z(q, s, \alpha) = Z\big(q(\cdot s), 1, 0\big). \tag{A.13}$$

Thus it suffices to prove the cases of $s = 1$ and $\alpha = 0$. We construct $Z(q, 1, 0)$ by a contraction mapping argument. Define the map

$$\Phi^q(z)(r) = L_0^{-1} \Pi[q \hat{\mathbf{E}}(z) - \mathbf{G_0}(z)](r), \tag{A.14}$$

where $\Pi = (\mathbf{V}^r)^{-1} P^{h(r)}$ is a projection of vector fields on $\mathbb{R}^+$ to $L^2(r\, dr)$ with the mapping $(\mathbf{V}^r)^{-1} : T_{\mathbf{h}(r)}\mathbb{S}^2 \to \mathbb{C}$ and the projection $P^{\mathbf{h}(r)} : \mathbb{R}^3 \to T_{\mathbf{h}(r)}\mathbb{S}^2$ defined in

Section 2.2; L_0^{-1} is the inverse map of L_0 and maps $L^2(r\,dr)$ to the X-subspace $h_1^\perp$; $\hat{\mathbf{E}}(z)$ is defined after (A.6), and $\mathbf{G_0}(z)$ is defined by (A.4).

We show that Φ^q is a contraction mapping in the class

$$\mathscr{A}_\delta = \left\{ z \in X : \|z\|_X \le 2C_1\delta \right\}, \quad C_1 = \|L_0^{-1}(\mathbf{V}^r)^{-1} P^{\mathbf{h}(r)}\|_{B(L^2,X)},$$

for sufficiently small $\delta > 0$. First,

$$\|\Phi^q(z)\|_X \le C_1\|q\|_2 + C\|\mathbf{G_0}\|_2 \le C_1\delta + C\|z\|_X^2.$$

Thus Φ^q maps $\mathscr{A}$ into itself if δ is sufficiently small. We now prove difference estimates for Φ^q. Suppose that $z_a, z_b \in \mathscr{A}$ are given, and let $\mathbf{v}_l = \mathbf{V}(z_l)$ and $\mathbf{e}_l = \hat{\mathbf{E}}(z_l)$, $l = a, b$. Also, define $\boldsymbol{\xi}_l$ by (A.1), and note that $\delta\boldsymbol{\xi} = \delta\mathbf{v}$. By Lemma A.1,

$$\|\delta\mathbf{v}\|_X + \|\delta\boldsymbol{\xi}\|_X + \|\delta\mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X.$$

We now estimate $\delta\mathbf{G_0}(z) = \mathbf{G_0}(z_a) - \mathbf{G_0}(z_b)$ in L_p, $p = 2, 4$ (we need $p = 4$ later):

$$\|\delta\mathbf{G_0}(z)\|_{L_p} \lesssim \|\delta\gamma_r\|_{L_p} + \left\|\frac{\delta\gamma}{r}\right\|_{L_p} + \|\delta(\xi_3\boldsymbol{\xi})\|_{L_p} \lesssim (\|z_a\|_X + \|z_b\|_X)\|\delta z\|_{X_p}. \tag{A.15}$$

Thus

$$\begin{aligned}\|\Phi^q(z_a) - \Phi^q(z_b)\|_X &\lesssim \|q\delta\mathbf{e} - \delta\mathbf{G_0}(z)\|_{L^2} \lesssim \|q\|_{L^2}\|\delta\mathbf{e}\|_{L^\infty} \\ &\quad + (\|z_a\|_X + \|z_b\|_X)\|\delta z\|_X \ll \|\delta z\|_X. \end{aligned}\tag{A.16}$$

Thus Φ^q is indeed a contraction mapping, and the function $Z(q, s, \alpha)$ exists.

We now consider the continuity. The continuity in s follows from (A.13), although it may not be Hölder continuous. For the continuity in q, let q_a and q_b be given, and let $z_l = Z(q_l, s, \alpha)$, $l = a, b$. An estimate similar to (A.16) shows

$$\|\delta z\|_X = \|\Phi^{q_a}(z_a) - \Phi^{q_b}(z_b)\|_X \lesssim \|\delta q\|_{L^2} + \varepsilon\|\delta z\|_X, \tag{A.17}$$

where $\varepsilon = \|q_a\|_{L^2} + \|q_b\|_{L^2} + \|z_a\|_X + \|z_b\|_X \ll 1$, and hence $\varepsilon\|\delta z\|_X$ can be absorbed to the left-hand side. This shows continuity in q in the L^2-norm. □

A.2. Evolution system of q, s, and α

By (A.1), the dynamics of $\mathbf{u}$ are completely determined by the dynamics of z, s, and α. Because of Lemma A.2, they are also completely determined by the dynamics of q, s, and α. The latter system is preferred by us since the q-equation is easier than the z-equation to estimate, and q lies in a more familiar space L^2 rather than z in X.

The equations for z and q are given by (2.12) and (2.22), respectively. However, since we choose the orthogonality condition (A.5) (i.e., $\langle h_1, z(t)\rangle = 0$, $\forall t$), the equations for s and α are different from (2.18).

We now specify the equations that we use. Let $\tilde{q} := e^{i(m+1)\theta+i\alpha}q$. Recall that $\nu\mathbf{e} = \nu_1\mathbf{e} + \nu_2 J^v\mathbf{e} = J^{\mathbf{v}}R\mathbf{v} = \widehat{\mathbf{k}} - v_3\mathbf{v}$, and recall that $\nu_r = -v_3(q + (m/r)\nu)$. By (2.22) and an integration by parts on the potential defined in (2.23), we obtain

$$i\tilde{q}_t + \Delta\tilde{q} = V\tilde{q}, \qquad V = V_1 - V_2 + \int_r^\infty \frac{2}{r'}V_2(r')\,dr', \tag{A.18}$$

where

$$V_1 := \frac{m(1+v_3)(mv_3 - m - 2)}{r^2} + \frac{mv_{3,r}}{r}, \qquad V_2 := \frac{1}{2}|q|^2 + \operatorname{Re}\frac{m}{r}\bar{\nu}q.$$

For s and α, condition (A.5) implies that $\langle h_1, \partial_t z(t)\rangle_X = 0$. Substituting in (2.12), we get

$$\langle h_1,\ (s^2\dot{\alpha} - ims\dot{s})(1+\gamma)h_1 + s^2\dot{\alpha}izh_3 - s\dot{s}rz_r\rangle_X = \langle h_1,\ -iNz + P\mathbf{F_1}\rangle_X\,. \tag{A.19}$$

Note that

$$\langle h_1,\ Nz\rangle_X = (L_0N_0h_1, L_0z)_{L^2}, \qquad \langle h_1,\ r\partial_r z\rangle_X = (rN_0h_1, z_r)_{L^2}.$$

Let $G_1 := \langle h_1, P\mathbf{F_1}\rangle_X = (N_0h_1, P\mathbf{F_1})_{L^2}$, where $N_0 := -\Delta_r + m^2/r^2$. By Lemma B.1 with $g = N_0h_1$,

$$G_1 = \int_0^\infty \Big(ig_r(-\gamma z_r + z\gamma_r) + \frac{m}{r}h_1g(-2\gamma_r - iz_2z_{1,r} + iz_1z_{2,r}) + \frac{m}{r}(h_1g)_r(\gamma^2 - iz_2z) + i\frac{m^2}{r^2}(2h_1^2 - 1)g\gamma z\Big)r\,dr.$$

Separating real and imaginary parts, we can rewrite (A.19) as a system for $\dot{\alpha}$ and $\dot{s}$:

$$(\|h_1\|_X^2 I + A)\begin{bmatrix} s^2\dot{\alpha} \\ -ms\dot{s}\end{bmatrix} = \vec{G}_2 := \begin{bmatrix}(L_0N_0h_1, L_0z_2)_2 \\ -(L_0N_0h_1, L_0z_1)_2\end{bmatrix} + \begin{bmatrix}\operatorname{Re} G_1 \\ \operatorname{Im} G_1\end{bmatrix}, \tag{A.20}$$

where

$$I = \begin{bmatrix} 1, & 0 \\ 0, & 1\end{bmatrix}, \qquad A = \begin{bmatrix} \langle h_1,\ \gamma h_1 - z_2h_3\rangle_X\,, & \frac{1}{m}(rN_0h_1,\ z_{1,r})_{L^2} \\ \langle h_1,\ z_1h_3\rangle_X\,, & \langle h_1,\ \gamma h_1\rangle_X + \frac{1}{m}(rN_0h_1, z_{2,r})_{L^2}\end{bmatrix}.$$

We have $\|A\|_{L^\infty} \lesssim \|z\|_X$.

We study the integral equation version of (A.18) and (A.20) for $\tilde{q}$, s, and α:

$$\tilde{q}(t) = e^{-it\Delta}\tilde{q}_0 - i\int_0^t e^{-i(t-\tau)\Delta}(V\tilde{q})(\tau)\,d\tau, \tag{A.21}$$

$$\begin{bmatrix} s(t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} s_0 \\ \alpha_0 \end{bmatrix} + \int_0^t \left\{ \begin{bmatrix} 0 & -(ms)^{-1} \\ s^{-2} & 0 \end{bmatrix} (\|h_1\|_X^2 I + A)^{-1}\vec{G}_2 \right\}(\tau)\,d\tau. \tag{A.22}$$

A.3. Contraction mapping and conclusion

Proof of Theorem 1.4

Let $q_0 \in L^2_{\mathrm{rad}}(\mathbb{R}^2)$, let $s_0 > 0$, and let $\alpha_0 \in \mathbb{R}$ be given with $\|q_0\|_{L^2} \le \delta$. For $\delta, \sigma > 0$ sufficiently small, we find a solution of (A.21) and (A.22) for $t \in I = [0, \sigma s_0^2]$.

We first construct the solution assuming $s_0 = 1$. The solution for general s_0 is obtained from rescaling,

$$\mathbf{u}(t,x) = \tilde{\mathbf{u}}\Big(\frac{t}{s_0^2}, \frac{x}{s_0}\Big),$$

where $\tilde{u}$ is the solution corresponding to initial data $\tilde{\mathbf{u}}_0(x) = \mathbf{u}_0(x/s_0)$ and $s(\tilde{\mathbf{u}}_0) = 1$.

Assuming that $s_0 = 1$, we define a (contraction) mapping in the following class:

$$\mathscr{A}_{\delta,\sigma} = \Big\{(\tilde{q}, s, \alpha) : I = [0,\sigma] \to L^2(\mathbb{R}^2) \times \mathbb{R}^+ \times \mathbb{R} : \|\tilde{q}\|_{\mathrm{Str}[I]} \le \delta; \\ \forall t,\ q(t) = e^{-i(m+1)\theta - i\alpha(t)}\tilde{q}(t) \in L^2_{\mathrm{rad}},\ s(t) \in [0.5, 1.5]\Big\} \tag{A.23}$$

for sufficiently small $\delta, \sigma > 0$. Here,

$$\|q\|_{\mathrm{Str}[I]} \equiv \|q\|_{L_t^\infty L_x^2[I] \cap L_t^4 L_x^4[I] \cap L_t^{8/3} L_x^8[I]}.$$

The map is defined as follows. Let $\tilde{q}_0 = e^{i(m+1)\theta + i\alpha_0} q_0$. Suppose that $Q = (\tilde{q}, s, \alpha)(t) \in \mathscr{A}_{\delta,\sigma}$ has been chosen. For each $t \in I$, let $q = e^{-i(m+1)\theta - i\alpha}\tilde{q}$, let $z = Z(q, s, \alpha)$ be defined by Lemma A.2, and let $\mathbf{v} = \mathbf{V}(z, s, \alpha)$ and $\mathbf{e} = \hat{\mathbf{E}}(z, s, \alpha)$ be defined by (A.1) and (A.6), respectively. We then substitute these functions into the right-hand sides of (A.21) and (A.22). The output functions are denoted as $\tilde{q}^\sharp(t)$, $s^\sharp(t)$, and $\alpha^\sharp(t)$. The map $Q \to \Psi(Q) = (\tilde{q}^\sharp, s^\sharp, \alpha^\sharp)$ is the (contraction) mapping.

The following estimates are shown in [11, Lem. 3.1]:

$$\|\tilde{q}^\sharp\|_{\mathrm{Str}[I]} \lesssim \|q_0\|_{L_x^2} + (\sigma^{1/2} + \|q\|^2_{L^4_{t,x}[I]})\|q\|_{L^4_{t,x}[I]}. \tag{A.24}$$

We also have $|\vec{G}_2| \lesssim \|z\|_X + \|z\|_X^4$, and thus

$$|s^\sharp(t) - 1| + |\alpha^\sharp(t) - \alpha_0| \lesssim \int_0^t |\vec{G}_2(\tau)|\,d\tau \lesssim \sigma\|q\|_{L_t^\infty L_x^2} + \sigma\|q\|^4_{L_t^\infty L_x^2}.$$

Therefore $\mathscr{A}_{\delta,\sigma}$ is invariant under the map Ψ if δ and σ are sufficiently small.

We now consider the more delicate difference estimate. Suppose that we have $Q_l = (\tilde{q}_l, s_l, \alpha_l)(t)$ for $l = a, b$. Let z_l, v_l, $\mathbf{e}_l$, $\tilde{q}_l^\sharp$, $s_l^\sharp$, and $\alpha_l^\sharp$ be defined respectively. Denote

$$\delta\tilde{q} = \tilde{q}_a(t,r) - \tilde{q}_b(t,r), \qquad \delta z = z_a\Big(t, \frac{r}{s_a}\Big) - z_b\Big(t, \frac{r}{s_b}\Big), \qquad \text{and so on.} \tag{A.25}$$

Note that we define δz in terms of r, not in terms of ρ (i.e., $\delta z \neq z_a(\rho) - z_b(\rho)$; see Rem. A.3 after the proof). In the rest of the proof, we denote

$$\|q\|_2 = \max_{a,b}(\|q_a\|_2, \|q_b\|_2), \qquad \|z\|_X = \max_{a,b}(\|z_a\|_X, \|z_b\|_X), \qquad \text{and so on.}$$

To start with, note that

$$\|z\|_{L_t^\infty X} \lesssim \delta, \qquad |\delta h_1| \lesssim |\delta s|\frac{h_1}{r}, \qquad |\delta h_3| \lesssim |\delta s|\frac{h_1^2}{r}, \qquad |\delta\gamma| \lesssim |z||\delta z|. \tag{A.26}$$

We first estimate $\delta\mathbf{e} = \mathbf{e}_a - \mathbf{e}_b = \hat{\mathbf{E}}(z_a, s_a, \alpha_a) - \hat{\mathbf{E}}(z_b, s_b, \alpha_b)$. By (A.7),

$$|\delta\mathbf{e}| \lesssim |\delta\alpha| + \left\|\hat{\mathbf{E}}\Big(z_a\Big(\frac{\cdot}{s_a}\Big)\Big) - \hat{\mathbf{E}}\Big(z_b\Big(\frac{\cdot}{s_b}\Big)\Big)\right\|_{L^\infty}.$$

By Lemma A.1, $\|\hat{\mathbf{E}}(z_a(\cdot/s_a)) - \hat{\mathbf{E}}(z_b(\cdot/s_b))\|_{L^\infty} \lesssim \|z_a(\cdot/s_a) - z_b(\cdot/s_b)\|_X = \|\delta z\|_X$. Thus

$$|\delta\mathbf{e}| \lesssim |\delta\alpha| + \|\delta z\|_X. \tag{A.27}$$

We next estimate $\|\delta z\|_X$. By (A.3),

$$\delta(L_0 z)\hat{\jmath} = \delta[se^{-\alpha R}q\mathbf{e}(r)] - \delta\mathbf{G_0}.$$

Here, $\delta(L_0 z) = L_0(r/s_a)z_a(r/s_a) - L_0(r/s_b)z_b(r/s_b)$ and $\delta\mathbf{G_0} = \mathbf{G_0}(z_a(r/s_a)) - \mathbf{G_0}(z_b(r/s_b))$. Rewrite

$$\delta(L_0 z) = D_1 + L_0\Big(\frac{r}{s_a}\Big)\delta_1 z,$$

where

$$D_1 = (\delta L_0)z_b\Big(\frac{r}{s_b}\Big), \qquad \delta_1 z = z_a\Big(\frac{r}{s_a}\Big) - \Pi_{s_a} z_b\Big(\frac{r}{s_b}\Big),$$

and Π_s is the projection removing $h_1(z/s)$: $\Pi_s f = f - (\langle h_1(\cdot/s), f\rangle_X / \langle h_1, h_1\rangle_X)h_1(\cdot/s)$. Here, we have used $L_0(r/s_a) = L_0(r/s_a)\Pi_{s_a}$. Since $L_0(r/s) =$

$s[\partial_r - (m/r)h_3(r/s)]$, we have $\delta L_0 \sim \delta s[L_0(r/s) - s(m^2/r^2)h_1^2(r/s)\cdot(r/s^2)]$, and hence

$$\|D_1\|_{L^2} \lesssim |\delta s|\cdot\|z\|_X.$$

Thus, taking $L_0(r/s_a)^{-1}$,

$$\|\delta_1 z\|_X \lesssim \left\|\delta[se^{-\alpha R}q\mathbf{e}(r)]\right\|_2 + \|\delta\mathbf{G_0}\|_2 + \|D_1\|_{L^2}.$$

We can decompose

$$\delta z = \delta_1 z + \delta_2 z, \qquad \delta_2 z = (1-\Pi_{s_a})z_b\Big(\frac{r}{s_b}\Big),$$

and we have

$$\|\delta_2 z\|_X \lesssim \Big\langle h_1\Big(\frac{z}{s_a}\Big) - h_1\Big(\frac{z}{s_b}\Big), z_b\Big(\frac{r}{s_b}\Big)\Big\rangle_X \le |\delta s|\|z\|_X.$$

Note that

$$\begin{aligned}|\delta\mathbf{G_0}| &\lesssim |\delta\mathbf{h}||\gamma_\rho| + \frac{\delta s}{r}(|\gamma|+|\boldsymbol{\xi}|^2) + |\delta\gamma_\rho| + \frac{1}{r}(|\delta\gamma|+|\boldsymbol{\xi}||\delta\boldsymbol{\xi}|)\\ &\lesssim |\delta s|\Big(|z||z_r| + \frac{|z|^2}{r}\Big) + |\delta z|\Big(|z_r| + \frac{|z|}{r}\Big) + |z||\delta z_r|.\end{aligned}$$

Thus

$$\|\delta\mathbf{G_0}\|_2 \lesssim |\delta s|\|z\|_X^2 + \|z\|_X\|\delta z\|_X.$$

Finally,

$$\left\|\delta[se^{-\alpha R}q\mathbf{e}(r)]\right\|_2 \lesssim (|\delta s| + |\delta\alpha| + \|\delta\mathbf{e}\|_{L^\infty})\cdot\|q\|_2 + \|\delta q\|_2.$$

Adding these estimates, and using (A.27) and $\|z\|_X \lesssim \|q\|_2$, we get

$$\|\delta z\|_X \lesssim (|\delta s| + |\delta\alpha| + \|\delta z\|_X)\cdot\|q\|_2 + \|\delta q\|_2.$$

Absorbing $\|\delta z\|_X\|q\|_2$ to the left-hand side, we get

$$\|\delta z\|_X \lesssim (|\delta s| + |\delta\alpha|)\cdot\|q\|_2 + \|\delta q\|_2. \tag{A.28}$$

We now estimate $\|\delta\tilde{q}^\sharp\|_{\mathrm{Str}[I]}$. Apply the Strichartz estimate to the difference of (A.21),

$$\begin{aligned}\|\delta\tilde{q}^\sharp\|_{\mathrm{Str}[I]} &\lesssim \|\delta(V\tilde{q})\|_{L^{4/3}_{t,x}} \lesssim \|V(\delta\tilde{q})\|_{L^{4/3}_{t,x}} + \|(\delta V)\tilde{q}\|_{L^{4/3}_{t,x}}\\ &\lesssim \|V\|_{L^2_{t,x}}\|\delta\tilde{q}\|_{L^4_{t,x}} + \|\delta V\|_{L^2_{t,x}+L_t^{8/3}L_x^{8/5}}\|\tilde{q}\|_{L^4_{t,x}\cap L_t^{8/3}L_x^8}.\end{aligned}$$

Recall that $V = V_1 - V_2 + \int^r (1/r')V_2$. By the 4-dimensional Hardy inequality, for each fixed t,

$$\|V\|_{L^2_x} \lesssim \|V_1\|_{L^2_x} + \|V_2\|_{L^2_x} \lesssim \left\|\frac{1+v_3}{r^2}\right\|_2 + \left\|\frac{v_{3,r}}{r}\right\|_2 + \|q\|_4^2 + \|q\|_4 \cdot \left\|\frac{\nu}{r}\right\|_4,$$

and since $v_3(r) = (h_3 + h_3\gamma + h_1 z_2)(r/s)$ and $|\nu| = |\widehat{\mathbf{k}} - v_3\mathbf{v}|$,

$$\left\|\frac{1+v_3}{r^2}\right\|_2 + \left\|\frac{v_{3,r}}{r}\right\|_2 \lesssim 1 + \|z\|_X + \left\|\frac{z}{r}\right\|_4 \cdot \|z\|_{X_4},$$

$$\left\|\frac{\nu}{r}\right\|_4^2 = \left\|\frac{1-v_3^2}{r^2}\right\|_2 \lesssim \left\|\frac{1+v_3}{r^2}\right\|_2.$$

Thus $\|V\|_{L^2_x} \lesssim 1 + \|q\|^2_{L^4_x}$, and hence $\|V\|_{L^2_{t,x}[I]} \lesssim \sigma^{1/2} + \|q\|^2_{L^4_{t,x}}$.

Denote $Y = L^2_{t,x} + L_t^{8/3}L_x^{8/5}$. By the Hardy inequality again,

$$\begin{aligned}\|\delta V\|_Y &\lesssim \|\delta V_1\|_Y + \|\delta V_2\|_Y \\ &\lesssim \left\|\frac{\delta v_3}{r^2}\right\|_Y + \left\|\frac{\partial_r \delta v_3}{r}\right\|_Y + \left(\|q\|_{L^4_{t,x}} + \left\|\frac{\nu}{r}\right\|_{L^4_{t,x}}\right)\|\delta q\|_{L^4_{t,x}} \\ &\quad + \|q\|_{L_t^{8/3}L_x^8} \cdot \left\|\frac{\delta\nu}{r}\right\|_{L_t^\infty L_x^2}.\end{aligned}$$

Note that $\nu = \mathbf{e}\cdot(\widehat{\mathbf{k}} - v_3\mathbf{v})$. Thus $\delta\nu = \delta\mathbf{e}\cdot(\widehat{\mathbf{k}} - v_3\mathbf{v}) - \mathbf{e}\cdot((\delta v_3)\mathbf{v} + v_3\delta\mathbf{v})$, and

$$\left\|\frac{\delta\nu}{r}\right\|_{L^2_x} \lesssim \|\delta\mathbf{e}\|_\infty \left\|\frac{1}{r}(\widehat{\mathbf{k}} - v_3\mathbf{v})\right\|_2 + \left\|\frac{1}{r}\delta\mathbf{v}\right\|_2.$$

Since $\|(1/r)(\widehat{\mathbf{k}} - v_3\mathbf{v})\|_2 \lesssim 1 + \|z\|_X^2 \lesssim 1$ and $\|(1/r)\delta\mathbf{v}\|_2 \lesssim |\delta\alpha|\|(\mathbf{h}+\boldsymbol{\xi})/r\|_2 + \|\delta\mathbf{h}/r\|_2 + \|\delta z/r\|_2$, we conclude using (A.27) and (A.28),

$$\left\|\frac{\delta\nu}{r}\right\|_{L^2_x} \lesssim |\delta s| + |\delta\alpha| + \|\delta q\|_2.$$

For $\delta v_3/r^2$ and $\partial_r\delta v_3/r$, since $v_3(r) = (h_3 + h_3\gamma + h_1 z_2)(r/s)$,

$$\frac{1}{r^2}|\delta v_3| \lesssim \frac{1}{r^2}(|\delta h_3| + |\delta h_1||z| + |\delta\gamma| + h_1|\delta z|) \lesssim \frac{h_1+|z|}{r}\left(|\delta s|\frac{h_1}{r} + \frac{|\delta z|}{r}\right),$$

$$\begin{aligned}\frac{1}{r}|\partial_r\delta v_3| &\lesssim |\delta s|\left(\frac{h_1(h_1+|z|)}{r^2} + \frac{h_1+h_1^2|z|}{r}|z_r|\right) \\ &\quad + \frac{h_1+h_1^2|z|}{r}\frac{|\delta z|}{r} + \frac{h_1+|z|}{r}|\partial_r\delta z|.\end{aligned}$$

We do not want to bound $(z/r)(\delta z/r)$ and $(z/r)\partial_r \delta z$ in L_x^2 since, otherwise, we would need a bound for $\|\delta z\|_{X_p}$, $p > 2$, which requires extra effort. We have

$$\begin{aligned}\Big\|\frac{\delta v_3}{r^2}\Big\|_Y + \Big\|\frac{\partial_r \delta v_3}{r}\Big\|_Y &\lesssim \|\delta s\|_{L_t^\infty}\Big\|\Big(\frac{h_1(h_1+|z|)}{r^2} + \frac{h_1+h_1^2|z|}{r}|z_r|\Big)\Big\|_{L^2_{t,x}} \\ &\quad + \Big\|\frac{h_1+h_1^2|z|}{r}\frac{|\delta z|}{r}\Big\|_{L^2_{t,x}} + \Big\|\frac{h_1+|z|}{r}\Big(\frac{|\delta z|}{r} + |\partial_r \delta z|\Big)\Big\|_{L_t^{8/3}L_x^{8/5}} \\ &\lesssim \|\delta s\|_{L_t^\infty} + \Big(1 + \Big\|\frac{z}{r}\Big\|_{L_t^{8/3}L_x^8}\Big)\|\delta z\|_{L_t^\infty X}.\end{aligned}$$

Using $\|z/r\|_{L_t^{8/3}L_x^8} \lesssim \|q\|_{L_t^{8/3}L_x^8} \lesssim \delta$ and (A.28), we conclude that

$$\|\delta\tilde{q}^\sharp\|_{\mathrm{Str}[I]} \lesssim (\sigma^{1/2} + \|\tilde{q}\|^2_{L^4_{t,x}})\|\delta\tilde{q}\|_{L^4_{t,x}} + \|\tilde{q}\|_{\mathrm{Str}[I]}(\|\delta s\|_{L_t^\infty} + \|\delta\alpha\|_{L_t^\infty} + \|\delta\tilde{q}\|_{L_t^\infty L_x^2}). \tag{A.29}$$

We now estimate $\delta s^\sharp$ and $\delta\alpha^\sharp$. Estimating the difference of (A.22),

$$\|\delta s^\sharp\|_{L^\infty(I)} + \|\delta\alpha^\sharp\|_{L^\infty(I)} \lesssim \int_I (|\delta s| + |\delta A|)|\vec{G}_2| + |\delta\vec{G}_2|\, d\tau.$$

Note that $|\vec{G}_2| \lesssim \|z\|_X + \|z\|_X^4$, note that

$$|\delta A| \lesssim \|\delta\mathbf{h}\|_X\|z\|_X + \|h_1\|_X\|\delta z\|_X \lesssim |\delta s|\|z\|_X + \|\delta z\|_X,$$

and note that

$$\begin{aligned}|\delta\vec{G}_2| &\lesssim \|\delta h\|_X\|z\|_X + \|h_1\|_X\|\delta z\|_X + |\delta G_1| \\ &\lesssim |\delta s|\|z\|_X + \|\delta z\|_X + (1+\|z\|_\infty)(\|z\|_\infty\|\partial_r\delta z\|_2 + \|\partial_r z\|_2\|\delta z\|_\infty) \\ &\quad + (\|z\|_\infty + \|z\|^3_\infty)\Big\|\frac{\delta z}{r}\Big\|_2.\end{aligned}$$

Thus

$$\begin{aligned}\|\delta s^\sharp\|_{L^\infty(I)} + \|\delta\alpha^\sharp\|_{L^\infty(I)} &\lesssim \int_I |\delta s|\|z\|_X + (1+\|z\|_X^3)\|\delta z\|_X\, d\tau \\ &\lesssim \sigma\|z\|_X\|\delta s\|_{L^\infty(I)} + \sigma\|\delta z\|_{L_t^\infty X}.\end{aligned} \tag{A.30}$$

Combining (A.28)–(A.30), we have proved that

$$\begin{aligned}&\|\delta\tilde{q}^\sharp\|_{\mathrm{Str}[I]} + \|\delta s^\sharp\|_{L^\infty(I)} + \|\delta\alpha^\sharp\|_{L^\infty(I)} \\ &\quad\lesssim (\sigma^{1/2} + \delta)(\|\delta\tilde{q}\|_{\mathrm{Str}[I]} + \|\delta s\|_{L^\infty(I)} + \|\delta\alpha\|_{L^\infty(I)}).\end{aligned} \tag{A.31}$$

Thus Ψ is a contraction mapping on $\mathscr{A}_{\delta,\sigma}$ if σ and δ are sufficiently small. We have therefore established the unique existence of a triplet $[s_W(t), \alpha_W(t), q_W(t)]$ solving the (s, α, q)-system. This yields a map $\mathbf{u}_W(t) \in C([0, T]; \Sigma_m)$.

If $\mathbf{u}_0 \in \dot{H}^2$, the a priori estimates in [11, Lem. 3.1] show that $\|\nabla \tilde{q}\|_{\mathrm{Str}[I]}$ is uniformly bounded, so $\mathbf{u}_W(t) \in C(I; \Sigma_m \cap \dot{H}^2)$.

If $\mathbf{u}_0^n \to \mathbf{u}_0$ in $\Sigma_m \cap \dot{H}^k$, $k = 1, 2$, a difference estimate similar to (A.31) shows

$$D^n \lesssim \|\tilde{q}_0^n - \tilde{q}_0\|_2 + (\sigma^{1/2} + \delta) D^n,$$

where $D^n = \|\tilde{q}^n - \tilde{q}\|_{\mathrm{Str}[I]} + \|s^n - s\|_{L^\infty(I)} + \|\alpha^n - \alpha\|_{L^\infty(I)}$. Thus $D_n \to 0$ as $n \to \infty$, and hence $\mathbf{u}_W^n \to \mathbf{u}_W$.

The energy $\mathcal{E}(\mathbf{u}_W(t))$ is conserved since $\mathcal{E}(\mathbf{u}_W(t)) = 4\pi m + \pi \|q(t)\|_{L_x^2}^2 = 4\pi m + \pi \|q_0\|_{L_x^2}^2$.

Finally, we must verify that $\mathbf{u}_W$ is a solution of the Schrödinger flow, as in Definition 1.2. To do this, approximate the initial data $\mathbf{u}_0$ in Σ_m by $\mathbf{u}_0^k$ with $\nabla \mathbf{u}_0^k \in H^{10}$ (say). By [22], there is a unique strong solution $\mathbf{u}_S^k(t)$ with initial data $\mathbf{u}_0^k$. The corresponding triple $[s_S^k(t), \alpha_S^k(t), q_S^k(t)]$ must satisfy the (s, α, q)-system. By uniqueness, $s_S^k(t) \equiv s_W^k(t)$, and so on, and so $\mathbf{u}_W^k(t) \equiv \mathbf{u}_S^k(t)$. By continuous dependence on $\dot{H}^1$-data, $\mathbf{u}_S^k$ converges to $\mathbf{u}_W$ in $C([0, T]; \Sigma_m)$ and, in particular, in $C([0, T]; L^2_{\mathrm{loc}})$. Finally, $\mathbf{u}_S^k$ satisfies the weak form of the Schrödinger flow (see Def. 1.2), and passing to the limit, so does $\mathbf{u}_W$. Dropping the subscript W ($\mathbf{u} := \mathbf{u}_W$), Theorem 1.4 is established. $\square$

We now consider Corollary 1.5.

Proof of Corollary 1.5
Suppose that T is the blowup time. By Theorem 1.4, for each $t < T$ we have $T - t \geq \sigma s(\mathbf{u}(t))^2$. Thus $s(\mathbf{u}(t)) \leq \sigma^{-1/2}\sqrt{T - t}$. If $k = 2$, by [11, Th. 2.1], $\|\mathbf{u}(t)\|_{\dot{H}^2} \geq C_2/s(\mathbf{u}(t)) \geq C_2 \sigma^{1/2}(T - t)^{-1/2}$. On the other hand, the $\dot{H}^2$-estimates of [11] show that the $\dot{H}^2$-norm can blow up only if $\liminf_{t \to T^-} s(t) = 0$. Thus $T^2_{\max} = T^1_{\max}$. Statement (ii) follows from Theorem 1.4 directly. Corollary 1.5 is established.

Remark A.3
(1) In Theorem 1.4, we did not try to prove continuity on data $\mathbf{u}_0$ in $\dot{H}^2$, which would require difference estimates in H^1 for $\tilde{q}$.
(2) In (A.25), we define δz in terms of r, not in terms of ρ (i.e., $\delta z \neq \tilde{\delta} z = z_a(\rho) - z_b(\rho)$). Indeed, in view of (A.3), since L_0 depends on ρ, it may seem natural to bound $\tilde{\delta} z$ using $L_0 \tilde{\delta} z \widehat{\jmath} = \delta[s e^{-\alpha R} q \mathbf{e}(s\rho)] + \tilde{\delta} \mathbf{G_0}$. However, to bound the right-hand side, we need to bound the difference $q_a \mathbf{e}_a(s_b \rho) - q_a \mathbf{e}_a(s_a \rho) = \int_{s_a}^{s_b} \rho \partial_r (q_a \mathbf{e}_a)(\sigma \rho)\, d\sigma$ for which $\|\mathbf{u}\|_{\dot{H}^2}$ is insufficient, and we need a weighted

norm of $\mathbf{u}$. The reason is that the dilation magnifies the difference when ρ is large. In addition, to bound δv_3 using $\tilde{\delta} z$ instead of δz, one needs a bound for z_{rr}.

(3) In the proof, we have avoided using $\|\delta z\|_{X_4}$ since its estimate requires $\|\delta \mathbf{e}\|_\infty$. We know how to control $\|\delta \mathbf{e}\|_\infty$ by $\|\delta z\|_X$, but we do not know if $\|\delta \mathbf{e}\|_\infty \lesssim \|\delta z\|_{X_4}$.

A.4. Small-energy case

The proof of Theorem 1.6 is similar to that of Theorem 1.4.

Proof of Theorem 1.6

When $m \geq 1$, the limits $\lim_{r\to 0} \mathbf{v}_0(r)$ and $\lim_{r\to\infty} \mathbf{v}_0(r)$ exist, and it is necessary that $\mathbf{u}_0(0) = \mathbf{u}_0(\infty)$. We may assume that $\mathbf{u}_0(0) = \mathbf{u}_0(\infty) = -\widehat{\mathbf{k}}$. In the proof of Theorem 1.4, we may redefine

$$\mathbf{h}(r) := -\widehat{\mathbf{k}}, \qquad \mathbf{v}(r) = (z_2, z_1, -1-\gamma)^T,$$

and the parameters s and α are no longer needed. The same proof—in particular, the difference estimate $\|\delta\tilde{q}^\sharp\|_{\mathrm{Str}[I]} \lesssim (\sigma^{1/2} + \delta)\|\delta\tilde{q}\|_{\mathrm{Str}[I]}$—then gives the local well-posedness. □

Note that this proof does not directly apply to the radial case since $\|\mathbf{u}(r)\|_{\dot{H}^1}$ no longer controls $\|z/r\|_2$.

B. Some lemmas

B.1. Computation of nonlinear terms

To find the equations for $\dot{s}$ and $\dot{\alpha}$, we need to compute $(g, (\mathbf{V}^{\mathbf{h}})^{-1} P^{\mathbf{h}} \mathbf{F_1})_{L^2}$ for $g = h_1$ or $g = N_0 h_1$. Here is the result.

LEMMA B.1

Recall that $\mathbf{F_1} = -2\gamma_r (m/r) h_1 \widehat{\jmath} + \xi \times (\Delta_r + (m^2/r^2) R^2 \xi)$, *and recall that* $(\mathbf{V}^{\mathbf{h}})^{-1} P^{\mathbf{h}} \mathbf{F_1} = \widehat{\jmath} \cdot \mathbf{F_1} + i(h \times \widehat{\jmath}) \cdot \mathbf{F_1}$. *For any suitable function* g,

$$\big(g, (\mathbf{V}^{\mathbf{h}})^{-1} P^{\mathbf{h}} \mathbf{F_1}\big)_{L^2} = \int_0^\infty \Big(i g_r(-\gamma z_r + z\gamma_r) + \frac{m}{r} h_1 g(-2\gamma_r - i z_2 z_{1,r} + i z_1 z_{2,r}) + \frac{m}{r}(h_1 g)_r(\gamma^2 - i z_2 z) + i\frac{m^2}{r^2}(2h_1^2 - 1) g\gamma z\Big) r\, dr. \tag{B.1}$$

Proof
Decompose

$$\int_0^\infty g(\mathbf{V^h})^{-1} P^{\mathbf{h}} \mathbf{F_1} r\,dr = \int -2g\frac{m}{r}h_1\gamma_r + \int gP(\xi\times\Delta_r\xi) + \int gP\Big(\xi\times\frac{m^2}{r^2}R^2\xi\Big) =: I_1+I_2+I_3.$$

Denote $[a,b,c]=a\widehat{\boldsymbol{\jmath}}+bh\times\widehat{\boldsymbol{\jmath}}+ch$. For any vector η,

$$P(\xi\times\eta)=[1,i,0]\cdot([z_1,z_2,\gamma]\times\eta)=([1,i,0]\times[z_1,z_2,\gamma])\cdot\eta=[i\gamma,-\gamma,-iz]\cdot\eta.$$

Since $h_r=(m/r)h_1h\times\widehat{\boldsymbol{\jmath}}$,

$$\partial_r[a,b,c]=\Big[a_r,\ b_r+\frac{m}{r}h_1c,\ c_r-\frac{m}{r}h_1b\Big].$$

Thus

$$\begin{aligned}
I_2&=\int g[i\gamma,-\gamma,-iz]\cdot\Delta_r[z_1,z_2,\gamma]\\
&=\int \partial_r[-ig\gamma,g\gamma,igz]\cdot\partial_r[z_1,z_2,\gamma]\\
&=\int\Big[-i(g\gamma)_r,(g\gamma)_r+\frac{m}{r}h_1igz,\ i(gz)_r-\frac{m}{r}h_1g\gamma\Big]\\
&\quad\times\Big[z_{1,r},\ z_{2,r}+\frac{m}{r}h_1\gamma,\ \gamma_r-\frac{m}{r}h_1z_2\Big]\\
&=\int g(-i\gamma_rz_{1,r}+\gamma_rz_{2,r}+iz_r\gamma_r)+\int g_r(-i\gamma z_{1,r}+\gamma z_{2,r}+iz\gamma_r)\\
&\quad+\int g_r\Big(\frac{m}{r}h_1\gamma^2-i\frac{m}{r}h_1z_2z\Big)+\int g\Big(-i\frac{m}{r}h_1z_2z_r+i\frac{m}{r}h_1zz_{2,r}\Big)\\
&\quad+\int g\frac{m^2}{r^2}h_1^2iz_1\gamma.
\end{aligned}$$

Note that the first integral is zero, and we have canceled two $\int g(m/r)h_1\gamma\gamma_r$. Also,

$$\begin{aligned}
I_3&=\int g[i\gamma,-\gamma,-iz]\cdot\frac{m^2}{r^2}R^2\xi\\
&=\int g(\gamma h_3-ih_1z,\ i\gamma,\ *)\cdot\frac{m^2}{r^2}(z_2h_3-\gamma h_1,\ z_1,\ 0)\\
&=\int\frac{m^2}{r^2}g(h_3^2\gamma z_2-h_1h_3\gamma^2+ih_1h_3z_2z+ih_1^2\gamma z-i\gamma z_1).
\end{aligned}$$

Summing up $I_1+I_2+I_3$, we get the lemma. □

B.2. Linear weighted L^2-estimate

LEMMA B.2

Let H be a self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the weighted resolvent estimate

$$\sup_{\mu\notin\mathbb{R};\ \phi\in L^2,\|\phi\|_{L^2}=1} \left\| \frac{1}{|x|}(H-\mu)^{-1}\frac{1}{|x|}\phi \right\|_{L^2} \lesssim 1.$$

Then for $f(x,t) \in (1/|x|)L^2$,

$$\left\| \frac{1}{|x|}\int_0^t e^{i(t-s)H} f(x,s)\,ds \right\|_{L^2_{x,t}(\mathbb{R}^n\times\mathbb{R})} \lesssim \||x|f\|_{L^2_{x,t}(\mathbb{R}^n\times\mathbb{R})}.$$

Proof

First, we have some simplifications. It suffices to prove the estimate for $f(x,t)$ compactly supported, and $f(x,t) \in (1/|x|)L^2_{x,t} \cap L^\infty_t L^2_x$ (by density). Also, it is enough to consider $t \geq 0$ (i.e., $f(x,t)$ supported in $\{t \geq 0\}$). Finally, we regularize the integral: set

$$F_\epsilon(x,t) := \frac{1}{|x|}\int_0^t e^{i(t-s)(H+i\epsilon)} f(x,s)\,ds.$$

We prove the estimate for F_ϵ with an ϵ-independent constant, and the lemma follows from this. Under our assumptions, F_ϵ is well defined as a $(1/|x|)L^2_x$-valued function of t, and

$$\int_0^\infty \||x|F_\epsilon(\cdot,t)\|_{L^2_x}\,dt < \infty.$$

Hence the Fourier transform of F_ϵ in t is well defined (as a $(1/|x|)L^2_x$ -valued function of τ):

$$\tilde{F}_\epsilon(x,\tau) := (2\pi)^{-1/2}\int_0^\infty e^{-it\tau}F_\epsilon(x,t)\,dt.$$

Changing the order of integration, we see

$$\begin{aligned}
\tilde{F}_\epsilon(x,\tau) &= \frac{1}{|x|}(2\pi)^{-1/2}\int_0^\infty dte^{-it\tau}\int_0^t dse^{i(t-s)(H+i\epsilon)}f(x,s) \\
&= \frac{1}{|x|}(2\pi)^{-1/2}\int_0^\infty dse^{-is(H+i\epsilon)}\int_s^\infty dte^{it(H-\tau+i\epsilon)}f(x,s) \\
&= \frac{1}{|x|}(2\pi)^{-1/2}(i)(H-\tau+i\epsilon)^{-1}\int_0^\infty dse^{-is\tau}f(x,s)\,ds \\
&= \frac{1}{|x|}(i)(H-\tau+i\epsilon)^{-1}\tilde{f}(x,\tau).
\end{aligned}$$

So using the weighted resolvent estimate gives

$$\|\tilde{F}_\epsilon\|_{L^2_x} \lesssim \||x|\tilde{f}(x,\tau)\|_{L^2_x},$$

and squaring and integrating in τ yields

$$\|\tilde{F}_\epsilon\|^2_{L^2_{x,\tau}} \lesssim \||x|\tilde{f}\|^2_{L^2_{x,\tau}} \lesssim \||x|f\|^2_{L^2_{x,t}}.$$

By a vector-valued version of the Plancherel theorem (see, e.g., [19, Chap. XIII.7]),

$$\|F_\epsilon\|^2_{L^2_{x,t}} = \|\tilde{F}_\epsilon\|^2_{L^2_{x,t}} \lesssim \||x|f\|^2_{L^2_{x,t}},$$

completing the proof. □

B.3. Proof of the splitting lemma

Here, we prove Lemma 4.1.

Proof of Lemma 4.1

For $\mathbf{u} = e^{m\theta R}\mathbf{v}(r) \in \Sigma_m$, $s > 0$, and $\alpha \in \mathbb{R}$, define

$$F(\mathbf{u}; s, \alpha) := \int_0^\infty (\widehat{\jmath} + i J^{\mathbf{h}(\rho)}\widehat{\jmath}) \cdot e^{-\alpha R}\mathbf{v}(s\rho)h_1(\rho)\rho\, d\rho \in \mathbb{C}.$$

Note that for $\mathbf{u}$ of the form (4.1), (4.2) is equivalent to $F(\mathbf{u}; s, \alpha) = 0$.

Suppose that $\mathcal{E}(\mathbf{u}) \le 4\pi m + \delta^2$. It is shown in [11] that if δ is sufficiently small, then there are $\hat{s}$, $\hat{\alpha}$, and $\hat{z}$ such that $\mathbf{u}(r,\theta) = e^{[m\theta+\hat{\alpha}]R}[(1+\hat{\gamma}(r/\hat{s}))\mathbf{h}(r/\hat{s}) + \mathbf{V}^{r/\hat{s}}(\hat{z}(r/\hat{s}))]$ and with $\|\hat{z}\|^2_X \lesssim \delta_1^2 := \mathcal{E}(\mathbf{u}(0)) - 4\pi m \le \delta^2$. (But $\hat{z}$ does not satisfy (4.2).)

It follows from this and the fact that $\rho h_1(\rho) \in L^2(\rho\, d\rho)$ for $m \ge 3$ that for some $\delta_0 > 0$, F is a C^1-map from

$$\left\{\mathbf{u} \in \Sigma_m \,\middle|\, \mathcal{E}(\mathbf{u}) \le 4\pi m + \delta_0^2\right\} \times (\mathbb{R}^+ \times \mathbb{R})$$

into $\mathbb{C}$. Furthermore, straightforward computations show that

$$F\left(e^{m\theta R}\mathbf{h}(r); 1, 0\right) = 0$$

and

$$\begin{pmatrix} \partial_s F\left(e^{m\theta R}\mathbf{h}(r); 1, 0\right) \\ \partial_\alpha F\left(e^{m\theta R}\mathbf{h}(r); 1, 0\right) \end{pmatrix} = \|h_1\|^2_{L^2}\begin{pmatrix} i \\ -1 \end{pmatrix}.$$

By the implicit function theorem, we can solve $F = 0$ to get $s = s(\mathbf{u})$ and $\alpha = \alpha(\mathbf{u})$ for $\mathbf{u}$ in an $\dot{H}^1$-neighborhood of the harmonic map $e^{m\theta R}\mathbf{h}(r)$.

Since $\|\hat{z}\|_X \lesssim \delta$, provided δ is chosen small enough (depending on the size of this neighborhood),

$$\hat{\mathbf{u}}(x) := e^{-\hat{\alpha}R}\mathbf{u}(\hat{s}x) = e^{m\theta R}\left[\left(1+\hat{\gamma}(r)\right)\mathbf{h}(r) + \mathbf{V}^r\hat{z}(r)\right]$$

lies in this neighborhood, yielding $s(\hat{\mathbf{u}})$ and $\alpha(\hat{\mathbf{u}})$ with $F(\hat{\mathbf{u}}; s(\hat{\mathbf{u}}), \alpha(\hat{\mathbf{u}})) = 0$. Furthermore,

$$|s(\hat{\mathbf{u}}) - 1| + |\alpha(\hat{\mathbf{u}})| \lesssim \|\hat{z}\|_X,$$

and so

$$\|z(\rho)\|_X = \left\|(\widehat{\jmath} + iJ^{\mathbf{h}(\rho)}\widehat{\jmath}) \cdot e^{-\alpha(\hat{\mathbf{u}})R}\hat{\mathbf{v}}\big(s(\hat{\mathbf{u}})\rho\big)\right\|_X \lesssim \|\hat{z}\|_X \lesssim \mathcal{E}(\mathbf{u}) - 4\pi m.$$

To complete the proof of the lemma, undo the rescaling: set $s(\mathbf{u}) := s(\hat{\mathbf{u}})/\hat{s}$, and set $\alpha(\mathbf{u}) := \alpha(\hat{\mathbf{u}}) + \hat{\alpha}$. □

References

[1] I. BEJENARU, *On Schrödinger maps*, Amer. J. Math. **130** (2008), 1033–1065. MR 2427007 542

[2] N. BURQ, F. PLANCHON, J. G. STALKER, and A. S. TAHVILDAR-ZADEH, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), 519–549. MR 2003358 545, 559, 561, 562

[3] ———, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*, Indiana Univ. Math. J. **53** (2004), 1665–1680. MR 2106340 545, 559, 560, 561

[4] K.-C. CHANG, W. Y. DING, and R. YE, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. **36** (1992), 507–515. MR 1180392 539

[5] N.-H. CHANG, J. SHATAH, and K. UHLENBECK, *Schrödinger maps*, Comm. Pure Appl. Math. **53** (2000), 590–602. MR 1737504 543, 544, 545, 552, 556

[6] M. CHRIST and A. KISELEV, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), 409–425. MR 1809116 562

[7] W. DING, "On the Schrödinger flows" in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, Higher Ed. Press, Beijing, 2002, 283–291. MR 1957040 539

[8] W. DING and Y. WANG, *Schrödinger flow of maps into symplectic manifolds*, Sci. China Ser. A **41** (1998), 746–755. MR 1633799 539

[9] ———, *Local Schrödinger flow into Kähler manifolds*, Sci. China Ser. A **44** (2001), 1446–1464. MR 1877231 542

[10] M. GRILLAKIS and V. STEFANOPOULOS, *Lagrangian formulation, energy estimates, and the Schrödinger map problem*, Comm. Partial Differential Equations **27** (2002), 1845–1877. MR 1941660 539

[11] S. GUSTAFSON, K. KANG, and T.-P. TSAI, *Schrödinger flow near harmonic maps*, Comm. Pure Appl. Math. **60** (2007), 463–499. MR 2290708 539, 540, 541, 542, 545, 547, 549, 553, 554, 555, 566, 572, 577, 581

[12] A. D. IONESCU and C. E. KENIG, *Low-regularity Schrödinger maps*, Differential Integral Equations **19** (2006), 1271–1300. MR 2278007 542

[13] J. KATO and H. KOCH, *Uniqueness of the modified Schrödinger map in $H^{3/4+\epsilon}(\mathbb{R}^2)$*, Comm. Partial Differential Equations **32** (2007), 415 – 429. MR 2304155 542

[14] T. KATO, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. **162** (1965/1966), 258 – 279. MR 0190801 561

[15] A. M. KOSEVICH, B. A. IVANOV, and A. S. KOVALEV, *Magnetic solitons*, Phys. Rep. **194** (1990), 117 – 238. 539

[16] H. MCGAHAGAN, *An approximation scheme for Schrödinger maps*, Comm. Partial Differential Equations **32** (2007), 375 – 400. MR 2304153 542

[17] A. NAHMOD, A. STEFANOV, and K. UHLENBECK, *On Schrödinger maps*, Comm. Pure Appl. Math. **56** (2003), 114 – 151; *Erratum*, Comm. Pure Appl. Math. **57** (2004), 833 – 839. MR 1929444 ; MR 2038118 542

[18] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975. MR 0493420 558

[19] ———, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978. MR 0493421 581

[20] I. RODNIANSKI and W. SCHLAG, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), 451 – 513. MR 2038194 559, 561

[21] I. RODNIANSKI and J. STERBENZ, *On the formation of singularities in the critical $O(3)$ σ-model*, preprint, arXiv:math/0605023v2 [math.AP] 545

[22] P.-L. SULEM, C. SULEM, and C. BARDOS, *On the continuous limit for a system of classical spins*, Comm. Math. Phys. **107** (1986), 431 – 454. MR 0866199 542, 577

[23] T. TAO, *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation*, Comm. Partial Differential Equations **25** (2000), 1471 – 1485. MR 1765155 559

[24] C.-L. TERNG and K. UHLENBECK, “Schrödinger flows on Grassmannians” in *Integrable Systems, Geometry, and Topology*, AMS/IP Stud. Adv. Math. **36**, Amer. Math. Soc., Providence, 2006, 235 – 256. MR 2222517 539

Gustafson
Department of Mathematics, University of British Columbia, Vancouver, British Columbia V6T 1Z2, Canada; gustaf@math.ubc.ca

Kang
Department of Mathematics, Sungkyunkwan University and Institute of Basic Science, Suwon 440-746, Republic of Korea; kkang@skku.edu

Tsai
Department of Mathematics, University of British Columbia, Vancouver, British Columbia V6T 1Z2, Canada; ttsai@math.ubc.ca