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Asymptotic Stability and Completeness in the Energy Space for Nonlinear Schrödinger Equations with Small Solitary Waves

Stephen Gustafson, Kenji Nakanishi, and Tai-Peng Tsai

1 Introduction

In this paper, we study a class of nonlinear Schrödinger equations (NLS) which admit families of small solitary wave solutions. We consider solutions which are small in the energy space H^1, and decompose them into solitary wave and dispersive wave components. The goal is to establish the asymptotic stability of the solitary wave and the asymptotic completeness of the dispersive wave. That is, we show that as $t \to \infty$, the solitary wave component converges to a fixed solitary wave, and the dispersive component converges strongly in H^1 to a solution of the free Schrödinger equation.

We briefly supply some background. Solutions of dispersive partial differential equations (with repulsive nonlinearities) tend to spread out in space, although they often have conserved L^2 mass. There has been extensive study of this phenomenon, usually referred to as *scattering theory*. These equations include Schrödinger equations, wave equations, and KdV equations. However, such equations can also possess solitary wave solutions which have localized spatial profiles that are constant in time (e.g., if the nonlinearity is attractive or if a linear potential is present). To understand the asymptotic dynamics of general solutions, it is essential to study the interaction between the solitary waves and the dispersive waves. The matter becomes more involved when the linearized operator around the solitary wave possesses multiple eigenvalues, which correspond to excited states. The interaction between eigenstates (mediated by the nonlinearity) is very delicate, and few results are known.

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For NLS with solitary waves, there are three types of results.

(1) Control of solutions in a finite time interval, which does not allow sufficient time for the excited state interaction to make a difference, and construction of all-time solutions with specified asymptotic behaviors (*scattering solutions*; see [10, 11]).

(2) Orbital stability of solitary waves. A solution stays close to the family of nonlinear bound states if it is initially close. This is usually proved by energy arguments; see, for example, [6, 14, 25, 38].

(3) Asymptotic stability of solitary waves. Here, one must assume that the spectrum of the linearized operator enjoys certain spectral properties (e.g., has only one eigenvalue or has multiple "well-placed" eigenvalues). Furthermore, *the initial data are typically assumed to be localized*, so that the dispersive wave has fast local decay. Even under restrictive spectral assumptions, only perturbation problems can be treated for large solitary waves (see [8, 9, 21, 24]; also [3, 4, 5] for 1D results), while more general results can be obtained for small solitary waves [22, 26, 27, 28, 32, 33, 34, 35, 36, 37].

In this paper, we study small solutions of the equation

$$i\partial_t\psi = (-\Delta + V)\psi + g(\psi), \quad \psi(0,\cdot) = \psi_0 \in H^1\left(\mathbb{R}^3\right), \tag{1.1}$$

with small data: $\|\psi_0\|_{H^1} \ll 1$ (this is equivalent to considering a nonlinearity multiplied by a small constant). Although we only consider the problem for $x \in \mathbb{R}^3$, the results and methods can be extended to spatial dimensions $d \geq 3$.

Here, $g(\psi)$ is either a pointwise nonlinearity or a Hartree-type (nonlocal) nonlinearity (or their sum), satisfying gauge covariance:

$$g\left(\psi e^{i\alpha}\right) = g(\psi)e^{i\alpha}, \quad \text{with } g\left(|\psi|\right) \in \mathbb{R}. \tag{1.2}$$

More detailed assumptions are given below. In either case, we can find a functional $G : H^1 \to \mathbb{R}$, satisfying $G(\psi e^{i\alpha}) = G(\psi)$ (gauge invariance), and

$$\partial_\varepsilon^0 G(\psi + \varepsilon\eta) := \frac{d}{d\varepsilon} G(\psi + \varepsilon\eta)\Big|_{\varepsilon=0} = \operatorname{Re}\left(g(\psi), \eta\right). \tag{1.3}$$

Here we denote the inner product in L^2 by

$$(a, b) := \int_{\mathbb{R}^3} \bar{a} b \, dx. \tag{1.4}$$

Under suitable assumptions, the L^2-norm $\|\psi(t)\|_{L^2}$ and the Hamiltonian

$$\frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla\psi|^2 + V|\psi|^2\right)dx + G(\psi) \tag{1.5}$$

are constant in time. Using these conserved quantities and the smallness of $\|\psi_0\|_{H^1}$, one can prove a uniform estimate $\sup_t \|\psi(t)\|_{H^1} \ll 1$ and obtain global well-posedness.

We do *not* assume that ψ_0 is localized ($\psi_0 \in L^1(\mathbb{R}^3)$, e.g., or in a weighted space), as is usually done. As a result, we cannot expect a time decay rate for L^p-norms ($p > 2$) of the dispersive part of the solution. However, the space H^1 is natural, as it is intimately related to the Hamiltonian structure, and persists globally in time (in contrast to weighted spaces, whose smallness persists only for a short time due to dispersion, and L^1, which may be instantaneously lost and so does not seem to have physical relevance). A related motivation comes from the situation where the linearized operator around a solitary wave has many "ill-placed" eigenvalues. In this case, the dispersive component tends to decay very slowly (unpublished work). It is thus essential to be able to remove the localization assumption on the data.

Asymptotic stability of solitons with initial data in the energy space was first established for generalized KdV equations [18, 19]. A main tool used is the almost monotonicity of the L^2-mass of the dispersion on the right side of the solitons, which is not available for NLS. The current paper provides such results for NLS. Actually, our result is stronger, since it describes the asymptotics in the strong topology of the energy space, while the above papers dealt with weak convergence or convergence on the right, which neglects anything propagating much slower than the main soliton(s). On the other hand, it is weaker since it only considers small solitons. In this aspect, note that a recent preprint of Tao [30] shows that global-in-time radially symmetric solutions of cubic focusing NLS on $\mathbb{R}^3$ have a time asymptotic decomposition in H^1 which is consistent with the solitary wave plus dispersive wave picture, although not quite that precise. His approach does not impose spectral assumptions and seems promising.

We assume that $-\Delta + V$ supports only one eigenvalue $e_0 < 0$, which is nondegenerate, and we denote by ϕ_0 the corresponding positive, normalized eigenfunction. More detailed assumptions on V are given below. (The one-eigenvalue assumption releases us from tracking excited state components, which usually decay more slowly than the dispersion and are harder to control. We will consider two-eigenvalue case in [15].) Under these assumptions, there exists a family of small "nonlinear bound states" $Q = Q[z]$, parameterized by small $z = (\phi_0, Q) \in \mathbb{C}$, which satisfy $Q[z] - z\phi_0 = o_{H^2 \cap W^{1,1}}(z) \perp \phi_0$ and solve the nonlinear eigenvalue problem

$$(-\Delta + V)Q + g(Q) = EQ, \quad E = E[z] = e_0 + o(z) \in \mathbb{R}. \tag{1.6}$$

See Lemma 2.1 for details. Gauge covariance is inherited by Q:

$$Q\left[ze^{i\alpha}\right] = Q[z]e^{i\alpha}, \tag{1.7}$$

and so $E[z] = E[|z|]$. The nonlinear bound states give rise to exact solitary-wave solutions $\psi(x,t) = Q(x)e^{-iEt}$ of (1.1). $Q[z]$ is differentiable in z if we regard it as a real vector

$$z = z_1 + iz_2 \longleftrightarrow (z_1, z_2) \in \mathbb{R}^2. \tag{1.8}$$

We will denote its z-derivatives by

$$D_1Q[z] := \frac{\partial}{\partial z_1}Q[z], \qquad D_2Q[z] := \frac{\partial}{\partial z_2}Q[z] \tag{1.9}$$

(we use the symbol D in order to distinguish them from space or time derivatives). Then $DQ[z]$ denotes the Jacobian matrix, regarded as an $\mathbb{R}$-linear map on $\mathbb{C}$:

$$DQ[z] : \mathbb{C} \longrightarrow \mathbb{C}, \quad DQ[z]w \longmapsto D_1Q[z]\operatorname{Re} w + iD_2Q[z]\operatorname{Im} w. \tag{1.10}$$

The gauge covariance of $Q[z]$ implies that

$$DQ[z]iz = iQ[z]. \tag{1.11}$$

Given a general solution $\psi(t)$ of (1.1), it is natural to decompose it into solitary wave and dispersive wave components:

$$\psi(t) = Q\left[z(t)\right] + \eta(t). \tag{1.12}$$

For any such decomposition, (1.1) yields an equation for η:

$$i\partial_t\eta = H[z]\eta + E[z]Q[z] - iDQ[z]\dot{z} + F_2(z,\eta), \tag{1.13}$$

where $H[z]$ denotes the *linearized operator* around $Q[z]$,

$$H[z]\eta := (-\Delta + V)\eta + \partial_\varepsilon^0 g(Q + \varepsilon\eta), \tag{1.14}$$

and F_2 collects terms which are higher-order in η:

$$F_2(z, \eta) := g(Q + \eta) - g(Q) - \partial_\varepsilon^0 g(Q + \varepsilon\eta). \tag{1.15}$$

The decomposition (1.12) is, of course, not unique. To specify the path $z(t)$ uniquely, we impose an orthogonality condition which will make η dispersive. Since the linearization destroys gauge invariance, the linearized operator $H[z]$ is not complex-linear. It is, however, symmetric if we regard $\mathbb{C}$ as $\mathbb{R}^2$ and use the reduced inner product

$$\langle a, b \rangle := \operatorname{Re}(a, b) = \int_{\mathbb{R}^3} (\operatorname{Re} a \operatorname{Re} b + \operatorname{Im} a \operatorname{Im} b) dx. \tag{1.16}$$

The symmetry of $H[z]$ follows from (1.14) and

$$\left\langle \partial_\varepsilon^0 g(Q + \varepsilon\eta), \zeta \right\rangle = \partial_\varepsilon^0 \left\langle g(Q + \varepsilon\eta), \zeta \right\rangle = \partial_\varepsilon^0 \partial_\delta^0 G(Q + \varepsilon\eta + \delta\zeta). \tag{1.17}$$

We will require η to belong to the following subspace.

Definition 1.1. The "*continuous spectral subspace*" $\mathcal{H}_c[z]$ is defined as

$$\mathcal{H}_c[z] := \left\{ \eta \in L^2 : \left\langle i\eta, D_1 Q[z] \right\rangle = \left\langle i\eta, D_2 Q[z] \right\rangle = 0 \right\}. \tag{1.18}$$

As we will show in Lemma 2.3, we can uniquely decompose $\psi(t)$ as

$$\psi(t) = Q\left[z(t)\right] + \eta(t), \quad \eta(t) \in \mathcal{H}_c\left[z(t)\right]. \tag{1.19}$$

The requirement $\eta(t) \in \mathcal{H}_c[z(t)]$ determines $z(t)$ uniquely. An evolution equation for $z(t)$ is derived from differentiating the relation $\langle i\eta, D_j Q[z] \rangle = 0$ with respect to t, and using (1.13) (see (3.17)). Our goal is to prove the asymptotic stability of $Q[z(t)]$ and the asymptotic completeness of $\eta(t)$.

Remarks 1.2. (1) The subspace $\mathcal{H}_c[z]$ is an invariant subspace of $i(H[z] - E[z])$, as follows from the relation

$$\left(H[z] - E[z]\right) D_j Q[z] = \left(D_j E[z]\right) Q[z] \tag{1.20}$$

(which is the result of differentiating (1.6)), together with (1.11). Restricting to $\mathcal{H}_c[z]$ eliminates nondecaying solutions of the linear equation $\partial_t \eta = -i(H[z] - E[z])\eta$ for fixed z.

(2) When $z \in \mathbb{R}^+$, $\mathcal{H}_c[z]$ is just the orthogonal complement of $\{Q, i(\partial/\partial|z|)Q\}$ in the inner product $\langle\cdot,\cdot\rangle$. This subspace is often used in the literature. The current definition using z, instead of its magnitude and phase, is more natural because we allow $z = 0$, for which case the phase is not well defined.

(3) Note that we impose a time-dependent condition $\eta(t) \in \mathcal{H}_c[z(t)]$ instead of simpler conditions such as $\eta(t) \in \mathcal{H}_c[0]$, that is, $(\eta(t), \phi_0) = 0$ (which is used in [22, 37]). The reason is the following. If we assume $(\eta(t), \phi_0) = 0$, then the equations for $\dot{z} + iEz$ yield

$$|\dot{z} + iEz| \lesssim \big|(\phi_0, A\eta)\big| + \big|(\phi_0, F_2)\big|, \tag{1.21}$$

where A is some linear operator. The term $(\phi_0, A\eta)$ is linear in η and hence is not integrable in time, in light of the estimate $\eta \in L_t^2 W^{1,6}$. Thus we cannot conclude that $|z|$ and $E[z]$ have limits as $t \to \infty$. This term drops out if we require $\eta(t) \in \mathcal{H}_c[z]$, and the equation for $\dot{z} + iEz$ (and hence $(d/dt)|z|$) becomes quadratic in η.

We now state precise assumptions on the potential V and on the nonlinearity g. We denote the usual Lorentz space by $L^{p,q} = (L^\infty, L^1)_{1/p,q}$ for $1 < p < \infty$ and $1 \le q \le \infty$ (see [2]). $W^{1,p}$ denotes the usual Sobolev space.

Assumption 1.3. V is a real-valued function belonging to $L^2 + L^\infty$. (We note that under this assumption, $-\Delta + V$ is a selfadjoint operator on L^2, with domain H^2. See, e.g., [23].) Its negative part $V_- := \max\{0, -V\}$ is further assumed to satisfy $\|V_-\|_{(L^2+L^\infty)(\{|x|>R\})} \to 0$ as $R \to \infty$. We suppose $-\Delta + V$ has only one eigenvalue $e_0 < 0$, and let ϕ_0 be a corresponding normalized eigenvector. e_0 is simple and ϕ_0 can be taken to be positive (see [23]). Denote the projections onto the discrete and continuous spectral subspaces of $-\Delta + V$ by

$$P_d = \phi_0(\phi_0, \cdot), \qquad P_c = 1 - P_d. \tag{1.22}$$

The following Strichartz estimates are assumed to hold:

$$\begin{aligned} &\left\|e^{it(\Delta-V)} P_c \phi\right\|_X \lesssim \|\phi\|_{H^1}, \\ &\left\|\int_{-\infty}^t e^{is(\Delta-V)} P_c F(s)\, ds\right\|_X \lesssim \|F\|_{L_t^2 W^{1,6/5}}, \end{aligned} \tag{1.23}$$

where $X := L_t^\infty H^1 \cap L_t^2 W^{1,6} \cap L_t^2 L^{6,2}$.

Remark 1.4. The Strichartz estimates of Assumption 1.3 hold when, for example,

$$\left|V(x)\right| \le C\left(1+|x|\right)^{-3-\varepsilon} \tag{1.24}$$

for some $\varepsilon > 0$, and the bottom of the continuous spectrum, zero, is neither an eigenvalue nor a resonance. Estimates without derivatives can be proved by applying the L^1-L^∞ decay estimate [13, 16, 39] to the endpoint Strichartz estimate [17], where the stronger estimate in the Lorentz space was actually proved. We need the Lorentz space $L^{6,2}$ estimate only to handle the critical case of the Hartree equation (with convolution potentials decaying like $1/|x|^2$). Estimates of the derivatives can be obtained by using the equivalence

$$\|\phi\|_{W^{1,p}} \sim \left\|H_1^{1/2}\phi\right\|_{L^p}, \quad H_1 = -\Delta + V + \|V\|_\infty + 1, \tag{1.25}$$

for $1 < p < \infty$, and the commutativity of $e^{it(\Delta-V)}$ with $H_1^{1/2}$. The equivalence can be shown by applying the complex interpolation for fractional powers [31, Section 1.15.3] to the equivalence in $W^{2,p}$, using the boundedness of imaginary powers H_1^{is}, which follows by [7] from the fact that e^{-tH_1} is a positivity-preserving contraction semigroup on L^p.

Assumption 1.5. The nonlinearity is assumed to be of one of the following two forms or their sum:

(a) $g : \mathbb{C} \to \mathbb{C}$ is a function satisfying gauge covariance (1.2) which, when restricted to $\mathbb{R}$, is twice-differentiable, with $g(0) = g'(0) = 0$, and

$$\left|g''(s)\right| \le C\left(s^{1/3} + s^3\right); \tag{1.26}$$

(b) $g(\psi) = (\Phi * |\psi|^2)\psi$, where Φ is a real potential, and

$$\Phi \in L^1 + L^{3/2,\infty}. \tag{1.27}$$

Remarks 1.6. (1) Examples of nonlinearities satisfying Assumption 1.5 include

$$g(\psi) = a|\psi|^{4/3}\psi + b|\psi|^4\psi + \left[\left(\frac{c}{|x|^{3-\varepsilon}} + \frac{d}{|x|^2}\right) * |\psi|^2\right]\psi, \tag{1.28}$$

where $a, b, c, d \in \mathbb{R}$, $0 < \varepsilon < 1$. Note that the small-data assumption allows the consideration of focusing problems globally in time, which could otherwise blow up for large enough data.

(2) Recall from Remarks 1.2(3) that, using a time-dependent decomposition, the equation for $\dot z + iEz$ becomes quadratic in η. Even in this case we need to use an L^2_t-type

Strichartz estimate for η in order to get convergence of $|z|$, since we cannot have better decay as long as we start with H^1 initial data. Thus the endpoint Strichartz estimates (1.23) are irreplaceable in our argument. As a bonus, they allow us to treat borderline nonlinearities, such as $g(\psi) = \pm|\psi|^4\psi$ and $g(\psi) = \pm|\psi|^{4/3}\psi$, which are not covered in the previous works [22, 26, 27].

(3) Although our primary interest is in the asymptotics in the energy space, our argument in this paper works in any Sobolev space H^s, $s \geq 1$, if the above assumptions on V are modified accordingly. Indeed, the control of the trajectory of the soliton does not require much regularity unless V is too rough (it suffices that V is a bounded multiplier from H^s to H^{s-2}). The estimate on the linear error terms for the perturbation is not affected by the regularity, and the remaining argument for higher-order error terms is quite similar to the low-energy scattering in H^s spaces (see, e.g., [20]), once we assume the Strichartz estimate for $-\Delta + V$ in H^s. In particular, by considering the problem in $H^2 \subset L^\infty$, one may treat arbitrarily high-power nonlinearities.

We can now state our main theorems.

Theorem 1.7 (asymptotic stability and completeness)**.** Let Assumptions 1.3 and 1.5 hold. Every solution ψ of (1.1) with data ψ_0 sufficiently small in H^1 can be uniquely decomposed as

$$\psi(t) = Q\big[z(t)\big] + \eta(t), \tag{1.29}$$

with differentiable $z(t) \in \mathbb{C}$ and $\eta(t) \in \mathcal{H}_c[z(t)]$ satisfying

$$\begin{aligned} &\|\eta\|_{L_t^2 W^{1,6} \cap L_t^\infty H^1} + \|z\|_{L_t^\infty} \lesssim \big\|\psi_0\big\|_{H^1}, \\ &\big\|\dot z + iE[z]z\big\|_{L_t^1 \cap L_t^\infty} \lesssim \big\|\psi_0\big\|_{H^1}^2. \end{aligned} \tag{1.30}$$

Moreover, there exist $z_+ \in \mathbb{C}$ with $\big||z_+| - |z(0)|\big| \lesssim \|\psi_0\|_{H^1}^2$, and $\eta_+ \in H^1 \cap \operatorname{Ran} P_c$ such that

$$z(t)\exp\left\{i\int_0^t E\big[z(s)\big]\,ds\right\} \longrightarrow z_+, \qquad \big\|\eta(t) - e^{it(\Delta - V)}\eta_+\big\|_{H^1} \longrightarrow 0 \tag{1.31}$$

as $t \to \infty$. In particular, $|z(t)| \to |z_+|$ and, if $z_+ \neq 0$, $\arg z(t) + \int_0^t E[z(s)]ds - \arg z_+ \to 0 \bmod 2\pi$. □

The corresponding result with no bound state was obtained in [16, 29] for small H^1 data and [12] for large data with no potential and $g(\psi) = +|\psi|^{m-1}\psi$. Results similar to Theorem 1.7 in the case of *localized* initial data ψ_0 and $g(\psi) = \lambda|\psi|^{m-1}\psi$ were first obtained for the case $\|\eta(0)\|_{H^1\cap L^1} \ll |z(0)|$ by Soffer and Weinstein [26, 27], and then extended to all ψ_0 small in H^1 and weighted L^2-spaces by Pillet and Wayne [22]. The latter work was extended to the 1D case in [37]. In all [22, 26, 27, 37], the solutions $\psi(t)$ are decomposed with respect to fixed selfadjoint linear operators. A time-dependent decomposition similar to (1.19) seems to have first appeared in [3].

Theorem 1.8 (nonlinear wave operator)**.** Let Assumptions 1.3 and 1.5 hold. There exists $\delta > 0$ such that for any $z_+ \in \mathbb{C}$ with $|z_+| = m_\infty \in [0, \delta]$ and $\eta_+ \in H^1 \cap \operatorname{Ran} P_c$ with $\|\eta_+\|_{H^1} \le \delta$, there is a global solution $\psi(t)$ of (1.1) satisfying the conclusion of Theorem 1.7 with the prescribed asymptotic data z_+ and η_+. □

We do not claim the uniqueness of $\psi(t)$.

Special cases of Theorem 1.8, further assuming $m_\infty \gg \|\eta_+\|_{H^1\cap L^1}$ or $m_\infty = 0$ with $\|\eta_+\|_{H^1\cap L^1} \ll 1$ for $g(\psi) = \pm|\psi|^2\psi$, were obtained in [33, 36].

In Theorems 1.7 and 1.8, one may replace $e^{it(\Delta - V)}\eta_+$ by $e^{it\Delta}\widetilde{\eta}_+$ with $\widetilde{\eta}_+ \in H^1$ if asymptotic completeness in H^1 of the wave operator between $-\Delta + V$ and $-\Delta$ holds. It holds, for example, if (1.24) holds.

Theorem 1.7 implies in particular that any small solution looks like a solitary wave for large time locally in space. But due to the fact that the data is not assumed to be localized, we cannot, in general, have a convergence rate for it. In fact, we have the following theorem.

Theorem 1.9 (examples of slow decay of dispersion)**.** Let Assumptions 1.3 and 1.5 hold. For any nonempty ball $B \subset \mathbb{R}^3$, there exists $\delta > 0$ for which the following holds. For any positive function $f(t)$ which goes to zero as $t \to \infty$, and any $z_+ \in \mathbb{C}$ with $|z_+| = m_\infty \in [0, \delta]$, there exists a solution $\psi(t)$ of (1.1) satisfying the conclusions of Theorem 1.7 and

$$\limsup_{t\to\infty} \left(\frac{\inf_{|z'|\le 2\delta} \left\|\psi(t) - Q[z']\right\|_{L^2(B)}}{f(t)} \right) = \infty. \tag{1.32}$$

□

Finally, we remark on the case of a large soliton with H^1 perturbation. In this case, the derivative of the phase $\arg z(t)$ does not have fast decay, and there is a commutator term in the equation of the dispersion which is no longer smaller than the dispersion itself. One needs to include this term in the linearized operator, which becomes time-dependent, and an analogue of the Strichartz estimates is not available yet. An extra difficulty arises when the soliton is allowed to move: the trajectory of a soliton cannot be treated as a perturbation of a constant-velocity movement.

2 Preliminaries

In this section, we give three lemmas concerning the nonlinear ground states.

Lemma 2.1 (nonlinear ground states)**.** There exists $\delta > 0$ such that for each $z \in \mathbb{C}$ with $|z| \le \delta$, there is a solution $Q[z] \in H^2 \cap W^{1,1}$ of (1.6) with $E = E[|z|] \in \mathbb{R}$ such that

$$Q[z] = z\phi_0 + q[z], \quad (q, \phi_0) = 0. \tag{2.1}$$

The pair (q, zE) is unique in the class

$$\|q\|_{H^2} \le \delta, \qquad |E - e_0| \le \delta. \tag{2.2}$$

Moreover, $Q[ze^{i\alpha}] = Q[z]e^{i\alpha}$, $Q[|z|]$ is real, and

$$\begin{aligned} &q[z] = o(z^2), \quad \text{in } H^2 \cap W^{1,1}, \\ &DQ[z] = (1, i)\phi_0 + o(z), \quad D^2Q[z] = o(1), \quad \text{in } H^2 \cap W^{1,1}, \\ &E[z] = e_0 + o(z), \qquad DE[z] = o(1), \end{aligned} \tag{2.3}$$

as $z \to 0$. □

A special case of this lemma is proved in [26, 27], referring to [1]. We will prove the lemma under weaker assumptions on V and g in the appendix.

The following is an immediate but useful corollary of this lemma.

Lemma 2.2 (continuous spectral subspace comparison)**.** There exists $\delta > 0$ such that for each $z \in \mathbb{C}$ with $|z| \le \delta$, there is a bijective operator $R[z] : \mathcal{H}_c[0] \to \mathcal{H}_c[z]$ satisfying

$$P_c|_{\mathcal{H}_c[z]} = R[z]^{-1}. \tag{2.4}$$

Moreover, $R[z] - I$ is compact and continuous in z in the operator norm on any space Y satisfying $H^2 \cap W^{1,1} \subset Y \subset H^{-2} + L^\infty$. □

We remark that no corresponding statement holds for the case of large solitary waves.

Proof. $R[z]$ is given by

$$R[z]\eta = \eta + \phi_0\alpha[z]\eta, \tag{2.5}$$

where the operator $\alpha[z] : \mathcal{H}_c[0] \to \mathbb{C}$ is defined by solving the equations

$$\langle \eta + \phi_0 \alpha[z]\eta, D_j Q[z] \rangle = 0, \quad j = 1, 2. \tag{2.6}$$

These are solvable due to the property $D_j Q[z] = (1, i)\phi_0 + o(z)$. Then $R[z]$ is obviously the inverse of P_c restricted onto $\mathcal{H}_c[z]$. For any $\eta \in \mathcal{H}_c[0]$, we have

$$|\alpha[z]\eta| \lesssim |\langle \eta, DQ[z] \rangle| \lesssim o(z)\|\eta\|_{H^{-2}+L^\infty}, \tag{2.7}$$

which implies compactness of $R[z] - I$ in Y. The continuity in z follows from that of $DQ[z]$. ■

Lemma 2.3 (best decomposition). There exists $\delta > 0$ such that any $\psi \in H^1$ satisfying $\|\psi\|_{H^1} \le \delta$ can be uniquely decomposed as

$$\psi = Q[z] + \eta, \tag{2.8}$$

where $z \in \mathbb{C}, \eta \in \mathcal{H}_c[z]$, and $|z| + \|\eta\|_{H^1} \lesssim \|\psi\|_{H^1}$. □

Proof. We look for a unique solution z of the equation $A(z) = 0$, where we define

$$A_j(z) := \langle i(\psi - Q[z]), D_j Q[z] \rangle, \quad j = 1, 2. \tag{2.9}$$

Let $n := \|\psi\|_{H^1}$. The Jacobian matrix of the map $z \mapsto A(z)$ is written as

$$\begin{aligned} D_j A_k(z) &= \langle -iD_j Q[z], D_k Q[z] \rangle + \langle i(\psi - Q[z]), D_j D_k Q[z] \rangle \\ &= j - k + o(n + |z|), \end{aligned} \tag{2.10}$$

by Lemma 2.1. Let $z_0 := (\phi_0, \psi)$. So $|z_0| \le n$. Then, from Lemma 2.1, we have

$$A(z_0) = \langle i(\psi - z_0\phi_0) + o(n^2), (1, i)\phi_0 + o(n) \rangle = o(n^2). \tag{2.11}$$

Now the result is an immediate consequence of the inverse function theorem. ■

3 Asymptotic stability and completeness

This section is devoted to a proof of Theorem 1.7. We will first estimate the nonlinearity in Section 3.1 and then prove the theorem in the subsequent subsection.

3.1 Estimates on the nonlinearity

Before starting the proofs of our theorems, we establish some nonlinear estimates, first for the pointwise nonlinearity and then for the convolution nonlinearity.

(I) Pointwise nonlinearity. Our assumption (1.26) implies that for $k = 0, 1, 2$,

$$\left|D^k g(z)\right| \lesssim \sum_{j=0}^{k} \left|g^{(j)}(|z|) z^{j-k}\right| \lesssim |z|^{7/3-k} + |z|^{5-k}. \tag{3.1}$$

The nonlinear term F_2, defined in (1.15), can be expanded by the mean value theorem as

$$F_2(\eta) = g(Q+\eta) - g(Q) - \partial_\varepsilon^0 g(Q+\varepsilon\eta) = \int_0^1 (1-\varepsilon)\partial_\varepsilon^2 g(Q+\varepsilon\eta)d\varepsilon. \tag{3.2}$$

Then we can estimate it as

$$\left|F_2\right| \lesssim \sup_{0<\varepsilon<1} \left|D^2 g(Q+\varepsilon\eta)\right||\eta|^2 \lesssim \left(1+|Q|+|\eta|\right)^4 |\eta|^2, \tag{3.3}$$

$$\left\|F_2\right\|_{L^1+L^\infty} \lesssim \left(1+\|Q\|_{L^6}+\|\eta\|_{L^6}\right)^4 \|\eta\|_{L^6}^2. \tag{3.4}$$

We also need to estimate $g(Q+\eta) - g(Q)$. By using the generalized Hölder inequality,

$$\begin{aligned}
\left\|g(Q+\eta) - g(Q)\right\|_{W^{1,6/5}} &\lesssim \left\|\left(Dg(Q+\eta) - Dg(Q)\right)\nabla Q\right\|_{L^{6/5}} + \left\|Dg(Q+\eta)\nabla\eta\right\|_{L^{6/5}} \\
&\lesssim \left(\|Q\|_{L^2} + \|\eta\|_{L^2}\right)^{1/3} \|\eta\|_{L^6}\|\nabla Q\|_{L^2} \\
&\quad + \left(\|Q\|_{L^6} + \|\eta\|_{L^6}\right)^3 \|\eta\|_{L^6}\|\nabla Q\|_{L^6} \\
&\quad + \left(\|Q\|_{L^2}^{4/3} + \|\eta\|_{L^2}^{4/3} + \|Q\|_{L^6}^4 + \|\eta\|_{L^6}^4\right)\|\nabla\eta\|_{L^6} \\
&\lesssim C\left(\|Q\|_{H^2} + \|\eta\|_{H^1}\right)\|\eta\|_{W^{1,6}},
\end{aligned} \tag{3.5}$$

where $C(s) \lesssim s^{4/3} + s^4$.

(II) Convolution nonlinearity. The nonlinear term F_2 has the following form:

$$F_2 = Q\Phi * |\eta|^2 + \eta\Phi * \left(2\,\mathrm{Re}(Q\bar{\eta}) + |\eta|^2\right). \tag{3.6}$$

By the generalized Young inequality in Lorentz spaces, we have, under the assumption (1.27),

$$\left\|F_2\right\|_{L^1+L^\infty} \lesssim \left(\|Q\|_{L^6} + \|\eta\|_{L^6}\right)\|\eta\|_{L^{6,2}}^2. \tag{3.7}$$

This is the only place where we need the Lorentz space $L^{6,2}$.

As for $g(Q+\eta)-g(Q)$, its gradient $\nabla(g(Q+\eta)-g(Q))$ is expanded into a sum of trilinear forms where one of three functions has the derivative and at least one of them is η. By the generalized Young inequality, we have

$$\left\|\Phi * (\psi_1\psi_2)\psi_3\right\|_{L^{6/5}} \lesssim \|\Phi\|_{L^1+L^{3/2,\infty}} \left\|\psi_{\sigma(1)}\right\|_{L^2} \left\|\psi_{\sigma(2)}\right\|_{L^2\cap L^6} \left\|\psi_{\sigma(3)}\right\|_{L^6} \tag{3.8}$$

for any permutation σ. So we may put an η or $\nabla\eta$ in L^6, another function without derivative in $L^2 \cap L^6$, and the remaining one in L^2. Hence we obtain

$$\left\|g(Q+\eta)-g(Q)\right\|_{W^{1,6/5}} \lesssim \|\eta\|_{W^{1,6}} \left(\|\eta\|_{H^1} + \|Q\|_{H^1}\right)^2. \tag{3.9}$$

3.2 Asymptotic stability and completeness

Now we prove our main result, Theorem 1.7.

Let $\psi(x,t)$ solve the nonlinear Schrödinger equation (1.1) with initial data

$$\psi(0,\cdot) = \psi_0, \qquad \left\|\psi_0\right\|_{H^1(\mathbb{R}^3)} \ll 1. \tag{3.10}$$

It is easy to prove local well-posedness in H^1 by using the Strichartz estimate (1.23) (the discrete spectral part does not bother us on finite intervals). The unique solution thereby obtained belongs to $L_t^\infty H^1 \cap L_t^2 W^{1,6}$.

Our argument below will yield time-global a priori estimates, so that the solution ψ exists and remains small in H^1 for all time. More precisely, we take $\delta' > 0$ much smaller than any δ in the previous lemmas, and take the initial data ψ_0 such that

$$\left\|\psi_0\right\|_{H^1} < \delta' \ll \delta. \tag{3.11}$$

We will show that

$$\|\psi\|_{L_t^\infty H^1[0,T]} < \delta \Longrightarrow \|\psi\|_{L_t^\infty H^1[0,T]} < \frac{\delta}{2}, \tag{3.12}$$

for any $T > 0$, provided δ and δ' were chosen sufficiently small. Then, by continuity in time, this bound and the solution together extend globally in time. In the argument below we will not specify explicitly the time interval. The assumption $\|\psi\|_{L_t^\infty H^1} < \delta$ allows us to use all of the previous lemmas.

By Lemma 2.3, we have the decomposition

$$\psi = Q\left[z(t)\right] + \eta(t), \quad \eta \in \mathcal{H}_c[z]. \tag{3.13}$$

The equation for η is given in (1.13). We now derive the evolution equation for $z(t)$. Differentiating the relation $\langle i\eta, D_j Q[z]\rangle = 0$ with respect to t and plugging (1.13) into that, we obtain

$$0 = \big\langle H\eta + EQ - iDQ\dot{z} + F_2, D_j Q\big\rangle + \big\langle i\eta, D_j DQ\dot{z}\big\rangle, \tag{3.14}$$

where H, E, and Q all depend on z (but this dependence is dropped from the notation). By the symmetry of H and (1.20), we have

$$\big\langle H\eta, D_j Q\big\rangle = \big\langle \eta, HD_j Q\big\rangle = \big\langle \eta, D_j(EQ)\big\rangle = \big\langle \eta, ED_j Q\big\rangle = \big\langle i\eta, ED_j DQiz\big\rangle, \tag{3.15}$$

where we used $\langle i\eta, DQ\rangle = 0$ and (1.11). By (1.11), we have

$$\big\langle EQ - iDQ\dot{z}, D_j Q\big\rangle = \big\langle DQ(iEz + \dot{z}), iD_j Q\big\rangle. \tag{3.16}$$

Thus we obtain

$$\sum_{k=1,2} \big(\big\langle iD_j Q, D_k Q\big\rangle + \big\langle i\eta, D_j D_k Q\big\rangle\big)(\dot{z} + iEz)_k = -\big\langle F_2, D_j Q\big\rangle. \tag{3.17}$$

The matrix on the left-hand side is the Jacobian matrix in (2.10), and so is estimated as

$$\big\langle iD_j Q, D_k Q\big\rangle + \big\langle i\eta, D_j D_k Q\big\rangle = j - k + o(\delta). \tag{3.18}$$

Inverting this matrix, we obtain

$$|\dot{z} + iEz| \lesssim \big|\big\langle F_2, DQ[z]\big\rangle\big| \lesssim \big\|F_2\big\|_{L^1 + L^\infty} \tag{3.19}$$

at any t. Applying estimates (3.4) and (3.7), we obtain

$$\|\dot{z} + iEz\|_{L^2} \lesssim \|\eta\|_{L_t^\infty L^{6,2}} \|\eta\|_{L_t^2 L^{6,2}} \lesssim \|\eta\|_{L_t^\infty H^1} \|\eta\|_{L_t^2 L^{6,2}} \tag{3.20}$$

and $\|\dot{z} + iEz\|_{L^\infty} \lesssim \|\eta\|_{L_t^\infty H^1}^2$, where we used the Sobolev embedding $H^1 \subset L^{6,2}$.

Next, we estimate η by writing (1.13) in the form

$$i\partial_t \eta = (-\Delta + V)\eta + F \tag{3.21}$$

with

$$F := g(Q + \eta) - g(Q) - iDQ(\dot{z} + iEz). \tag{3.22}$$

Denote $\eta_c := P_c\eta$, where $P_c = 1 - \phi_0(\phi_0, \cdot)$ is defined in (1.22). The Strichartz estimates applied to (3.21) and Lemma 2.2 yield

$$\|\eta\|_X \lesssim \|\eta_c\|_X \lesssim \|\eta(0)\|_{H^1} + \|P_cF\|_{L_t^2 W^{1,6/5}} \lesssim \|\psi_0\|_{H^1} + \|F\|_{L_t^2 W^{1,6/5}}, \tag{3.23}$$

where

$$X := L_t^\infty H^1 \cap L_t^2 W^{1,6} \cap L_t^2 L^{6,2}. \tag{3.24}$$

By Lemma 2.1 and the estimates (3.5) and (3.9) in Section 3.1, we obtain

$$\|F\|_{L_t^2 W^{1,6/5}} \lesssim \|\dot z + iEz\|_{L^2} + \delta\|\eta\|_{L_t^2 W^{1,6}}. \tag{3.25}$$

From (3.20), (3.23), and (3.25), we deduce that

$$\|\eta\|_X + \|\dot z + iEz\|_{L^2}^{1/2} + \|F\|_{L_t^2 W^{1,6/5}} \lesssim \|\psi_0\|_{H^1} < \delta', \tag{3.26}$$

if we take δ sufficiently small. Choosing δ' even smaller, we obtain the desired bootstrapping estimate (3.12), and so the solution, as well as all the estimates, extends globally.

Moreover, we have

$$\|\partial_t|z|\|_{L^1} \le \|\dot z + iEz\|_{L^1} \lesssim \|\eta\|_{L_t^2 L^{6,2}}^2 \lesssim \|\psi_0\|_{H^1}^2, \tag{3.27}$$

so $|z(t)|$ and $E[z(t)] = E[|z(t)|]$ converge as $t \to \infty$.

Finally, we prove that η is asymptotically free. We have the integral equation

$$\eta_c(t) = e^{it(\Delta - V)}\left[\eta_c(0) - i\int_0^t e^{-is(\Delta-V)} P_cF(s)ds\right]. \tag{3.28}$$

By the Strichartz estimate, for any $T > S > 0$, we have

$$\left\|\int_S^T e^{-is(\Delta-V)} P_cF(s)ds\right\|_{H^1} \lesssim \|F\|_{L_t^2 W^{1,6/5}[S,T]} \longrightarrow 0, \tag{3.29}$$

as $T > S \to \infty$, by the Lebesgue dominated convergence theorem and the finiteness of $\|F\|_{L_t^2 W^{1,6/5}(0,\infty)}$. Thus the integral in (3.28) converges in H^1 as $t \to \infty$, and we obtain

$$\lim_{t\to\infty} e^{-it(\Delta-V)}\eta_c(t) = \eta_c(0) - i\int_0^\infty e^{-is(\Delta-V)} P_cF(s)ds =: \eta_+. \tag{3.30}$$

In particular, $\eta_c(t)$ converges to 0 weakly in H^1. Then Lemma 2.2 implies that $\eta_d(t) = (R[z(t)] - I)\eta_c(t)$ converges to 0 strongly in H^1. Therefore we conclude that

$$\|\eta(t) - e^{it(\Delta-V)}\eta_+\|_{H^1} \longrightarrow 0. \tag{3.31}$$

4 Nonlinear wave operator

In this section, we prove Theorem 1.8. We will construct the desired solution by first assigning the asymptotic data at large finite time T and then taking the weak limit as $T \to \infty$. Recall that $m_\infty, \|\eta_+\|_{H^1} \le \delta$. For any $T > 0$, we define ψ^T to be a solution of (1.1) such that for the decomposition

$$\psi^T = Q\left[z^T\right] + \eta^T, \quad \eta^T \in \mathcal{H}_c\left[z^T\right], \tag{4.1}$$

and the modified parameter

$$\tilde{z}^T(t) = z^T(t) \exp\left\{ i \int_0^t E\left[z^T(s)\right] ds \right\}, \tag{4.2}$$

we have

$$\tilde{z}^T(T) = z_+, \qquad P_c \eta^T(T) = e^{iT(\Delta - V)} \eta_+. \tag{4.3}$$

First we need to show that such a solution does exist. For that purpose, we fix T for a moment and consider a family of solutions $\{\psi_\alpha\}_\alpha$ to (1.1) for $\alpha \in S = \{z \in \mathbb{C} \mid |z| = |z_+|\}$ with the initial condition at $t = T$:

$$\psi_\alpha(T) = Q[\alpha] + R[\alpha]\eta_+, \tag{4.4}$$

where the operator $R[\alpha]$ is given by Lemma 2.2. We decompose $\psi_\alpha = Q[z_\alpha] + \eta_\alpha$ as before and define $\tilde{z}_\alpha$ from z_α in the same way as in (4.2). It suffices to show that $\tilde{z}_\alpha(T) = z_+$ for some $\alpha \in S$. Since the phase factor

$$\int_0^T E\left[z_\alpha(s)\right] ds \tag{4.5}$$

is continuous from $\alpha \in S \to \mathbb{R}$, the mapping

$$\alpha \in S \longmapsto \tilde{z}_\alpha(T) \in S \tag{4.6}$$

has degree (winding number) 1 and so is surjective to S. Thus the solution ψ^T exists and is well defined. Theorem 1.7 implies that ψ^T satisfies

$$\left\|\psi^T\right\|_{L_t^2 W^{1,6} \cap L^\infty H^1} \lesssim \delta, \qquad \left\|\partial_t \tilde{z}^T\right\|_{L^1(T,\infty)} \le \left\|(\dot{z} + iEz)^T\right\|_{L^1(T,\infty)} \lesssim \delta^2, \tag{4.7}$$

and the integral equation

$$P_c\eta^T(t) = e^{it(\Delta - V)}\eta_+ - i\int_T^t e^{i(t-s)(\Delta-V)}P_cF^T(s)ds, \tag{4.8}$$

where F^T is as defined in (3.22). By the same argument as in Section 3.2, we deduce that for any $S \le T$,

$$\begin{aligned}
\|\eta^T\|_{L_t^2W^{1,6}[S,\infty]} &\lesssim \|P_c\eta^T\|_{L_t^2W^{1,6}[S,\infty]} \\
&\lesssim \|e^{it(\Delta-V)}\eta_+\|_{L_t^2W^{1,6}[S,\infty]} + \|F^T\|_{L_t^2W^{1,6/5}[S,\infty]}, \\
\|F^T\|_{L_t^2W^{1,6/5}[S,\infty]} &\lesssim \|(\dot z + iEz)^T\|_{L^2[S,\infty]} + \delta\|\eta^T\|_{L_t^2W^{1,6}[S,\infty]} \\
&\lesssim \delta\|\eta^T\|_{L_t^2W^{1,6}[S,\infty]}.
\end{aligned} \tag{4.9}$$

Therefore, when δ is sufficiently small, we have

$$\|\eta^T\|_{L_t^2W^{1,6}[S,\infty]} \lesssim \|e^{it(\Delta-V)}\eta_+\|_{L_t^2W^{1,6}[S,\infty]}. \tag{4.10}$$

Applying the Strichartz estimate once again, we get

$$\|P_c\eta^T - e^{it(\Delta-V)}\eta_+\|_{L_t^\infty H^1\cap L_t^2W^{1,6}\cap L_t^2L^{6,2}[S,\infty]} \lesssim \delta\|e^{it(\Delta-V)}\eta_+\|_{L_t^2W^{1,6}[S,\infty]}. \tag{4.11}$$

Now we take the limit $T \to \infty$. By (4.7), z^T is equicontinuous on $\mathbb{R}$ and so is $\tilde z^T$ on the extended real line $[-\infty,\infty]$. From the equations, η^T is equicontinuous in $C(\mathbb{R}, \text{w-}H^1)$ and so is $P_c\eta^T$ in $C([-\infty,\infty], \text{w-}H^1)$ by (4.11). Then, by Lemma 2.2, $\eta^T = R[z^T]P_c\eta^T$ is also equicontinuous in $C([-\infty,\infty], \text{w-}H^1)$. Therefore η^T and z^T are convergent along some subsequence in the following topology:

$$\begin{aligned}
&\eta^T \longrightarrow \eta^\infty \quad \text{in } (C^0 \cap L^\infty)(\text{w-}H^1) \cap \text{w-}L_t^2W^{1,6}, \\
&z^T \longrightarrow z^\infty \quad \text{in } C^0(\mathbb{R}), \qquad \tilde z^T \longrightarrow \tilde z^\infty \quad \text{in } L^\infty(\mathbb{R}).
\end{aligned} \tag{4.12}$$

This implies the convergence of ψ^T itself:

$$\psi^T = Q[z^T] + \eta^T \longrightarrow Q[z^\infty] + \eta^\infty =: \psi^\infty \tag{4.13}$$

in $C(\text{w-}H^1)\cap\text{w-}L_t^2W^{1,6}$ on any finite time interval. Extracting a subsequence if necessary, we may assume that the nonlinearity $g(\psi^T)$ also converges in $\text{w-}L_t^2W^{1,6/5}$ on any finite time interval. Then the local convergence of ψ^T in $C(L^p)$ for $p < 6$ implies that the limit of the nonlinearity is the desired $g(\psi^\infty)$. Hence we deduce that ψ^∞ is a solution to (1.1) belonging to $C(\mathbb{R}; H^1) \cap L^2_{loc}(W^{1,6})$. From the uniform convergence of $\tilde{z}^T$ to $\tilde{z}^\infty$, we have

$$\lim_{T\to\infty} \tilde{z}^\infty(T) = \lim_{T\to\infty} \tilde{z}^T(T) = z_+. \tag{4.14}$$

From the weak convergence uniform in time, we get

$$\begin{aligned} &\|P_d\eta^\infty(t)\|_{H^1} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \\ &\|P_c\eta^\infty - e^{it(\Delta-V)}\eta_+\|_{L^\infty_t H^1[S,\infty]} \lesssim \|e^{i(\Delta-V)t}\eta_+\|_{L^2_t W^{1,6}[S,\infty]} \longrightarrow 0 \quad \text{as } S \longrightarrow \infty. \end{aligned} \tag{4.15}$$

Thus ψ^∞ is a solution with the desired asymptotic profile.

5 Examples of slow decay of dispersion

In this section, we prove Theorem 1.9.

For a fixed ball $B \subset \mathbb{R}^3$, choose $\xi_0 \in H^1$ satisfying

$$\|\xi_0\|_{H^1} = 1, \quad \|\xi_0\|_{L^2(B)} > 0, \quad (\phi_0, \xi_0) = (\phi_0, \xi_0)_{L^2(B)} = 0, \tag{5.1}$$

where the first inner product is in $L^2(\mathbb{R}^3)$ as before. The constants below depend on B and ξ_0. We use a small parameter $\varepsilon > 0$ to control the size of solution.

We define an increasing sequence of times T_j inductively as follows. Let $T_j > 1$ and, for $j > 1$, assume that we have defined T_k for $k < j$. By the Strichartz estimate, there exists $T > \max_{k<j} T_k$ such that

$$\sum_{k<j,\,\pm} \|e^{\pm i(\Delta-V)(t-T_k)}\xi_0\|_{L^2_t W^{1,6}[T,\infty]} < \varepsilon 2^{-j}, \tag{5.2}$$

$$\sup_{t>T} f(t) \le \varepsilon^2 2^{-2j}. \tag{5.3}$$

Then we can choose $T_j > T$ such that

$$\sum_{k<j,\,\pm} \|e^{\pm i(\Delta-V)(T_j-T_k)}\xi_0\|_{W^{1,6}} < \varepsilon 2^{-j}. \tag{5.4}$$

We define the final data by

$$\eta_+ := \sum_{j>0} \varepsilon 2^{-j} e^{i(\Delta-V)(-T_j)}\xi_0, \tag{5.5}$$

and the asymptotic profile of the dispersive part is given by

$$\eta_\ell := \sum_j \varepsilon 2^{-j} e^{i(\Delta - V)(t - T_j)} \xi_0. \tag{5.6}$$

Let $\psi(t) = Q[z(t)] + \eta(t)$ be a solution furnished by Theorem 1.8, corresponding to η_+ and z_+ of size at most ε. By (4.11) and (5.2), we have

$$\|P_c \eta(T_j) - \eta_\ell(T_j)\|_{H^1} \lesssim \varepsilon \|\eta_\ell\|_{L_t^2 W^{1,6}[T_j, \infty]} \lesssim \varepsilon^2 2^{-j}. \tag{5.7}$$

By (5.4), we have

$$\|\eta_\ell(T_j) - \varepsilon 2^{-j} \xi_0\|_{L^6} \lesssim \varepsilon^2 2^{-j}. \tag{5.8}$$

Then, by Lemma 2.2, we have

$$\|P_d \eta(T_j)\|_{L^6} \lesssim \varepsilon \|P_c \eta(T_j)\|_{L^6} \lesssim \varepsilon^2 2^{-j}. \tag{5.9}$$

Thus we obtain

$$\|\eta(T_j) - \varepsilon 2^{-j} \xi_0\|_{L^2(B)} \lesssim \|\eta(T_j) - \varepsilon 2^{-j} \xi_0\|_{L^6} \lesssim \varepsilon^2 2^{-j}. \tag{5.10}$$

By Lemma 2.1, we have

$$\|Q[z(T_j)] - Q[z] - (z(T_j) - z)\phi_0\|_{L^2(B)} \lesssim o(\varepsilon)|z(T_j) - z|. \tag{5.11}$$

Since $\phi_0 > 0$ everywhere, we may assume, by choosing ε sufficiently small, that

$$\|Q[z(T_j)] - Q[z] - (z(T_j) - z)\phi_0\|_{L^2(B)} < \varepsilon \|(z(T_j) - z)\phi_0\|_{L^2(B)}. \tag{5.12}$$

Since ξ_0 and ϕ_0 are orthogonal in $L^2(B)$, we obtain, from (5.10) and (5.12),

$$\|\eta(T_j) + Q[z(T_j)] - Q[z]\|_{L^2(B)} \gtrsim \varepsilon 2^{-j} \|\xi_0\|_{L^2(B)} - \varepsilon^2 2^{-j} \gtrsim \varepsilon 2^{-j}, \tag{5.13}$$

provided ε is sufficiently small. Thus we obtain, by (5.3),

$$\inf_z \|\psi(T_j) - Q[z]\|_{L^2(B)} \gtrsim \varepsilon 2^{-j} \gtrsim 2^j f(T_j). \tag{5.14}$$

Appendix

Nonlinear bound states

In this appendix, we prove Lemma 2.1.

For the linear potential V, we may weaken Assumption 1.3 to include only those parts which are relevant for existence of nonlinear bound states.

Assumption A.1. We suppose $V \in L^2 + L^\infty$ with $\|V_-\|_{L^2+L^\infty}(|x| > R) \to 0$ as $R \to \infty$, and that $e_0 < 0$ is a simple eigenvalue of $-\Delta + V$ (we do not need the Strichartz estimates and we do not need e_0 to be the only eigenvalue or even the ground state).

We need the nonlinearity g to be just superquadratic. Thus, we may replace Assumption 1.5 by the following.

Assumption A.2. g is as in Assumption 1.5, but, in the pointwise case, is only required to satisfy (instead of (1.26))

$$g''(s) = o(1) \quad \text{as } s \in \mathbb{R} \longrightarrow 0^+. \tag{A.1}$$

Proof of Lemma 2.1 under Assumptions A.1 and A.2. For each z, we look for a solution $Q = z\phi_0 + q$ and $E = e_0 + e'$ of (1.6) with $(\phi_0, q) = 0$ small and $e' \in \mathbb{R}$ small. Then

$$(-\Delta + V)q + g\big(z\phi_0 + q\big) = e'\big(z\phi_0 + q\big) + e_0 q. \tag{A.2}$$

Taking projections on the ϕ_0 and $\phi_0^\perp$ directions, we get

$$\begin{aligned} e'z &= \big(\phi_0, g\big(z\phi_0 + q\big)\big), \\ H_0 q &= -P_c g\big(z\phi_0 + q\big) + e'q, \end{aligned} \tag{A.3}$$

where we denote $H_0 := -\Delta + V - e_0$. The right sides are of order $o(z)$. We will use a contraction mapping argument to solve for $q = o(z^2)$ in H^2 and $e' = o(z)$ uniquely, for sufficiently small z. Differentiating by z, we obtain the equations for higher derivatives:

$$\begin{aligned} zDe' + e'J &= \big(\phi_0, Dg(Q)\big), \\ zD^2e' + JDe' &= \big(\phi_0, D^2g(Q)\big), \\ H_0 Dq &= -P_c Dg(Q) + qDe' + e'Dq, \\ H_0 D^2 q &= -P_c D^2 g(Q) + qD^2e' + DqDe' + e'D^2q, \\ Dg(Q) &= g'(Q)\big(J\phi_0 + Dq\big), \\ D^2g(Q) &= g''(Q)\big(J\phi_0 + Dq\big)^2 + g'(Q)D^2q, \end{aligned} \tag{A.4}$$

where we have omitted subscripts for D and $J := Dz = (1, i)$, and some constant coefficients.

Assumption A.1 implies that $-\Delta + V$ is selfadjoint on L^2 with domain H^2, so $\phi_0 \in H^2$. Furthermore, the assumption $\|V_-\|_{(L^2+L^\infty)(\{|x|>R\})} \to 0$ implies that $-\Delta + V$ is a relatively compact perturbation of $-\Delta + V_+$. So, by the Weyl theorem, the essential spectrum of $-\Delta + V$ is contained in $[0, \infty)$. In particular, e_0 is an isolated point of the spectrum. Since it is a simple eigenvalue, we have

$$H_0^{-1} : L_\perp^2 \longrightarrow H_\perp^2 \text{ bounded,} \tag{A.5}$$

where $H_\perp^2$ and $L_\perp^2$ denote the Sobolev spaces restricted to the orthogonal complement of ϕ_0.

Now we can solve the equations in the closed convex set

$$K := \left\{(q, e') \in H_\perp^2 \times \mathbb{R} \mid \|q\|_{H^2} \le |z|^2,\ |e'| \le |z|\right\} \tag{A.6}$$

for sufficiently small $z \in \mathbb{C}$ (the case $z = 0$ is trivial). Indeed, we define the map $M : (q_0, e_0') \mapsto (q_1, e_1')$ by

$$\begin{aligned} g_0 &:= g(z\phi_0 + q_0), \\ ze_1' &:= (\phi_0, g_0), \\ q_1 &:= H_0^{-1}\big(-P_c g_0 + e_0' q_0\big). \end{aligned} \tag{A.7}$$

Then, under Assumption A.2, we have the easy estimates

$$\begin{aligned} &|ze_1'| \lesssim \|g_0\|_{H^{-2}} \le o\Big(\|z\phi_0 + q_0\|_{H^2}^2\Big) \lesssim o(z^2), \\ &\|q_1\|_{H^2} \lesssim \|g_0\|_{L^2} + |e_0'|\|q_0\|_{H^2} \lesssim o(z^2), \end{aligned} \tag{A.8}$$

which implies that M maps K into K. Let $(e_{j+2}', q_{j+2}) := M(e_j', q_j)$ and $g_j := g(z\phi_0 + q_j)$ for $j = 0, 1$. Similarly, we can estimate the difference

$$\begin{aligned} |z(e_2' - e_3')| &\lesssim \|g_0 - g_1\|_{H^{-2}} \le o(\|z\phi_0 + q_j\|_{H^2})\|q_0 - q_1\|_{H^2} \lesssim o(z)\|q_0 - q_1\|_{H^2}, \\ \|q_2 - q_3\|_{H^2} &\lesssim \|g_0 - g_1\|_{L^2} + |e_0' - e_1'|\|q_0\|_{L^2} + |e_1'|\|q_0 - q_1\|_{H^1} \\ &\lesssim |z|\Big(|e_0' - e_1'| + \|q_0 - q_1\|_{H^2}\Big), \end{aligned} \tag{A.9}$$

which implies that M is a contraction on K and hence has a unique fixed point in K.

Suppose now there is a solution (q, e') in the class $K' = \{(q, e') : \|q\|_{H^2} \le \gamma,\ |e'| \le \gamma\}$. We have

$$\|q\|_{H^2} \lesssim \left\|g(z\phi_0 + q)\right\|_{L^2} + |e'|\|q\|_{L^2} \lesssim o(1)|z|^2 + o(1)\|q\|_{H^2}. \tag{A.10}$$

Thus $\|q\|_{H^2} \lesssim o(1)|z|^2$. It follows that $|ze'| \lesssim \|g(z\phi_0+q)\|_{H^{-2}} \lesssim o(z^2)$, and hence $(q, e') \in K$. This shows the uniqueness in the class K'.

Let (q, e') be the unique solution and $Q := z\phi_0 + q$. Since the equation becomes real-valued when $z \in \mathbb{R}$, the unique solution $Q[z]$ is also real-valued.

By the same argument as above, we have

$$\begin{aligned}
&|z||De'| \lesssim o(z) + \left\|Dg(Q)\right\|_{H^{-2}},\\
&\|Dq\|_{H^2} \lesssim \left\|Dg(Q)\right\|_{L^2} + |z|^2|De'| + |z|\|Dq\|_{H^2},\\
&\left\|Dg(Q)\right\|_{L^2} \le o(\|Q\|_{H^2})\left\|J\phi_0 + Dq\right\|_{H^2} \le o(z)\big(1 + \|Dq\|_{H^2}\big),
\end{aligned} \tag{A.11}$$

which imply that $De' = o(1)$ and $\|Dq\|_{H^2} = o(z)$. Similarly, we have

$$\begin{aligned}
&|z|\left|D^2e'\right| \lesssim o(1) + \left\|D^2g(Q)\right\|_{H^{-2}},\\
&\left\|D^2q\right\|_{H^2} \lesssim \left\|D^2g(Q)\right\|_{L^2} + o(z^2)\left|D^2e'\right| + o(z)|De'| + o(z)\left\|D^2q\right\|_{H^2},\\
&\left\|D^2g(Q)\right\|_{L^2} \lesssim o(1)\left\|J\phi_0 + Dq\right\|_{H^2}^2 + o(z)\left\|D^2q\right\|_{H^2},
\end{aligned} \tag{A.12}$$

which imply that $|D^2e'| = o(1/z)$ and $\|D^2q\|_{H^2} = o(1)$.

Next we establish the estimate in $W^{1,1}$. Actually, following [1], we will obtain an exponentially weighted energy estimate. Let

$$\mathcal{E}(\psi, \phi) := (\nabla\psi, \nabla\phi) + \int V\bar{\psi}\phi\, dx \tag{A.13}$$

denote the bilinear form associated to $-\Delta + V$. It is defined on $H^1 \times H^1$. Set

$$b := \lim_{R\to\infty} \inf\left\{\mathcal{E}(\varphi, \varphi) \mid \varphi \in H^1,\ \|\varphi\|_2 = 1,\ \varphi(x) = 0 \text{ for } |x| < R\right\}. \tag{A.14}$$

Suppose $b < 0$. Then there exists a sequence φ_R satisfying $\|\varphi_R\|_2 = 1$, $\varphi_R(x) = 0$ for $|x| < R$, and $\mathcal{E}(\varphi_R, \varphi_R) < \delta$ for some fixed $\delta < 0$. It is easy to check that φ_R is bounded in H^1. Since it converges weakly to 0 as $R \to \infty$, by the assumption $\|V_-\|_{(L^2+L^\infty)(\{|x|>R\})} \to 0$, the negative part $\int V_-|\varphi_R|^2 dx$ of the energy converges to 0, a contradiction. Thus $b \ge 0$. In other words, there exists $\delta(R)$ with $\delta(R) \to b \ge 0$ as $R \to \infty$, such that for any $\varphi \in H^1$

satisfying $\varphi(x) = 0$ for $|x| < R$, we have

$$\mathcal{E}(\varphi, \varphi) \geq \delta(R)\|\varphi\|_2^2. \tag{A.15}$$

Now we apply this inequality with localized exponential weight χ_R defined by

$$\chi_R(x) = \begin{cases} e^{\varepsilon(|x|-R)} - 1 & (R < |x| < 2R), \\ e^{\varepsilon(3R-|x|)} - 1 & (2R < |x| < 3R), \\ 0, & \text{else}, \end{cases} \tag{A.16}$$

with some fixed small $\varepsilon > 0$. Then we have

$$|\nabla\chi_R| \lesssim \varepsilon(\chi_R + 1). \tag{A.17}$$

From (A.15), we have, for any $\varphi \in H^1$,

$$\begin{aligned} \delta(R)\|\chi_R\varphi\|_2^2 &\leq \mathcal{E}(\chi_R\varphi, \chi_R\varphi) \\ &= \mathcal{E}(\chi_R^2\varphi, \varphi) + \int |\varphi\nabla\chi_R|^2 dx \\ &= (\chi_R^2\varphi, H_0\varphi) + e_0\|\chi_R\varphi\|_2^2 + \int |\varphi\nabla\chi_R|^2 dx. \end{aligned} \tag{A.18}$$

Recall $H_0 := -\Delta + V - e_0$. By using (A.17), we obtain for sufficiently large R and small ε,

$$\|\chi_R\varphi\|_2^2 \lesssim (\chi_R^2\varphi, H_0\varphi) + \varepsilon^2\|\varphi\|_2^2. \tag{A.19}$$

Also, the Sobolev norm is estimated by

$$\begin{aligned} \|\chi_R\nabla\varphi\|_2^2 &\leq \|\nabla(\chi_R\varphi)\|_2^2 + \|\varphi\nabla\chi_R\|_2^2 \\ &\lesssim \mathcal{E}(\chi_R\varphi, \chi_R\varphi) + \|\chi_R\varphi\|_2^2 + \varepsilon^2\|\varphi\|_2^2 \\ &\lesssim (\chi_R^2\varphi, H_0\varphi) + \varepsilon^2\|\varphi\|_2^2. \end{aligned} \tag{A.20}$$

Thus we obtain the key relation

$$\|\chi_R\varphi\|_{H^1}^2 \lesssim (\chi_R\varphi, \chi_R H_0\varphi) + \|\varphi\|_2^2 \tag{A.21}$$

for any $\varphi \in H^1$, sufficiently large R, and small ε. Now we substitute each of $\varphi = \phi_0, q, Dq, D^2q$ into (A.21) and use the equations they satisfy. We find

$$\|\chi_R\phi_0\|_{H^1}^2 \lesssim \|\phi_0\|_2^2 = 1, \tag{A.22}$$

and under Assumption A.2, we easily obtain

$$\begin{aligned}\|\chi_R q\|_{H^1}^2 &\lesssim \|Q^{-1}g(Q)\|_{\mathcal{B}(H^1;H^{-1})}\\ &\quad\times\left(\|\chi_R Q\|_{H^1}\|\chi_R q\|_{H^1}+\|Q\|_{H^1}\|\chi_R q\|_{H^1}\|\chi_R\phi_0\|_{H^1}\right)+\|q\|_2^2 \\ &\lesssim \left(o(z^2)+|z|\|\chi_R q\|_{H^1}\right)\|\chi_R q\|_{H^1}+o(z^4),\end{aligned}\tag{A.23}$$

which implies that $\|\chi_R q\|_{H^1} = o(z^2)$. Similar estimates hold for $\chi_R Dq$ and $\chi_R D^2 q$. It follows that each of these functions is bounded in $W^{1,1}$. ■

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Stephen Gustafson: Mathematics Department, University of British Columbia, Vancouver, BC, Canada V6T 1Z2
E-mail address: gustaf@math.ubc.ca

Kenji Nakanishi: Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
E-mail address: n-kenji@math.nagoya-u.ac.jp

Tai-Peng Tsai: Mathematics Department, University of British Columbia, Vancouver, BC, Canada V6T 1Z2
E-mail address: ttsai@math.ubc.ca