IMRN International Mathematics Research Notices
2002, No. 31

Relaxation of Excited States in Nonlinear Schrödinger Equations

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1 Introduction

Consider the nonlinear Schrödinger equation

$$i\partial_t\psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \quad \psi(t=0) = \psi_0, \tag{1.1}$$

where V is a smooth localized real potential, $\lambda = \pm 1$, and $\psi = \psi(t,x) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is a wave function. The goal of this paper is to study the asymptotic dynamics of the solution for initial data ψ_0 near some *nonlinear excited state*.

Recall that for any solution $\psi(t) \in H^1(\mathbb{R}^3)$ the L^2-norm and the Hamiltonian

$$\mathcal{H}[\psi] = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V|\psi|^2 + \frac{1}{4}\lambda|\psi|^4 \, dx \tag{1.2}$$

are constants for all t. The global well-posedness for small solutions in $H^1(\mathbb{R}^3)$ can be proven using these conserved quantities and a continuity argument.

We assume that the linear Hamiltonian $H_0 := -\Delta + V$ has two simple eigenvalues $e_0 < e_1 < 0$ with normalized eigenfunctions ϕ_0, ϕ_1. We further assume that

$$e_0 < 2e_1. \tag{1.3}$$

The nonlinear bound states to Schrödinger equation (1.1) are solutions to the equation

$$(-\Delta + V)Q + \lambda|Q|^2Q = EQ. \tag{1.4}$$

Received 15 January 2002. Revision received 1 April 2002.
Communicated by Carlos Kenig.

They are critical points to the Hamiltonian $\mathcal{H}[\phi]$ defined in (1.2) subject to the constraint that the L^2-norm of ψ is fixed. For any bound state $Q = Q_E$, $\psi(t) = Qe^{-iEt}$ is a solution to the nonlinear Schrödinger equation.

We may obtain two families of such bound states by standard bifurcation theory, corresponding to the two eigenvalues of the linear Hamiltonian. For any E sufficiently close to e_0 so that $E - e_0$ and λ have the same sign, there is a unique positive solution $Q = Q_E$ to (1.4) which decays exponentially as $x \to \infty$ (see Lemma 2.1). We call this family the *nonlinear ground states* and we refer to it as $\{Q_E\}_E$. Similarly, there is a *nonlinear excited state* family $\{Q_{1,E_1}\}_{E_1}$ for E_1 near e_1. We will abbreviate them as Q and Q_1. From Lemma 2.1, we also have $\|Q_E\| \sim |E - e_0|^{1/2}$ and $\|Q_{1,E_1}\| \sim |E_1 - e_1|^{1/2}$.

It is well known that the family of nonlinear ground states is stable in the sense that if

$$\inf_{\Theta,E} \left\|\psi(t) - Q_E e^{i\Theta}\right\|_{L^2} \tag{1.5}$$

is small for $t = 0$, it remains so for all t, see [9]. Let $\|\cdot\|_{L^2_{loc}}$ denote a local L^2-norm (a precise choice will be made later on). We expect that this difference actually approaches zero in local L^2-norm, that is,

$$\lim_{t\to\infty} \inf_{\Theta,E} \left\|\psi(t) - Q_E e^{i\Theta}\right\|_{L^2_{loc}} = 0. \tag{1.6}$$

If $-\Delta + V$ has only one bound state, it is proven in [4, 8, 13, 14] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. Suppose now that $-\Delta + V$ has multiple bound states, say, two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1. It is proven in [16] that the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. (See also [1, 2] for the one-dimensional case and [15] for nonlinear Klein-Gordon equations.)

Denote by L^2_r the weighted L^2 spaces (r may be positive or negative)

$$L^2_r(\mathbb{R}^3) \equiv \left\{\phi \in L^2(\mathbb{R}^3) : \langle x\rangle^r \phi \in L^2(\mathbb{R}^3)\right\}. \tag{1.7}$$

The space for initial data we shall use is

$$Y \equiv H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3). \tag{1.8}$$

We use L^2_{loc} to denote $L^2_{r_1}$. The parameter $r_1 > 3$ is fixed and will be determined in Lemma 2.2. We now state the assumptions in [16] on the potential V.

Assumption A0. $-\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has 2 simple eigenvalues $e_0 < e_1 < 0$, with normalized eigenvectors ϕ_0 and ϕ_1.

Assumption A1 (resonance condition). Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. We assume that $2e_{01} > |e_0|$, that is, $e_0 < 2e_1$. Let

$$\gamma_0 := \lim_{\sigma\to 0+} \operatorname{Im}\left(\phi_0\phi_1^2, \frac{1}{H_0 + e_0 - 2e_1 - \sigma i}\mathbf{P}_c^{H_0}\phi_0\phi_1^2\right). \tag{1.9}$$

Since the expression is quadratic, we have $\gamma_0 \geq 0$. We assume, for some $s_0 > 0$, that

$$\inf_{|s|<s_0}\lim_{\sigma\to 0+} \operatorname{Im}\left(\phi_0\phi_1^2, \frac{1}{H_0 + e_0 - 2e_1 + s - \sigma i}\mathbf{P}_c^{H_0}\phi_0\phi_1^2\right) \geq \frac{3}{4}\gamma_0 > 0. \tag{1.10}$$

We shall use $0i$ to replace σi and the limit $\lim_{\sigma\to 0+}$ later on.

Assumption A2. For λQ_E^2 sufficiently small, the bottom of the continuous spectrum to $-\Delta + V + \lambda Q_E^2$, 0, is not a generalized eigenvalue, that is, not a resonance. Also, we assume that V satisfies the following assumptions: there is a small $\sigma > 0$ such that,

$$\left|\nabla^\beta V(x)\right| \leq C\langle x\rangle^{-5-\sigma}, \quad \text{for } |\beta| \leq 2. \tag{1.11}$$

Also, the functions $(x\cdot\nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with a $-\Delta$-bound < 1:

$$\left\|(x\cdot\nabla)^k V\phi\right\|_2 \leq \sigma_0\|-\Delta\phi\|_2 + C\|\phi\|_2, \quad \sigma_0 < 1,\ k = 0, 1, 2, 3. \tag{1.12}$$

Thus, the $W^{k,p}$ estimates for the wave operator $W_{H_0} = \lim_{t\to\infty} e^{itH_0}e^{it(\Delta+E)}$ in [18] hold for $k \leq 2$.

Assumption A2 contains some standard conditions to assure that most tools in linear Schrödinger operators apply. These conditions are certainly not optimal. The main assumption in Assumptions A0, A1, and A2 is the condition $2e_{01} > |e_0|$ in Assumption A1. The rest of Assumption A1 are just generic assumptions. This condition states that the excited state energy is closer to the continuum spectrum than to the ground state energy. It guarantees that twice the excited state energy of $H_0 - e_0$ becomes a resonance in the continuum spectrum (of $H_0 - e_0$). This resonance produces the main relaxation mechanism. If this condition fails, the resonance occurs in higher order terms and a proof of relaxation will be much more complicated. Also, the rate of decay will be different.

The main result in [16] concerning the relaxation of the ground states can be summarized in the following theorem.

Theorem A. Suppose that suitable assumptions on V hold. Then there are small constants $\varepsilon_0, n_0 > 0$ such that, if the initial data ψ_0 satisfies $\|\psi_0 - ne^{i\Theta_0}\phi_0\|_Y \le \varepsilon_0^2 n^2$ for some $n \le n_0$ and some $\Theta_0 \in \mathbb{R}$, then there exists an E_∞ and a function $\Theta(t)$ such that $\|Q_{E_\infty}\|_Y - n = O(\varepsilon_0^2 n)$, $\Theta(t) = -E_\infty t + O(\log t)$, and

$$\left\|\psi(t) - Q_{E_\infty} e^{i\Theta(t)}\right\|_{L^2_{\mathrm{loc}}} \le C(1+t)^{-1/2}. \tag{1.13}$$

□

This theorem settles the question of asymptotic profile near ground states. (Notice that ε_0 depends on n, which was not emphasized in [16]. A stronger version removing this restriction on ε_0 is given in Theorem 4.1.) Suppose that the initial data ψ_0 is now near some nonlinear excited state. From the physical ground, we expect that ψ_t will eventually decay to some ground state unless the initial data ψ_0 is exactly a nonlinear excited state. We call this the *strong relaxation property*. For comparison, we define a weaker property, the *generic relaxation property*, as follows. Denote the space of initial data by X. Let X_1 (X_0 resp.) be the set of initial data such that the asymptotic profiles are given by some nonlinear excited (ground resp.) states. We shall say that the dynamics satisfy the generic relaxation property if X_1 has "measure zero." This concept depends on a notion of measure which should be specified in each context.

With this definition, the strong relaxation property means that X_1 is exactly the set of nonlinear excited states. In particular, X_1 is finite dimensional. We first note that the strong relaxation property is false. For any nonlinear excited state Q_1, define X_{1,Q_1} to be the set of initial data converging to Q_1 asymptotically. It is proven in [17] that for any given nonlinear excited state Q_1, X_{1,Q_1} contains a finite codimensional set. Thus our goal is to establish some weaker statement such as the generic relaxation property. This is the first step toward a classification of asymptotic dynamics of the nonlinear Schrödinger equation.

In order to state the main result, we first decompose the wave function using the eigenspaces of the Hamiltonian H_0 as

$$\psi = \underline{x}\phi_0 + \underline{y}\phi_1 + \underline{\xi}, \quad \underline{\xi} = \mathbf{P}_c^{H_0}\psi. \tag{1.14}$$

For initial data near excited states, this decomposition contains an error of order y^3 and it is difficult to read from (1.14) whether the wave function is exactly an excited state. Thus we use the decomposition

$$\psi = x\phi_0 + Q_1(y) + \xi, \tag{1.15}$$

where

$$y = \underline{y}, \qquad x = \underline{x} - \big(\phi_0, Q_1(y)\big), \qquad \xi = \underline{\xi} - \mathbf{P}_c Q_1(y). \tag{1.16}$$

Here we have used the convention that

$$Q_1(y) := Q_1(m)e^{i\Theta}, \quad m = |y|, \; me^{i\Theta} = y. \tag{1.17}$$

We shall prove in Section 2 that for ψ with sufficiently small Y-norm (1.8), such a decomposition exists and is unique. Thus we assume that $\psi_0 = x_0\phi_0 + Q_1(y_0) + \xi_0$ is sufficiently small in Y.

Theorem 1.1. Suppose the assumptions on V given above hold. There is a small constant $n_0 > 0$ such that the following holds. Let $\psi(t,x)$ be a solution of (1.1) with the initial data ψ_0 satisfying

$$\begin{aligned} &\|\psi_0\|_Y = n, \quad 0 < n \le n_0, \qquad |y_0| \ge \frac{1}{2}n, \\ &|x_0| \ge 2ne^{-(n^{-1/4})}, \qquad |x_0| \ge \varepsilon_2^{-1} n^2 \|\xi_0\|_Y, \end{aligned} \tag{1.18}$$

where $\varepsilon_2 > 0$ is a small constant to be fixed later in the proof. Let

$$n_1 = \left(|x_0|^2 + \frac{1}{2}|y_0|^2\right)^{1/2} \sim n. \tag{1.19}$$

Then, there exist an E_∞ with $\|Q_{E_\infty}\|_Y \sim n_1$ and a function $\Theta(t) = -E_\infty t + O(\log t)$ such that

$$C_1(1+t)^{-1/2} \le \|\psi(t) - Q_{E_\infty} e^{i\Theta(t)}\|_{L^2_{\mathrm{loc}}} \le C_2(1+t)^{-1/2}, \tag{1.20}$$

for some constants C_1 and C_2 depending on n. □

Condition (1.18) can be interpreted as follows: the excited state component, y_0, should account for at least half the mass of the initial data. (Here 1/2 can be replaced by any fixed small number.) *Under this condition, if the ground state component*, x_0, *is not extremely small compared with the continuum component* ξ_0, *then the dynamics relax to some ground state*. The condition $|x_0| \ge 2ne^{-(n^{-1/4})}$ is a very mild assumption to make sure that x_0 is not incredibly small.

It is instructive to compare our result with the linear stability analysis of [5, 6, 10, 11]. In our setup, the main result in [6] states that the linearized operator around a nonlinear excited state is structurally stable if $e_0 < 2e_1$ and unstable if $e_0 > 2e_1$. Hence, the excited states considered in this article are unstable and are expected to decay under generic perturbations. The instability of the excited state stated in Theorem 1.1 is thus consistent with the linear analysis. Notice that Theorem 1.1 tracks the dynamics for all time including the time regime when the dynamics are far away from the excited states. Furthermore, *for all initial data considered in Theorem 1.1*, *the relaxation rate to the asymptotic ground state is exactly of order* $t^{-1/2}$, a rate very different from the standard linear Schrödinger equations.

In view of the linear analysis, the existence [17] of (nonlinear) stable directions for excited states is a more surprising result. For the linear stable case, that is, $e_0 > 2e_1$, the only rigorous result is the existence [17] of (nonlinear) stable directions in this case. Although the linear analysis states that all directions are linearly stable, on physics ground we still expect excited states remain generically unstable.

We now explain the main idea of the proof for Theorem 1.1. The relaxation mechanism can be divided into three time regimes.

(1) The initial layer. The component of the wave function in the continuum spectrum direction gradually disperses away; the components in the bound states directions do not change much.

(2) The transition regime. Transition from the excited state to the ground state takes place in this interval. The component along the ground state grows in this regime; that along the excited state is slightly more complicated. We can further divide this time regime into two intervals. In part (i), the component along the excited state does not change much. In part (ii), it decreases steadily and eventually becomes smaller than the component along the ground state.

(3) Stabilization. The ground state dominates and is stable. Both the excited states and dispersive part gradually decay.

In different time regimes, the dominant terms are different and we have to linearize the dynamics according to the dominant terms. In the first time region, $\psi(t)$ is near an excited state, and it is best to use operator linearized around the excited state. In the third time regimes, $\psi(t)$ is near a ground states, and it is best to use operator linearized around a ground state. For the transition regimes, the dynamics are far away from both excited and ground states and we use the linear Hamiltonian H_0.

Besides technical problems associated with changing coordinate systems in different time intervals, there is an intrinsic difficulty related to the time reversibility of the Schrödinger equation. Imagine that we are now ready to show that our dynamics is in the third time regime and will stabilize around some nonlinear ground state. If we take the wave function ψ_t at this time and time reverse the dynamics, then the dynamics will drive this wave function back to the initial state near some excited state. The time reversed state ψ_t and the wave function ψ_t itself will satisfy the same estimates in the usual Sobolev or L_p senses. However, their dynamics are completely different: one stabilizes to a ground state; the other back to near an excited state. This suggests that ψ_t carries information concerning the time direction and this information will not show up if we measure it by the usual estimates.

This time reversal difficulty manifests itself in the technical proofs as follows. We shall see that, when the third time regime begins, the dispersive part is not well-localized and its L^2-norm can be larger than that of the bound states—both violate the conditions for approaching the ground states in [16]. To resolve this issue, we need to extract information which are time-direction sensitive so that even though the dispersive part may be large, it is irrelevant since it is *out-going*. Though the concept of out-going wave is known for linear Schrödinger equations, it is difficult to implement it for nonlinear Schrödinger equations. *We have, however, succeeded in defining a notion of "out-going estimates" which provides sufficient time-direction related information to control the asymptotic evolution.*

Resonance induced decay and growth. To illustrate the mechanism of resonance induced decay and growth, we consider the problem in the coordinates with respect to the linear Hamiltonian $H_0 = -\Delta + V$,

$$\psi(t) = x(t)\phi_0 + y(t)\phi_1 + \xi(t), \quad \xi(t) = \mathbf{P}_c^{H_0}\psi(t). \tag{1.21}$$

The nonlinear term $\psi^2\bar{\psi}$ can be split into a sum of many terms using this decomposition. However, there is only one important nonlinear term in the equation for each component: (assume that $\lambda = 1$)

$$i\dot{x} = e_0 x + \Big(\phi_0, (y\phi_1)^2\bar{\xi}\Big) + \cdots, \tag{1.22}$$

$$i\dot{y} = e_1 y + \big(\phi_1, 2(x\phi_0)(\bar{y}\phi_1)\xi\big) + \cdots, \tag{1.23}$$

$$i\partial_t \xi = H_0\xi + \mathbf{P}_c^{H_0}\bar{x}y^2\phi_0\phi_1^2 + \cdots. \tag{1.24}$$

From (1.22), we know that $u(t) = e^{ie_0 t}x(t)$ has less oscillation of lower order than $x(t)$. Hence we say that $x(t)$ has a *phase factor* $-e_0$. Similarly, $y(t)$ has a phase factor $-e_1$. The nonlinear term $\bar{x}y^2\phi_0\phi_1^2$ has a phase factor $e_0 - 2e_1$, which, due to assumption (1.3), is the only term in $\psi^2\bar{\psi}$ with a negative phase factor. It gives a term in ξ:

$$\xi(t) = \bar{x}y^2(t)\Phi + \cdots, \quad \Phi = \frac{1}{H_0 + e_0 - 2e_1 - 0i}\mathbf{P}_c^{H_0}\phi_0\phi_1^2. \tag{1.25}$$

Notice that Φ is complex and its imaginary part is analogous to the Fermi golden rule. This was extensively studied in [12, 15]. Substituting this term into (1.22) and (1.23), we have

$$\begin{aligned} i\dot{x} &= i\gamma_0 |y|^4 x + \cdots, \\ i\dot{y} &= -2i\gamma_0 |x|^2 |y|^2 y + \cdots, \end{aligned} \tag{1.26}$$

with γ_0 given in (1.9). In (1.26) we have omitted two types of irrelevant terms:

(1) terms with same phase factors as x or y, for example, $e_0 x$ and $|y|^2 x$ in (1.22). Since their coefficients are real, they disappear when we consider the equations for $|x|$ and $|y|$;

(2) terms with different phase factors, for example, $\bar{x}y^2$ in (1.22). Since these terms have different phases, their contribution averaging over time will be small. This can be made precise by the Poincaré normal form.

From (1.26) we obtain the decay of y and the growth of x as well as the three time regimes mentioned previously. However, it should be warned that this setup is only suitable when both x and y are of similar sizes.

2 The initial layer and the transition regimes: the setup

We now outline the basic strategy for the initial layer and the transition regimes. We first review the properties of the bound state families.

2.1 Nonlinear bound states

The basic properties of nonlinear bound state families can be summarized in the following lemma from [16].

Lemma 2.1. Suppose that $-\Delta + V$ satisfies Assumptions A0 and A2. There is a small constant $n_0 > 0$ such that the following hold. For any E between e_0 and $e_0 + \lambda n_0^2$, there is a nonlinear ground state Q_E solving (1.4). The nonlinear ground state Q_E is real, local, smooth, $\lambda^{-1}(E - e_0) > 0$, and

$$Q_E = n\phi_0 + O(n^3), \quad n \approx C\left[\lambda^{-1}(E - e_0)\right]^{1/2}, \; C = \left(\int \phi_0^4 \, dx\right)^{-1/2}. \tag{2.1}$$

Moreover, we have $R_E \equiv \partial_E Q_E = O(n^{-2}) Q_E + O(n) = O(n^{-1})$ and $\partial_E^2 Q_E = O(n^{-3})$. If we define $c_1 \equiv (Q, R)^{-1}$, then $c_1 = O(1)$ and $\lambda c_1 > 0$.

There is also a family of nonlinear excited states $\{Q_{E_1}\}_{E_1}$ for E_1 between e_1 and $e_1 + \lambda n_0^2$ satisfying similar properties: $Q_{E_1} = m\phi_1 + O(m^3)$ solves (1.4) with $m \sim C[\lambda^{-1}(E_1 - e_1)]^{1/2}$, and so forth. □

This lemma can be proven using standard perturbation argument, see [16]. For the purpose of this paper, we prefer to use the value $m = (\phi_1, Q_1)$ as the parameter and refer to the family of excited states as $Q_1(m)$. It is straightforward to compute the leading corrections of $Q_1(m)$ via standard perturbation argument used in proving Lemma 2.1. Thus we can write Q_1 as

$$\begin{aligned} &Q_1(m) = m\phi_1 + q(m), \quad q(m) \perp \phi_1, \\ &q(m) = m^3 q_3 + q^{(5)}(m), \quad q^{(5)}(m) = O(m^5), \end{aligned} \tag{2.2}$$

where $q_3 = -\lambda(H_0 - e_1)^{-1}\pi\phi_1^3$, and π is the projection

$$\pi h = h - (\phi_1, h)\phi_1. \tag{2.3}$$

Similarly, we can also expand $E_1(m)$ in m as

$$E_1(m) = e_1 + E_{1,2}m^2 + E_{1,4}m^4 + E_1^{(6)}(m), \quad E_1^{(6)}(m) = O(m^6). \tag{2.4}$$

Moreover, we can differentiate the relation of $Q_1(m)$ with respect to m to get

$$Q_1'(m) = \frac{d}{dm}Q_1 = \phi_1 + q'(m), \quad q'(m) = \frac{d}{dm}q(m) = O(m^2), \; q'(m) \perp \phi_1. \tag{2.5}$$

2.2 Equations

In the first and second time regimes, we write

$$\psi(t) = x(t)\phi_0 + Q_1(m(t))e^{i\Theta(t)} + \xi(t), \tag{2.6}$$

where $\xi \in \mathbf{H}_c(H_0)$, see (1.15). If we write $\Theta(t) = \theta(t) - \int_0^t E_1(m(s))ds$, we can write $y(t)$ as

$$y(t) = me^{i\Theta} = m\exp\left\{i\theta(t) - i\int_0^t E_1(m(s))\,ds\right\}. \tag{2.7}$$

Denote the part orthogonal to ϕ_1 by $h = x\phi_0 + \xi$. From Schrödinger equation (1.1), h satisfies the equation

$$i\partial_t h = H_0 h + G + \Lambda, \tag{2.8}$$

where

$$\begin{aligned} G &= \lambda|\psi|^2\psi - \lambda Q_1^3 e^{i\Theta} \\ &= \lambda Q_1^2\left(e^{i2\Theta}\bar{h} + 2h\right) + \lambda Q_1\left(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2\right) + \lambda|h|^2 h, \\ \Lambda &= \left(\dot{\theta}Q_1 - i\dot{m}Q_1'\right)e^{i\Theta}. \end{aligned} \tag{2.9}$$

Since $m(t)$ and $\theta(t)$ are chosen so that $h(t) \perp \phi_1$ for all t, we have $0 = (\phi_1, i\partial_t h(t)) = (\phi_1, G + (\dot{\theta}Q_1 - i\dot{m}Q_1')e^{i\Theta})$. Hence $m(t)$ and $\theta(t)$ satisfy

$$\dot{m} = \left(\phi_1, \operatorname{Im} Ge^{-i\Theta}\right), \qquad \dot{\theta} = -\frac{1}{m}\left(\phi_1, \operatorname{Re} Ge^{-i\Theta}\right). \tag{2.10}$$

We also have the equation for y:

$$\begin{aligned} i\dot{y} &= i\dot{m}e^{i\Theta} - \left(\dot{\theta} - E_1(m)\right)me^{i\Theta} \\ &= E_1(m)y + e^{i\Theta}\left(i\dot{m} - m\dot{\theta}\right) \\ &= E_1(m)y + \left(\phi_1, G\right). \end{aligned} \tag{2.11}$$

Here we have used (2.10). Denote $\Lambda_\pi = \pi\Lambda$. We can decompose equation (2.8) for h into equations for x and ξ (2.12). Summarizing, the original Schrödinger equation is equivalent to

$$\begin{aligned} i\dot{x} &= e_0 x + \left(\phi_0, G + \Lambda_\pi\right), \\ i\dot{y} &= E_1(m)y + \left(\phi_1, G\right), \\ i\partial_t\xi &= H_0\xi + \mathbf{P}_c\left(G + \Lambda_\pi\right). \end{aligned} \tag{2.12}$$

Clearly, x has an oscillation factor $e^{-ie_0 t}$, and, since $E_1(m) \sim e_1$, y has a factor $e^{-ie_1 t}$. Hence we define

$$x = e^{-ie_0 t}u, \qquad y = e^{-ie_1 t}v. \tag{2.13}$$

Together with the integral form of the equation for ξ, we have

$$\dot{u} = -ie^{ie_0 t}\left(\phi_0, G + \Lambda_\pi\right), \tag{2.14}$$

$$\dot{v} = -ie^{ie_1 t}\left[\left(E_1(m) - e_1\right)y + \left(\phi_1, G\right)\right], \tag{2.15}$$

$$\xi(t) = e^{-iH_0 t}\xi_0 + \int_0^t e^{-iH_0(t-s)}\,\mathbf{P}_c^{H_0} G_\xi(s)\,ds, \quad G_\xi = i^{-1}\left(G + \Lambda_\pi\right). \tag{2.16}$$

This is the system we shall study.

2.3 Basic estimates and decompositions

It is useful to decompose various terms according to orders in n so that we can identify their contributions. We now proceed to do this for G, Λ_π, $E(|y|)$, and $\xi(t)$. We expect that $x, y = O(n)$ and $\xi = O(n^3)$ locally.

For G. Recall that G is given by

$$G = \lambda Q_1^2\left(e^{i2\Theta}\bar{h} + 2h\right) + \lambda Q_1\left(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2\right) + \lambda|h|^2h, \tag{2.17}$$

with $h = x\phi_0 + \xi$ and $Q_1 = Q_1(|y|)$. From the decomposition (2.2) of $Q_1 = |y|\phi_1 + |y|^3q_3 + q^{(5)}(|y|)$, we decompose G as

$$\begin{aligned} G &= \lambda\left(y^2\phi_1^2 + 2y^3\bar{y}\phi_1q_3\right)\bar{h} + \lambda\left(|y|^2\phi_1^2 + 2|y|^4\phi_1q_3\right)2h \\ &\quad + \lambda\left(y\phi_1 + y^2\bar{y}q_3\right)2|h|^2 + \lambda\left(\bar{y}\phi_1 + y\bar{y}^2q_3\right)h^2 + \lambda|h|^2h + (*), \end{aligned} \tag{2.18}$$

where $(*) = \lambda[2|y|\phi_1q^{(5)} + (|y|^3q_3 + q^{(5)})^2](e^{i2\Theta}\bar{h} + 2h) + \lambda q^{(5)}(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2)$ with $q^{(5)} = q^{(5)}(|y|)$. We then substitute $h = x\phi_0 + \xi$ to obtain

$$G = G_3 + G_5 + G_7, \tag{2.19}$$

where

$$G_3 = \lambda\left(y^2\bar{x} + 2|y|^2x\right)\phi_0\phi_1^2 + \lambda\left(2|x|^2y + x^2\bar{y}\right)\phi_0^2\phi_1 + \lambda|x|^2x\phi_0^3, \tag{2.20}$$

$$\begin{aligned} G_5 &= \lambda\left(2y^3\bar{y}\bar{x} + 4|y|^4x\right)\phi_0\phi_1q_3 + \lambda\left(2|x|^2y^2\bar{y} + x^2y\bar{y}^2\right)\phi_0^2q_3 \\ &\quad + \lambda\left(x\phi_0 + y\phi_1\right)^2\bar{\xi} + 2\lambda\left|\left(x\phi_0 + y\phi_1\right)\right|^2\xi, \end{aligned} \tag{2.21}$$

$$\begin{aligned} G_7 &= \lambda\left[2|y|\phi_1q^{(5)}(|y|) + \left(|y|^3q_3 + q^{(5)}(|y|)\right)^2\right]\left(e^{i2\Theta}\bar{h} + 2h\right) \\ &\quad + \lambda q^{(5)}(|y|)\left(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2\right) \\ &\quad + \lambda\left(2y^3\bar{y}\phi_1q_3\bar{\xi} + 4|y|^4\phi_1q_3\xi\right) + \lambda\left(y\phi_1 2|\xi|^2 + \bar{y}\phi_1\xi^2\right) \\ &\quad + \lambda y^2\bar{y}q_3 2\left(x\phi_0\bar{\xi} + \bar{x}\phi_0\xi + |\xi|^2\right) + \lambda y\bar{y}^2q_3\left(2x\phi_0\xi + \xi^2\right) \\ &\quad + \lambda\phi_0\left(\bar{x}\xi^2 + 2x|\xi|^2\right) + \lambda|\xi|^2\xi. \end{aligned} \tag{2.22}$$

Note that $G_3 = O(n^3)$, $G_5 = O(n^5)$, and $G_7 = O(n^7)$. If we use the convention that

$$f \lesssim g_1 + g_2 + \cdots \tag{2.23}$$

for $\|f\| \le C\|g_1\| + \|g_2\| + \cdots$ for some suitable norms, we have

$$\begin{aligned} G &\lesssim n^2 x + n^2 \xi + \xi^3, \\ G_5 &\lesssim n^4 x + n^2 \xi, \\ G_7 &\lesssim n^6 x + n^4 \xi + n\xi^2 + \xi^3. \end{aligned} \tag{2.24}$$

It is crucial to observe that *no term in* G_3 *is of order* y^3. This is due to our setup emphasizing the role of nonlinear excited states. The price we pay is the introduction of terms involving q_3 and $q^{(5)}$.

We now identify the main oscillation factors of various terms. For example, $y^2\bar{x} = e^{i(-2e_1+e_0)t} v^2 \bar{u}$, and its factor is $-2e_1 + e_0$. For terms in G_3 the corresponding phase factors are given as follows:

$$\begin{array}{ccccc} y^2\bar{x} & |y|^2 x & |x|^2 y & x^2\bar{y} & |x|^2 x \\ -2e_1 + e_0 & -e_0 & -e_1 & -2e_0 + e_1 & -e_0. \end{array} \tag{2.25}$$

From the spectral assumption $|e_0| > 2|e_1|$, $-2e_1 + e_0$ is the only negative phase factor. Hence it is the only term of order n^3 that has resonance effect when we compute the main part of ξ. Also, since $|x|^2 y$ has the same phase as y, it will be resonant in the y-equation. Similarly, $|y|^2 x$ and $|x|^2 x$ have same phase as x and will be resonant in x-equation.

For Λ_π and $E(m)$. Recall that $\Lambda_\pi = \pi(\dot\theta Q_1 - i\dot{m}Q_1')e^{i\Theta}$. Since $\dot\theta = O(n^{-1}\|G\|_{\mathrm{loc}})$ and $\dot{m} = O(\|G\|_{\mathrm{loc}})$,

$$\|\Lambda_\pi(s)\| = O(\dot\theta)O(\pi Q_1) + O(\dot m)O(\pi Q_1') \le Cn^2\|G\|_{\mathrm{loc}}. \tag{2.26}$$

To find out the main part of Λ_π, we substitute equation (2.10) for $\dot m$ and $\dot\theta$ to obtain ($m = |y|$),

$$\begin{aligned} \Lambda_\pi &= \pi(\dot\theta Q_1 - i\dot m Q_1')e^{i\Theta} \\ &= -\left\{ \left(\phi_1, \frac{G}{2}\right) m^{-1}\pi Q_1 + \left(\phi_1, \frac{\bar G}{2}\right) m^{-1}\pi Q_1 e^{2i\Theta} \right\} \\ &\quad - i\left\{ \left(\phi_1, \frac{G}{2i}\right)\pi Q_1' + \left(\phi_1, \frac{\bar G}{2i}\right)\pi Q_1' e^{2i\Theta} \right\}. \end{aligned} \tag{2.27}$$

Since $G = G_3 + (G_5 + G_7)$ and $\pi Q_1(m) = m^3 q_3 + q^{(5)}(m)$ by (2.2), we have $\pi Q_1'(m) = 3m^2 q_3 + O(m^4)$, and the main part of Λ_π is (also recall that $y = me^{i\Theta}$)

$$\begin{aligned}\Lambda_{\pi,5} &= -\frac{1}{2}\{(\phi_1, G_3)|y|^2 q_3 + (\phi_1, \bar{G}_3) y^2 q_3\} \\ &\quad - \frac{1}{2}\{(\phi_1, G_3) 3|y|^2 q_3 + (\phi_1, \bar{G}_3) 3y^2 q_3\} \\ &= -2q_3(\phi_1, G_3|y|^2 + \bar{G}_3 y^2).\end{aligned} \tag{2.28}$$

Let $\Lambda_{\pi,7} = \Lambda_\pi - \Lambda_{\pi,5}$. We have

$$\Lambda_\pi = \Lambda_{\pi,5} + \Lambda_{\pi,7}, \tag{2.29}$$

$$\begin{aligned}\Lambda_{\pi,5} &\lesssim \|G_3\|_{\text{loc}}|y|^2 \lesssim n^4 x, \\ \Lambda_{\pi,7} &\lesssim \|G_5 + G_7\|_{\text{loc}}|y|^2 + \|G\|_{\text{loc}}|y|^4.\end{aligned} \tag{2.30}$$

The frequency $E(m)$ is already decomposed in (2.4).

For ξ. Recall the equation for ξ in (2.16), $\xi(t) = e^{-iH_0 t}\xi_0 + \int_0^t e^{-iH_0(t-s)} \mathbf{P}_c^{H_0} G_\xi(s)\, ds$, $G_\xi = i^{-1}(G + \Lambda_\pi)$. Since $\|\Lambda_\pi\| \le Cn^2\|G\|_{\text{loc}}$, the main terms in $G_\xi = i^{-1}(G+\Lambda_\pi)$ is $i^{-1}G_3$. We now compute the first term $\lambda y^2 \bar{x}\phi_0\phi_1^2$ in G_3 using integration by parts:

$$\begin{aligned}&- i\lambda \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c y^2 \bar{x}\phi_0\phi_1^2\, ds \\ &= -i\lambda e^{-iH_0 t}\int_0^t e^{i(H_0-0i)s}\, e^{i(e_0-2e_1)s}\, \nu^2 \bar{u}\mathbf{P}_c\phi_0\phi_1^2\, ds \\ &= -i\lambda e^{-iH_0 t}\left\{\left[\frac{1}{i(H_0 - 0i + e_0 - 2e_1)} e^{iH_0 s} e^{i(e_0-2e_1)s}\, \nu^2\bar{u}\mathbf{P}_c\phi_0\phi_1^2\right]_0^t \right.\\ &\qquad \left. - \int_0^t \frac{1}{i(H_0 - 0i + e_0 - 2e_1)} e^{iH_0 s} e^{i(e_0-2e_1)s}\, \frac{d}{ds}(\nu^2\bar{u})\mathbf{P}_c\phi_0\phi_1^2\, ds\right\} \\ &= y^2\bar{x}\Phi_1 - e^{-iH_0 t} y^2 \bar{x}(0)\Phi_1 - \int_0^t e^{-iH_0(t-s)}\, e^{i(e_0-2e_1)s}\, \frac{d}{ds}(\nu^2\bar{u})\Phi_1\, ds,\end{aligned} \tag{2.31}$$

where

$$\Phi_1 = -\frac{\lambda}{H_0 - 0i + e_0 - 2e_1}\mathbf{P}_c\phi_0\phi_1^2. \tag{2.32}$$

This term, with the phase factor $e_0 - 2e_1$, is the only one in G_3 having a negative phase factor (see (2.25)). Since $-(e_0 - 2e_1)$ is in the continuous spectrum of H_0, $H_0 + e_0 - 2e_1$ is not invertible, and needs a regularization $-0i$. We choose $-0i$, not $+0i$, so that the term $e^{-iH_0 t} y^2 \bar{x}(0)\Phi_1$ decays as $t \to \infty$ (see Lemma 2.2).

We can integrate all terms in G_3 and obtain the main terms of $\xi(t)$ as

$$\xi^{(2)}(t) = y^2\bar{x}\Phi_1 + |y|^2 x\Phi_2 + |x|^2 y\Phi_3 + x^2\bar{y}\Phi_4 + |x|^2 x\Phi_5, \tag{2.33}$$

where

$$\begin{aligned}\Phi_2 &= -\frac{2\lambda}{H_0 - e_0}\mathbf{P}_c\phi_0\phi_1^2, \qquad \Phi_3 = -\frac{2\lambda}{H_0 - e_1}\mathbf{P}_c\phi_0^2\phi_1,\\ \Phi_4 &= -\frac{\lambda}{H_0 - 2e_0 + e_1}\mathbf{P}_c\phi_0^2\phi_1, \qquad \Phi_5 = -\frac{\lambda}{H_0 - e_0}\mathbf{P}_c\phi_0^3.\end{aligned} \tag{2.34}$$

The rest of $\xi(t)$ is

$$\begin{aligned}\xi^{(3)}(t) &= e^{-iH_0t}\xi_0 - e^{-iH_0t}\xi^{(2)}(0) - \int_0^t e^{-iH_0(t-s)}\mathbf{P}_c G_4\,ds\\ &\quad + \int_0^t e^{-iH_0(t-s)}\mathbf{P}_c\big(G_\xi - i^{-1}G_3 - i^{-1}\lambda|\xi|^2\xi\big)\,ds\\ &\quad + \int_0^t e^{-iH_0(t-s)}\mathbf{P}_c\big(i^{-1}\lambda|\xi|^2\xi\big)\,ds\\ &\equiv \xi_1^{(3)}(t) + \xi_2^{(3)}(t) + \xi_3^{(3)}(t) + \xi_4^{(3)}(t) + \xi_5^{(3)}(t),\end{aligned} \tag{2.35}$$

The integrand G_4 in $\xi_3^{(3)}(t)$ consists of the remainders from the integration by parts:

$$\begin{aligned}G_4 &= e^{i(e_0-2e_1)s}\frac{d}{ds}\big(v^2\bar{u}\big)\Phi_1 + e^{i(-e_0)s}\frac{d}{ds}\big(|v|^2u\big)\Phi_2\\ &\quad + e^{i(-e_1)s}\frac{d}{ds}\big(|u|^2v\big)\Phi_3 + e^{i(-2e_0+e_1)s}\frac{d}{ds}\big(u^2\bar{v}\big)\Phi_4\\ &\quad + e^{i(-e_0)s}\frac{d}{ds}\big(u^2\bar{u}\big)\Phi_5.\end{aligned} \tag{2.36}$$

The integrands of $\xi_4^{(3)}(t)$ and $\xi_5^{(3)}(t)$ are higher order terms in G_ξ which we did not integrate. Here we single out $\xi_5^{(3)}(t)$ since $|\xi|^2\xi$ is a nonlocal term. Thus we have the following decomposition for ξ:

$$\xi(t) = \xi^{(2)}(t) + \xi^{(3)}(t) = \xi^{(2)} + \left(\xi_1^{(3)} + \cdots + \xi_5^{(3)}\right). \tag{2.37}$$

2.4 Linear estimates

We summarize some results on linear decay estimates.

Lemma 2.2 (decay estimates for e^{-itH_0}). Let the space dimension be 3. For $q \in [2,\infty]$ and $q' = q/(q-1)$,

$$\left\|e^{-itH_0}\mathbf{P}_c^{H_0}\phi\right\|_{L^q} \le C|t|^{-3(1/2-1/q)}\,\|\phi\|_{L^{q'}}. \tag{2.38}$$

For sufficiently large r_1, we have

$$\lim_{\sigma\to 0+} \left\| \langle x\rangle^{-r_1} e^{-itH_0} \frac{1}{(H_0+e_0-2e_1-\sigma i)^k} \mathbf{P}_c^{H_0} \langle x\rangle^{-r_1} \phi \right\|_{L^2} \le C\langle t\rangle^{-3/2} \|\phi\|_{L^2}, \tag{2.39}$$

where $k = 1, 2$. □

The decay estimate (2.38) is contained in [7, 18]; the estimate (2.39) is taken from [15, 16]. The estimate (2.39) holds only if we take $\sigma \to 0+$, not $\sigma \to 0-$.

3 The initial layer and the transition regimes: the estimates

We wish to show the following picture for the solution $\psi(t)$: in the initial layer regime, the dispersive part gradually disperses away, while the sizes of the bound states do not change much. In the transition regime, the *original* dispersive part becomes negligible, while the ϕ_0-components of $\psi(t)$ increases and the ϕ_1-component decreases.

Recall the orthogonal decomposition, $\psi(t) = \underline{x}\phi_0 + \underline{y}\phi_1 + \underline{\xi}$, in (1.14). We have $|\underline{x}(t)|^2 + |\underline{y}(t)|^2 + \|\underline{\xi}(t)\|_{L^2}^2 = \|\psi(t)\|_{L^2}^2 \le n^2$. If we decompose ψ via (1.15), that is, $\psi(t) = x\phi_0 + Q_1(y) + \xi$, we have $y = \underline{y}$, $x = \underline{x} + O(y^3)$, and $\xi = \underline{\xi} + O(y^3)$. Thus

$$|x(t)|, |y(t)|, \|\xi(t)\|_{L^2} \le \frac{5}{4} n, \qquad \|\xi_0\|_Y \le 4n. \tag{3.1}$$

We define L^2_{loc} and L^1_{loc}-norms by

$$\|f\|_{L^2_{loc}} = \left\|\langle x\rangle^{-r_0} f\right\|_{L^2}, \quad \|f\|_{L^1_{loc}} = \left\|\langle x\rangle^{-2r_0} f\right\|_{L^1}, \quad r_0 > 3. \tag{3.2}$$

Let n_0 and ε_0 be the small constants to be given in Theorem 4.1. By choosing a smaller n_0, we may assume that $n_0 \le (\varepsilon_0/2)^8$. Define

$$\varepsilon := \min\left\{ \frac{\varepsilon_0}{2}, \left(\log\left(\frac{2n}{|x_0|}\right)\right)^{-1/2} \right\}. \tag{3.3}$$

Since $n \le n_0 \le (\varepsilon_0/2)^8$ and $\log(2n/|x_0|) \le n^{-1/4}$, we have $\varepsilon \ge n^{1/8}$.

The following proposition is the main result for the dynamics in the initial layer and the transition regimes.

Proposition 3.1. Suppose that V satisfies the assumptions given in Section 1. Let $\psi(t, x)$ be a solution of (1.1) with the initial data ψ_0 satisfying (1.18). Let $\varepsilon_3 > 0$ be a sufficiently small constant to be fixed later. Let $t_0 = \varepsilon_3 n^{-4}$. Then there exist t_1 and t_2 such that, for some constant $C > 0$, we have

$$t_0 \le t_1 \le \frac{1.01}{\gamma_0 n^4} \log \frac{2n}{|x_0|}, \qquad C\big(n^4\varepsilon^2\big)^{-1} \le t_2 - t_1 \le 10100\big(\gamma_0 n^4 \varepsilon^2\big)^{-1}, \tag{3.4}$$

and the following estimates hold.

(i) For $0 \le t \le t_2$,

$$\big|x(t)\big| \ge \frac{3}{4} \sup_{0\le s\le t} \big|x(s)\big|, \tag{3.5}$$

$$\begin{aligned} &\big\|\xi(t)\big\|_{L^4} \le C_2 n^2 t^{1/4}\big|x(t)\big| + C_2\big\|\xi_0\big\|_Y \langle t\rangle^{-3/4}, \\ &\big\|\xi^{(3)}(t)\big\|_{L^2_{loc}} \le C_2 n^{7/2}\big|x(t)\big| + C_2\big\|\xi_0\big\|_Y \langle t\rangle^{-3/2}, \end{aligned} \tag{3.6}$$

where the constant C_2 will be specified in (3.27) of Section 3.1.

(ii) (Initial layer) For $0 \le t \le t_0$,

$$\begin{aligned} &\frac{1}{2}|x_0| \le \big|x(t)\big| \le \frac{3}{2}|x_0|, \\ &0.99\big|y_0\big| \le \big|y(t)\big| \le 1.01\big|y_0\big|. \end{aligned} \tag{3.7}$$

(iii) Recall $n_1 = (|x_0|^2 + (1/2)|y_0|^2)^{1/2}$ defined in (1.19). We have

$$\big|x(t_1)\big| \ge 0.01 n, \qquad \big|x(t_2)\big| \ge 0.99 n_1, \qquad \frac{1}{2}\varepsilon n \le \big|y(t_2)\big| \le 2\varepsilon n. \tag{3.8}$$

□

By (1.18), $\log(2n/|x_0|) \le n^{-1/4}$. Hence (3.4) implies that

$$t_2 \le C n^{-4} \log \frac{2n}{|x_0|} + C\varepsilon^{-2} n^{-4} \le C_3 n^{-4-1/4}, \tag{3.9}$$

for some constant C_3.

We will prove these estimates using (1.18), (3.1), and a continuity argument. Hence we can assume the following weaker estimates: for $0 \le t \le t_2$:

$$\begin{aligned} &\big|x(t)\big| \ge \frac{1}{2} \sup_{0\le s\le t} \big|x(s)\big|, \\ &\big|x(t)\big| \le 2|x_0| \quad \text{for } t < t_0, \\ &\big\|\xi(t)\big\|_{L^4} \le 2C_2 n^2 t^{1/4}\big|x(t)\big| + 2C_2\big\|\xi_0\big\|_Y \langle t\rangle^{-3/4}, \\ &\big\|\xi^{(3)}(t)\big\|_{L^2_{loc}} \le 2C_2 n^{7/2}\big|x(t)\big| + 2C_2\big\|\xi_0\big\|_Y \langle t\rangle^{-3/2}. \end{aligned} \tag{3.10}$$

By continuity, if we prove Proposition 3.1 assuming these weaker estimates, we have proved the proposition itself. We shall see also that estimates (3.10) will be used only in estimating higher order terms.

Recall from (2.26) that the local term Λ_π satisfies $\|\Lambda_\pi\|_r \le Cn^2\|G\|_{L^1_{loc}}$ for any r. Thus we have

$$\left|\dot{u}(t)\right| \lesssim \|G\|_{L^1_{loc}}, \qquad \left|\dot{v}(t)\right| \lesssim \|G\|_{L^1_{loc}} + |y|^3. \tag{3.11}$$

The following lemma provides estimates for G assuming the estimate (3.10).

Lemma 3.2. Let $G = G_3 + G_5 + G_7$ be as given by (2.19), (2.20), (2.21), and (2.22). Suppose that n is sufficiently small and the estimate (3.10) holds for $t \le C_3 n^{-4-1/4}$. Then the following estimates for G hold:

$$\left\|G(t)\right\|_{L^{4/3}\cap L^1} \le C_4 n^2\left|x(t)\right| + C(C_2)n^2\left\|\xi_0\right\|_Y \langle t\rangle^{-3/2}; \tag{3.12}$$

$$\left\|(G - G_3)(t)\right\|_{L^{4/3}\cap L^1} \le C_4 n^{7/2}\left|x(t)\right| + C(C_2)n^2\left\|\xi_0\right\|_Y \langle t\rangle^{-3/2}; \tag{3.13}$$

$$\left\|G_7(t)\right\|_{L^1_{loc}} \le C_4 n^{11/2}\left|x(t)\right| + C(C_2)n^2\left\|\xi_0\right\|_Y \langle t\rangle^{-3/2}; \tag{3.14}$$

where C_4 is a constant independent of C_2 and $C(C_2)$ denotes constants depending on C_2. Moreover, (3.12) and (3.13) remain true if we replace G by G_ξ, and $(G-G_3)$ by $(G_\xi - i^{-1}G_3)$. Furthermore,

$$\left\|G_\xi(t)\right\|_{L^1} \le C_5 n^3. \tag{3.15}$$

□

By the assumption (1.18), when $t > t_0$ the last term $Cn^2\|\xi_0\|_Y\langle t\rangle^{-3/2}$ is smaller and can be removed. The proof of this lemma is a straightforward application of the Hölder and Schwarz inequalities.

Proof. We first consider the nonlocal term $\lambda|\xi|^2\xi$ in G. Since $t_2 \le C_3 n^{-4-1/4}$, by (3.10) we have $\|\xi(s)\|_{L^4} \le Cn^{15/16}|x(s)| + C\|\xi_0\|_Y\langle s\rangle^{-3/4}$. Also, using (3.1) and the Hölder inequality, we have

$$\left\||\xi|^2\xi(s)\right\|_{L^{4/3}} \le C\left\|\xi(s)\right\|_{L^4}^3 \le C\left(n^{15/16}\left|x(s)\right|\right)^3 + C\left\|\xi_0\right\|_Y^3\langle s\rangle^{-9/4}; \tag{3.16}$$

$$\begin{aligned}\left\||\xi|^2\xi(s)\right\|_{L^1} &\le C\left\|\xi(s)\right\|_{L^2}\left\|\xi(s)\right\|_{L^4}^2 \\ &\le Cn\left\{\left(n^{15/16}\left|x(s)\right|\right)^2 + \left\|\xi_0\right\|_Y^2\langle s\rangle^{-3/2}\right\} \\ &\le Cn^{4-1/8}\left|x(s)\right| + Cn^2\left\|\xi_0\right\|_Y\langle s\rangle^{-3/2}.\end{aligned} \tag{3.17}$$

Hence this nonlocal term satisfies (3.12), (3.13). Moreover, to prove (3.14), we can bound $\||\xi|^2\xi(s)\|_{L^1_{loc}}$ as follows:

$$\begin{aligned}\||\xi|^2\xi(s)\|_{L^1_{loc}} &\le C\|\xi(s)\|_{L^2_{loc}}\|\xi(s)\|_{L^4}^2 \\ &\le C\big(n^2|x| + \|\xi_0\|_Y\langle s\rangle^{-3/2}\big)\Big\{\big(n^{15/16}|x|\big)^2 + \|\xi_0\|_Y^2\langle s\rangle^{-3/2}\Big\} \\ &\le Cn^{6-1/8}|x(s)| + Cn^2\|\xi_0\|_Y\langle s\rangle^{-3/2}.\end{aligned} \tag{3.18}$$

For the local terms $G-\lambda|\xi|^2\xi = G_3+G_5+(G_7-\lambda|\xi|^2\xi)$, all L^p-norms are equivalent. We can read from the explicit expressions of G the following estimates:

$$\begin{aligned}&G_3 \lesssim n^2x,\\ &G_5 \lesssim n^4x + n^2\xi,\\ &G_7 - \lambda|\xi|^2\xi \lesssim n^6x + n^4\xi + n\xi^2.\end{aligned} \tag{3.19}$$

To estimate ξ in the local terms we can use $\|\xi\|_{L^2_{loc}}$. For example,

$$\begin{aligned}&\|\bar{y}\phi_1\xi^2\|_{L^{4/3}} \le C|y|\|\phi_1\langle x\rangle^{r_0}\|_{L^4}\|\xi\|_{L^4}\|\xi\|_{L^2_{loc}} \le Cn^2\big(n^2|x| + \|\xi_0\|_Y\langle s\rangle^{-3/2}\big),\\ &\|\bar{y}\phi_1\xi^2\|_{L^1} \le C|y|\|\phi_1\langle x\rangle^{2r_0}\|_{L^\infty}\|\xi\|_{L^2_{loc}}^2 \le Cn\big(n^2|x| + \|\xi_0\|_Y\langle s\rangle^{-3/2}\big)^2.\end{aligned} \tag{3.20}$$

Together with the explicit expressions of G and G_3, similar arguments show (3.12), (3.13), and (3.14).

Since $G_\xi = i^{-1}(G+\Lambda_\pi)$ and $\|\Lambda_\pi\| \le Cn^2\|G\|_{L^1_{loc}}$ by (2.26), (3.12) and (3.14) hold, if we replace G and $G-G_3$ by G_ξ and $G_\xi - i^{-1}G_3$. Equation (3.15) follows from the above and (3.18). Thus the lemma is proved. ■

3.1 Estimates of the dispersive part

We now prove the estimates for ξ in Proposition 3.1 by using (3.1), (3.10), and Lemmas 2.2 and 3.2.

Step 1 (L^4-norm). Recall equation (2.16) for ξ: $\xi(t) = e^{-iH_0t}\xi_0 + \int_0^t e^{-iH_0(t-s)}\mathbf{P}_c^{H_0}G_\xi(s)\,ds$, $G_\xi = i^{-1}(G+\Lambda_\pi)$. By (3.10) and Lemmas 2.2 and 3.2, we have

$$\begin{aligned}\|\xi(t)\|_{L^4} &\le \|e^{-iH_0t}\xi_0\|_{L^4} + \int_0^t C|t-s|^{-3/4}\|G_\xi(s)\|_{4/3}\,ds\\ &\le C\|\xi_0\|_Y\langle t\rangle^{-3/4} + \int_0^t C|t-s|^{-3/4}\big(C_4n^2|x|(s) + C(C_2)n^2\|\xi_0\|_Y\langle s\rangle^{-3/2}\big)\,ds\\ &\le C_{2,1}\|\xi_0\|_Y\langle t\rangle^{-3/4} + C_{2,1}n^2t^{1/4}|x(t)| + C(C_2)n^2\|\xi_0\|_Y\langle t\rangle^{-4/3},\end{aligned} \tag{3.21}$$

where $C_{2,1}$ is some explicit constant.

Step 2 (L^2_{loc}-norm). Recall the decomposition (2.37): $\xi = \xi^{(2)} + \xi^{(3)} = \xi^{(2)} + (\xi_1^{(3)} + \cdots + \xi_5^{(3)})$. We will estimate the L^2_{loc}-norm of each term.

(1) $\xi^{(2)}$. Since $\Phi_1 \in L^2_{loc}$, and $\Phi_j \in L^2$, $j > 1$, we have

$$\left\|\xi^{(2)}(t)\right\|_{L^2_{loc}} \le C_{2,3} C n^2 |x(t)|, \tag{3.22}$$

for some explicit constant $C_{2,3}$.

(2) $\xi_1^{(3)}$. We have

$$\left\|\xi_1^{(3)}(t)\right\|_{L^2_{loc}} \le C_{2,4} \|\xi_0\|_Y \langle t\rangle^{-3/2}, \tag{3.23}$$

for some explicit constant $C_{2,4}$ by the $L^{p',p}$ estimate of e^{-itH_0} in Lemma 2.2.

(3) $\xi_2^{(3)}$. By the linear estimate (2.39) in Lemma 2.2 we have, for some constant $C_{2,5}$,

$$\left\|\xi_2^{(3)}(t)\right\|_{L^2_{loc}} \le C_{2,5} n^2 |x_0| \langle t\rangle^{-3/2}. \tag{3.24}$$

(4) $\xi_3^{(3)}$. To estimate $\xi_3^{(3)}(t) = -\int_0^t e^{-iH_0(t-s)} \mathbf{P}_c G_4 ds$ with G_4 defined in (2.36), we need estimates (3.11) for $\dot{u}$ and $\dot{v}$ and the linear estimate (2.39) in Lemma 3.2. Hence

$$\begin{aligned}
\left\|\xi_3^{(3)}(t)\right\|_{L^2_{loc}} &\le \int_0^t \left\|e^{-iH_0(t-s)} \mathbf{P}_c G_4\right\|_{L^2_{loc}} ds \\
&\le C \int_0^t \langle t-s\rangle^{-3/2} \left(n^2|\dot{u}| + n|x\dot{v}|\right) ds \quad \text{using (2.39)} \\
&\le C \int_0^t \langle t-s\rangle^{-3/2} \left(n^2\|G\|_{L^{4/3}} + n^4|x|\right) ds \quad \text{using (3.11)} \\
&\le C \int_0^t \langle t-s\rangle^{-3/2} \left(n^4|x| + C(C_2) n^4 \|\xi_0\|_Y \langle s\rangle^{-3/2}\right) ds \quad \text{using (3.12)} \\
&\le C_{2,6} n^4 |x(t)| + C(C_2) n^4 \|\xi_0\|_Y \langle t\rangle^{-3/2}.
\end{aligned} \tag{3.25}$$

(5) $\xi_4^{(3)} + \xi_5^{(3)}$. We write $\xi_4^{(3)} + \xi_5^{(3)} = \int_0^t e^{-iH_0(t-s)} \mathbf{P}_c G_{\xi,5}(s) ds$, where $G_{\xi,5}(s) := (G_\xi - i^{-1} G_3)(s)$. By Lemmas 3.2 and 2.2, we have for $t > 1$

$$\begin{aligned}
&\left\| (\xi_4^{(3)} + \xi_5^{(3)})(t) \right\|_{L^2_{\mathrm{loc}}} \\
&\quad \le \int_0^{t-1} \left\| e^{-iH_0(t-s)} \mathbf{P}_c G_{\xi,5}(s) \right\|_{L^8} ds + \int_{t-1}^{t} \left\| e^{-iH_0(t-s)} \mathbf{P}_c G_{\xi,5}(s) \right\|_{L^4} ds \\
&\quad \le C \int_0^{t-1} C|t-s|^{-3/2} \left\| G_{\xi,5}(s) \right\|_{8/7} ds + \int_{t-1}^{t} C|t-s|^{-3/4} \left\| G_{\xi,5}(s) \right\|_{4/3} ds \\
&\quad \le C \left(\int_0^{t-1} |t-s|^{-3/2} + \int_{t-1}^{t} |t-s|^{-3/4} \right) \\
&\qquad \times \left(C_4 n^{7/2} |x(s)| + C(C_2) n^2 \|\xi_0\| \langle s \rangle^{-3/2} \right) ds \\
&\quad \le C_{2,7} n^{7/2} |x(t)| + C(C_2) n^2 \|\xi_0\|_Y \langle t \rangle^{-3/2},
\end{aligned} \tag{3.26}$$

for some explicit constant $C_{2,7}$. If $t < 1$, we can bound the L^2_{loc}-norm by the L^4-norm. Hence, the last estimate for $t < 1$ follows from the estimate in Step 1.

We have obtained estimates on ξ involving explicit constants $C_{2,1}, \ldots, C_{2,7}$ and $C(C_2)$. We now define the constant C_2 in (3.6) to be

$$C_2 \equiv C_{2,1} + \cdots + C_{2,7}. \tag{3.27}$$

Since all terms involving $C(C_2)$ have some extra n factor, ξ satisfies the estimates in (3.6) provided that n is sufficiently small.

Summarizing, we have proved the following lemma.

Lemma 3.3. If n is sufficiently small, there is an explicit constant C_2 such that, if (3.10) holds in $[0, t]$ for some $t \le C_3 n^{-4-1/4}$, then the estimates (3.6) of $\xi(t)$ in Proposition 3.1 also hold in $[0, t]$. $\square$

Note that, in the proof, we only used (3.1), (3.10), and Lemmas 2.2 and 3.2. The information we need on the size of bound states is in (3.1) and the first estimate of (3.10). Since (3.1) is always true, we only need to ensure that the first estimate in (3.10) holds.

3.2 Normal form for equations of bound states

We now compute the Poincaré normal form for the bound states. This normal form will be used to estimate the bound states components x and y in Section 3.3.

Recall that

$$x(t) = e^{-ie_0 t} u(t), \qquad y(t) = e^{-ie_1 t} v(t). \tag{3.28}$$

Lemma 3.4. Suppose that (3.10) holds. There are perturbations μ of u and ν of v, to be defined in (3.74), satisfying

$$\big|u(t)-\mu(t)\big|+\big|v(t)-\nu(t)\big| \le C_6 n^2 \big|x(t)\big|, \tag{3.29}$$

such that

$$\begin{aligned} \dot{\mu} &= \big(c_1|\mu|^2+c_2|\nu|^2\big)\mu+\big(c_3|\mu|^4+c_4|\mu|^2|\nu|^2+c_5|\nu|^4\big)\mu+g_u,\\ \dot{\nu} &= \big(c_6|\mu|^2+c_7|\nu|^2\big)\nu+\big(c_8|\mu|^4+c_9|\mu|^2|\nu|^2+c_{10}|\nu|^4\big)\nu+g_v. \end{aligned} \tag{3.30}$$

Here g_u and g_v are error terms to be defined in (3.76). All coefficients $c_1,\dots,c_{10}$ are of order one and, except c_5 and c_9, purely imaginary. We have

$$\operatorname{Re} c_5 = \gamma_0, \qquad \operatorname{Re} c_9 = -2\gamma_0, \tag{3.31}$$

where $\gamma_0 > 0$ is defined in (1.9). Moreover, we can write g_v as

$$g_v = -iE^{(6)}\left(|y|\right)\nu + \widetilde{g}_v, \tag{3.32}$$

where $E^{(6)}\left(|y|\right) = O(|y|^6)$ is defined in (2.4), and

$$\big|g_u(t)\big|+\big|\widetilde{g}_v(t)\big| \le C_6 n^{11/2}\big|x(t)\big| + C_6 n^2 \big\|\xi_0\big\|_Y \langle t\rangle^{-3/2}, \tag{3.33}$$

for some explicit constant C_6. □

Proof. Recall (2.14) and (2.15) for u and v, $\dot{u} = -ie^{ie_0t}(\phi_0, G+\Lambda_\pi)$, $\dot{v} = -ie^{ie_1t}[(E_1(m)-e_1)y+(\phi_1,G)]$. Using the decompositions (2.29) for Λ_π and (2.4) for $E_1(m)$, we can decompose the equations for u and v according to orders in n:

$$\begin{aligned} \dot{u} &= -ie^{ie_0t}\big(\phi_0, G_3\big) - ie^{ie_0t}\big(\phi_0, G_5+\Lambda_{\pi,5}\big) - ie^{ie_0t}\big(\phi_0, G_7+\Lambda_{\pi,7}\big)\\ &\equiv R_{u,3}+R_{u,5}+R_{u,7}, \end{aligned} \tag{3.34}$$

$$\begin{aligned} \dot{v} &= -ie^{ie_1t}\big[\big(\phi_1, G_3\big)+E_{1,2}|y|^2y\big] - ie^{ie_1t}\big[\big(\phi_1, G_5\big)+E_{1,4}|y|^4y\big]\\ &\quad - ie^{ie_1t}\Big[\big(\phi_1, G_7\big)+E_1^{(6)}\left(|y|\right)y\Big]\\ &\equiv R_{v,3}+R_{v,5}+R_{v,7}. \end{aligned} \tag{3.35}$$

Using (2.30) and Lemma 3.2, which assumes (3.10), we have

$$
\begin{aligned}
|R_{u,5}| &\lesssim \|G_5\|_{L^1_{loc}} + \|\Lambda_{\pi,5}\|_{L^1} \lesssim n^4|x| + n^2\|\xi_0\|_Y \langle t\rangle^{-3/2}, && (3.36)\\
|R_{u,7}| &\lesssim \|G_7\|_{L^1_{loc}} + \|\Lambda_{\pi,7}\|_{L^1} \lesssim n^{11/2}|x| + n^2\|\xi_0\|_Y \langle t\rangle^{-3/2}, && (3.37)\\
|R_{v,5}| &\lesssim \|G_5\|_{L^1_{loc}} + |y|^5 \lesssim n^5 + n^2\|\xi_0\|_Y \langle t\rangle^{-3/2}, && (3.38)\\
|R_{v,7}| &\lesssim \|G_7\|_{L^1_{loc}} + |y|^7 \lesssim n^{13/2} + n^2\|\xi_0\|_Y \langle t\rangle^{-3/2}. && (3.39)
\end{aligned}
$$

We first integrate $R_{u,3}$ and $R_{v,3}$ in Step 1, and then $R_{u,5}$ and $R_{v,5}$ in Step 2.

Step 1 (integration of terms of order n^3). In equation (3.34) of u, terms of order n^3 are contained in $R_{u,3} = -ie^{ie_0t}(\phi_0, G_3)$. The resonant terms from G_3 are $|y|^2x$ and $|x|^2x$, whose phases cancel out the factor e^{ie_0t}. The other three terms of order n^3 in G_3 have different frequencies and can be removed using integration by parts. By (2.20) we have

$$
\begin{aligned}
\dot u &= -ie^{ie_0t}(\phi_0, G_3) + R_{u,5} + R_{u,7}\\
&= c_1|u|^2u + c_2|v|^2u + \frac{d}{dt}(u_1^-) + g_{u,1} + R_{u,5} + R_{u,7},
\end{aligned}
\tag{3.40}
$$

where

$$
\begin{aligned}
c_1 &= -i\lambda(\phi_0^2, \phi_0^2), \qquad c_2 = -i2\lambda(\phi_0^2, \phi_1^2),\\
u_1^- &= -\Bigg(\lambda\phi_0, \frac{e^{i(2e_0-2e_1)t}v^2\bar u}{2e_0-2e_1}\phi_0\phi_1^2 + \frac{e^{i(e_0-e_1)t}\,2|u|^2v}{e_0-e_1}\phi_0^2\phi_1\\
&\qquad + \frac{e^{i(-e_0+e_1)t}u^2\bar v}{-e_0+e_1}\phi_0^2\phi_1\Bigg),\\
g_{u,1} &= \left(\lambda\phi_0, \frac{e^{i(2e_0-2e_1)t}\frac{d}{dt}(v^2\bar u)}{2e_0-2e_1}\phi_0\phi_1^2 + \frac{e^{i(e_0-e_1)t}\frac{d}{dt}(2|u|^2v)}{e_0-e_1}\phi_0^2\phi_1\right.\\
&\qquad \left. + \frac{e^{i(-e_0+e_1)t}\frac{d}{dt}(u^2\bar v)}{-e_0+e_1}\phi_0^2\phi_1\right).
\end{aligned}
\tag{3.41}
$$

In the equation of v, (3.35), the terms of order n^3 are in $R_{v,3} = -ie^{ie_1t}[(\phi_1, G_3) + E_{1,2}|y|^2y]$. There is only one resonant term in G_3, namely, $|x|^2y$. Another resonant term of order n^3 is from the term $E_{1,2}|y|^2y$. The other four terms of order n^3 in G_3 have different frequencies and can be integrated. We thus have

$$
\begin{aligned}
\dot v &= -ie^{ie_1t}\big[(\phi_1, G_3) + E_{1,2}|y|^2y\big] + R_{v,5} + R_{v,7}\\
&= c_6|u|^2v + c_7|v|^2v + \frac{d}{dt}(v_1^-) + g_{v,1} + R_{v,5} + R_{v,7},
\end{aligned}
\tag{3.42}
$$

where

$$
\begin{aligned}
& c_6 = -i2\lambda\left(\phi_0^2, \phi_1^2\right), \qquad c_7 = -iE_{1,2}, \\
& v_1^- = -\left(\lambda\phi_1, \frac{e^{i(-e_1+e_0)t}v^2\bar{u}}{-e_1+e_0}\phi_0\phi_1^2 + \frac{e^{i(e_1-e_0)t}2|v|^2u}{e_1-e_0}\phi_0\phi_1^2\right. \\
& \qquad \left. + \frac{e^{i(2e_1-2e_0)t}u^2\bar{v}}{2e_1-2e_0}\phi_0^2\phi_1 + \frac{e^{i(e_1-e_0)t}|u|^2u}{e_1-e_0}\phi_0^3\right), \\
& g_{v,1} = \left(\lambda\phi_1, \frac{e^{i(-e_1+e_0)t}\frac{d}{dt}\left(v^2\bar{u}\right)}{-e_1+e_0}\phi_0\phi_1^2 + \frac{e^{i(e_1-e_0)t}\frac{d}{dt}\left(2|v|^2u\right)}{e_1-e_0}\phi_0\phi_1^2\right. \\
& \qquad \left. + \frac{e^{i(2e_1-2e_0)t}\frac{d}{dt}\left(u^2\bar{v}\right)}{2e_1-2e_0}\phi_0^2\phi_1 + \frac{e^{i(e_1-e_0)t}\frac{d}{dt}\left(|u|^2u\right)}{e_1-e_0}\phi_0^3\right).
\end{aligned}
\tag{3.43}
$$

We now define

$$
u_1 = u - u_1^-, \qquad v_1 = v - v_1^-. \tag{3.44}
$$

The equations for u_1 and v_1 are

$$
\begin{aligned}
\dot{u}_1 &= c_1|u|^2u + c_2|v|^2u + g_{u,1} + R_{u,5} + R_{u,7} \\
&= c_1|u_1|^2u_1 + c_2|v_1|^2u_1 + g_{u,2} + g_{u,1} + R_{u,5} + R_{u,7}, \\
g_{u,2} &= c_1\left(|u|^2u - |u_1|^2u_1\right) + c_2\left(|v|^2u - |v_1|^2u_1\right), \\
\dot{v}_1 &= c_6|u|^2v + c_7|v|^2v + g_{v,1} + R_{v,5} + R_{v,7} \\
&= c_6|u_1|^2v_1 + c_7|v_1|^2v_1 + g_{v,2} + g_{v,1} + R_{v,5} + R_{v,7}, \\
g_{v,2} &= c_6\left(|u|^2v - |u_1|^2v_1\right) + c_7\left(|v|^2v - |v_1|^2v_1\right).
\end{aligned}
\tag{3.45}
$$

We have finished the integration of order n^3 terms. Note that both u_1^- and v_1^- enter the equations of u_1 and v_1. This is the reason we compute their normal form together.

Observe that

$$
\left|u_1^-\right| + \left|v_1^-\right| \lesssim n^2|u|. \tag{3.46}
$$

We now decompose $g_{u,1}$, $g_{v,1}$, $g_{u,2}$, and $g_{v,2}$ according to their orders in n. We want to write them as sum of order n^5 and order n^7 terms. We first claim that $g_{u,1}$ and $g_{v,1}$ are of the form

$$
g_{u,1} = e^{ie_0t}g_{u,1,5} + g_{u,1,7}, \qquad g_{v,1} = e^{ie_1t}g_{v,1,5} + g_{v,1,7}, \tag{3.47}
$$

where $g_{u,1,7}$ and $g_{v,1,7}$ are order n^7 terms, and $g_{u,1,5}$ and $g_{v,1,5}$ are explicit homogeneous polynomials of degree 5 in $x, \bar{x}, y, \bar{y}$ with purely imaginary coefficients. Moreover, every term in $g_{u,1,5}$ has a factor x or $\bar{x}$. For example, the first term in $g_{u,1}$ is

$$\begin{aligned} &Ce^{i(2e_0-2e_1)t}\frac{d}{dt}(v^2\bar{u}) \\ &\quad = Ce^{i(2e_0-2e_1)t}\big(2\bar{u}v\dot{v}+v^2\bar{\dot{u}}\big) = Ce^{ie_0t}\big(2\bar{x}ye^{-ie_1t}\dot{v}+y^2e^{ie_0t}\bar{\dot{u}}\big) \\ &\quad = Ce^{ie_0t}\big(2\bar{x}ye^{-ie_1t}\big[R_{v,3}+R_{v,5}+R_{v,7}\big]+y^2e^{ie_0t}\big[\,\overline{R_{u,3}}+\overline{R_{u,5}}+\overline{R_{u,7}}\,\big]\big), \end{aligned} \tag{3.48}$$

where $C = (\lambda\phi_0, (2e_0-2e_1)^{-1}\phi_0\phi_1^2)$ is real. The leading terms of order n^5 are

$$\begin{aligned} &Ce^{ie_0t}\big(2\bar{x}ye^{-ie_1t}R_{v,3}+y^2e^{ie_0t}\overline{R_{u,3}}\,\big) \\ &\quad = Ce^{ie_0t}\big[-2\bar{x}yi\big[(\phi_1,G_3)+E_{1,2}|y|^2y\big]+|y|^2i(\phi_0,G_3)\big]. \end{aligned} \tag{3.49}$$

They are e^{ie_0t} times a polynomial of degree 5 in x, $\bar{x}$, y, $\bar{y}$ with purely imaginary coefficients. The rest belongs to $g_{u,1,7}$. Repeating the same calculation for all terms in $g_{u,1}$ and collecting terms of order n^5, we obtain $g_{u,1,5}$. The rest is $g_{u,1,7}$.

From the estimates of (3.36), (3.37), (3.38), and (3.39), we can bound $g_{u,1,7}$ by

$$\big|g_{u,1,7}(t)\big|+\big|g_{v,1,7}(t)\big| \lesssim n^6|u|+n^4\big\|\xi_0\big\|_Y\langle t\rangle^{-3/2}. \tag{3.50}$$

Similarly, we can write $g_{u,2}$ and $g_{v,2}$ as

$$g_{u,2} = e^{ie_0t}g_{u,2,5}+g_{u,2,7}, \qquad g_{v,2} = e^{ie_1t}g_{v,2,5}+g_{v,2,7}, \tag{3.51}$$

where $g_{u,1,5}$ and $g_{v,1,5}$ are explicit homogeneous polynomials of degree 5 in $x, \bar{x}, y, \bar{y}$ with purely imaginary coefficients, and $g_{u,2,7}$ and $g_{v,2,7}$ are order n^7 terms satisfying

$$\big|g_{u,2,7}(t)\big|+\big|g_{v,2,7}(t)\big| \lesssim n^6|u|. \tag{3.52}$$

Here we have used (3.46). Moreover, every term in $g_{u,2,5}$ and $g_{v,2,5}$ has a factor x or $\bar{x}$. We consider the first term in $g_{u,2}$ as an example. Using $u-u_1 = u_1^-$,

$$\begin{aligned} |u|^2u-|u_1|^2u_1 &= u^2\bar{u}-\big(u-u_1^-\big)^2\Big(\bar{u}-\overline{u_1^-}\Big) \\ &= u^2\overline{\big(u_1^-\big)}+2|u|^2u_1^-+O\Big(u\big|u_1^-\big|^2\Big) \\ &= e^{ie_0t}\big[x^2\overline{\big(e^{-ie_0t}u_1^-\big)}+2|x|^2e^{-ie_0t}u_1^-\big]+O\Big(u\big|u_1^-\big|^2\Big). \end{aligned} \tag{3.53}$$

The terms in the bracket belong to $g_{u,2,5}$. Since $e^{-ie_0t}u_1^-$ is a polynomial of degree 3 in x, $\bar{x}$, y, and $\bar{y}$ with real coefficients and c_1 in $g_{u,2}$ is purely imaginary, the above expression is of the desired form.

Summarizing, we can write

$$\begin{aligned} g_{u,1} + g_{u,2} &= e^{ie_0 t}\widetilde{R}_{u,5} + g_{u,3}, \\ g_{\nu,1} + g_{\nu,2} &= e^{ie_1 t}\widetilde{R}_{\nu,5} + g_{\nu,3}, \end{aligned} \tag{3.54}$$

where $\widetilde{R}_{u,5} = g_{u,1,5} + g_{u,2,5}$ and $\widetilde{R}_{\nu,5} = g_{u,1,5} + g_{u,2,5}$ are explicit homogeneous polynomials of degree 5 in x, $\bar{x}$, y, and $\bar{y}$ with *purely imaginary* coefficients. Moreover, every term in $\widetilde{R}_{u,5}$ and $\widetilde{R}_{\nu,5}$ has a factor x or $\bar{x}$. Also, $g_{u,3} = g_{u,1,7} + g_{u,2,7}$ and $g_{\nu,3} = g_{\nu,1,5} + g_{\nu,2,5}$. From assumption (3.10), we have

$$\left|\widetilde{R}_{u,5}\right| + \left|\widetilde{R}_{\nu,5}\right| \lesssim n^4|x|, \tag{3.55}$$

$$\left|g_{u,3}\right| + \left|g_{\nu,3}\right| \lesssim n^6|x| + n^4\left\|\xi_0\right\|_Y \langle t\rangle^{-3/2}. \tag{3.56}$$

The final equations for u_1 and ν_1 are

$$\dot{u}_1 = c_1|u_1|^2u_1 + c_2|\nu_1|^2u_1 + \left(R_{u,5} + e^{ie_0 t}\widetilde{R}_{u,5}\right) + \left(R_{u,7} + g_{u,3}\right), \tag{3.57}$$

$$\dot{\nu}_1 = c_6|u_1|^2\nu_1 + c_7|\nu_1|^2\nu_1 + \left(R_{\nu,5} + e^{ie_1 t}\widetilde{R}_{\nu,5}\right) + \left(R_{\nu,7} + g_{\nu,3}\right). \tag{3.58}$$

Step 2 (integration of terms of order n^5). We now integrate terms of order n^5. In u_1-equation (3.57) we have $R_{u,5} + e^{ie_0 t}\widetilde{R}_{u,5}$, where $R_{u,5}$ is from the decomposition of original equation (3.34) and $\widetilde{R}_{u,5}$ is from the error terms $g_{u,1} + g_{u,2}$. Similarly, terms of order n^5 in ν_1-equation (3.58) is $R_{\nu,5} + e^{ie_1 t}\widetilde{R}_{\nu,5}$. Observe that they are either of the form $x^\alpha y^\beta$ with $|\alpha| + |\beta| = 5$, or of the form $xy\xi$. Also note that there are two sources in $R_{u,5}$: G_5 and $\Lambda_{\pi,5}$. Among all these terms the main term is G_5.

We have already studied $\widetilde{R}_{u,5}$ and $\widetilde{R}_{\nu,5}$. We next consider Λ_π. Recall (2.29) and (2.28) that are $\Lambda_\pi = \Lambda_{\pi,5} + \Lambda_{\pi,7}$ and $\Lambda_{\pi,5} = -2q_3(\phi_1, G_3|y|^2 + \bar{G}_3 y^2)$. Thus $\Lambda_{\pi,5}$ is a homogeneous polynomial in x, $\bar{x}$, y, and y of degree 5 with purely real functions as coefficients. Therefore the term $-ie^{ie_0 t}(\phi_0, \Lambda_{\pi,5})$ in $\dot{u}$-equation (3.34) gives only polynomials with purely imaginary coefficients and a phase $e^{ie_0 t}$.

We now consider G_5, which is given by

$$\begin{aligned} G_5 = {} & \lambda\left(2y^3\bar{y}\bar{x} + 4|y|^4x\right)\phi_0\phi_1q_3 + \lambda\left(2|x|^2y^2\bar{y} + x^2y\bar{y}^2\right)\phi_0^2q_3 \\ & + \lambda\left(x\phi_0 + y\phi_1\right)^2\bar{\xi} + 2\lambda\left|\left(x\phi_0 + y\phi_1\right)\right|^2\xi. \end{aligned} \tag{3.59}$$

Recall the decomposition $\xi = \xi^{(2)} + \xi^{(3)}$, where

$$\xi^{(2)}(t) = y^2\bar{x}\Phi_1 + |y|^2x\Phi_2 + |x|^2y\Phi_3 + x^2\bar{y}\Phi_4 + |x|^2x\Phi_5, \tag{3.60}$$

with Φ_1 the only function with nonzero imaginary part, see (2.32), (2.33), (2.34), and (2.35). Write

$$\Phi_1 = \Phi_{1,R} + i\Phi_{1,I}, \tag{3.61}$$

with both $\Phi_{1,R}$ and $\Phi_{1,I}$ real. Denote the part of $\xi^{(2)}(t)$ with real coefficients by

$$\xi_R^{(2)}(t) = y^2\bar{x}\Phi_{1,R} + |y|^2 x\Phi_2 + |x|^2 y\Phi_3 + x^2\bar{y}\Phi_4 + |x|^2 x\Phi_5. \tag{3.62}$$

We can write $\xi = y^2\bar{x}i\Phi_{1,I} + \xi_R^{(2)} + \xi^{(3)}$. Thus we can further decompose G_5 as

$$G_5 = G_{5,1} + G_{5,2} + G_{5,3}, \tag{3.63}$$

where

$$\begin{aligned} G_{5,1} &= \big(x\phi_0 + y\phi_1\big)^2\bar{y}^2 x(-i)\Phi_{1,I} + 2\big|\big(x\phi_0 + y\phi_1\big)\big|^2 y^2\bar{x}i\Phi_{1,I}, \\ G_{5,2} &= \lambda\big(2y^3\bar{y}\bar{x} + 4|y|^4 x\big)\phi_0\phi_1 q_3 + \lambda\big(2|x|^2y^2\bar{y} + x^2y\bar{y}^2\big)\phi_0^2 q_3 \\ &\qquad + \lambda\big(x\phi_0 + y\phi_1\big)^2\overline{\xi_R^{(2)}} + 2\lambda\big|\big(x\phi_0 + y\phi_1\big)\big|^2\xi_R^{(2)}, \\ G_{5,3} &= \lambda\big(x\phi_0 + y\phi_1\big)^2\overline{\xi^{(3)}} + 2\lambda\big|\big(x\phi_0 + y\phi_1\big)\big|^2\xi^{(3)}. \end{aligned} \tag{3.64}$$

The term $G_{5,3}$ will be shown to be smaller than $G_{5,1}$ and $G_{5,2}$. Although $G_{5,1}$ and $G_{5,2}$ are of the same size, $G_{5,2}$ consists of monomials in x, $\bar{x}$, y, and $\bar{y}$ with *real* functions as coefficients, while $G_{5,1}$ with purely imaginary coefficients. The reason that $G_{5,1}$ has purely imaginary coefficients is due to the resonance of some linear combination of eigenvalues with the continuum spectrum of H_0 appearing in the form $(H_0 - 0i - 2e_1 + e_0)^{-1}$.

The only resonant term in u-equation from $G_{5,1}$ is $|y|^4x$ (from $y^2\bar{\xi}$):

$$-ie^{ie_0t}\big(\phi_0, (y\phi_1)^2\bar{y}^2x(-i)\Phi_{1,i}\big) = -\big(\phi_0\phi_1^2, \Phi_{1,i}\big)|v|^4u, \tag{3.65}$$

and the only resonant term in v-equation from $G_{5,1}$ is $|x|^2|y|^2y$ (from $x\bar{y}\xi$):

$$-ie^{ie_1t}\big(\phi_1, 2(x\phi_0)(\bar{y}\phi_1)y^2\bar{x}i\Phi_{1,i}\big) = 2\big(\phi_0\phi_1^2, \Phi_{1,i}\big)|u|^2|v|^2v. \tag{3.66}$$

Note that their coefficients only differ by a factor -2. By (2.32) and (1.9),

$$-\big(\phi_0\phi_1^2, \Phi_{1,i}\big) = \operatorname{Im}\left(\lambda\phi_0\phi_1^2, \frac{1}{H_0 - 0i - 2e_1 + e_0}\mathbf{P}_c\lambda\phi_0\phi_1^2\right) = \gamma_0 > 0. \tag{3.67}$$

Together with the definitions of $R_{u,5}$ and $R_{\nu,5}$ in (3.34), we can rewrite

$$
\begin{aligned}
R_{u,5} + e^{ie_0 t}\widetilde{R}_{u,5} &= e^{ie_0 t}\big[\widetilde{R}_{u,5} - i\big(\phi_0, G_{5,1} + G_{5,2} + \Lambda_{\pi,5}\big)\big] - ie^{ie_0 t}\big(\phi_0, G_{5,3}\big),\\
R_{\nu,5} + e^{ie_1 t}\widetilde{R}_{\nu,5} &= e^{ie_1 t}\big[\widetilde{R}_{\nu,5} - i\big(\phi_1, G_{5,1} + G_{5,2} + E_{1,4}|y|^4 y\big)\big] - ie^{ie_1 t}\big(\phi_1, G_{5,3}\big).
\end{aligned}
\tag{3.68}
$$

As in Step 1, we now integrate by parts the nonresonant terms inside the square brackets. The resonant terms cannot be integrated and we shall only collect them. This procedure is the same as in Step 1 and we only summarize the conclusion: we can write

$$
\begin{aligned}
R_{u,5} + e^{ie_0 t}\widetilde{R}_{u,5} &= \big(c_3|u|^4 + c_4|u|^2|\nu|^2 + c_5|\nu|^4\big)u\\
&\quad + \frac{d}{dt}\big(u_2^-\big) + g_{u,4} - ie^{ie_0 t}\big(\phi_0, G_{5,3}\big),\\
R_{\nu,5} + e^{ie_1 t}\widetilde{R}_{\nu,5} &= \big(c_8|u|^4 + c_9|u|^2|\nu|^2 + c_{10}|\nu|^4\big)\nu\\
&\quad + \frac{d}{dt}\big(\nu_2^-\big) + g_{\nu,4} - ie^{ie_1 t}\big(\phi_1, G_{5,3}\big).
\end{aligned}
\tag{3.69}
$$

Here c_3, c_4, c_5, c_8, c_9, c_{10} are constants of order one; u_2^- and ν_2^- are two homogeneous polynomials in u and ν of degree 5; and $g_{u,4}$ and $g_{\nu,4}$ are the integration remainders satisfying

$$
\big|u_2^-\big| + \big|u_2^-\big| = O\big(u^5 + u\nu^4\big), \tag{3.70}
$$

$$
\big|g_{u,4}\big| + \big|g_{\nu,4}\big| \le n^6|x| + n^4\big\|\xi_0\big\|\langle t\rangle^{-3/2}. \tag{3.71}
$$

Furthermore, except c_5 and c_9, all other constants are purely imaginary. The real parts of c_5 and c_9 are from $G_{5,1}$ and they are given explicitly by

$$
\operatorname{Re} c_5 = \gamma_0, \qquad \operatorname{Re} c_9 = -2\gamma_0. \tag{3.72}
$$

We can now write the equations for u and ν as

$$
\begin{aligned}
\dot u_1 &= c_1|u_1|^2 u_1 + c_2|\nu_1|^2 u_1 + \big(c_3|u|^4 + c_4|u|^2|\nu|^2 + c_5|\nu|^4\big)u\\
&\quad + \frac{d}{dt}\big(u_2^-\big) + g_{u,4} - ie^{ie_0 t}\big(\phi_0, G_{5,3}\big) + g_{u,3} + R_{u,7},\\
\dot \nu_1 &= c_6|u_1|^2 \nu_1 + c_7|\nu_1|^2 \nu_1 + \big(c_8|u|^4 + c_9|u|^2|\nu|^2 + c_{10}|\nu|^4\big)\nu\\
&\quad + \frac{d}{dt}\big(\nu_2^-\big) + g_{\nu,4} - ie^{ie_1 t}\big(\phi_1, G_{5,3}\big) + g_{\nu,3} + R_{\nu,7}.
\end{aligned}
\tag{3.73}
$$

We now define

$$
\begin{aligned}
\mu &\equiv u_1 - u_2^- = u - u_1^- - u_2^-,\\
\nu &\equiv \nu_1 - \nu_2^- = \nu - \nu_1^- - \nu_2^-.
\end{aligned}
\tag{3.74}
$$

We have

$$\begin{aligned}\dot{\mu} &= c_1|u_1|^2u_1 + c_2|\nu_1|^2u_1 + \big(c_3|u|^4 + c_4|u|^2|\nu|^2 + c_5|\nu|^4\big)u \\ &\quad + g_{u,4} - ie^{ie_0t}\big(\phi_0, G_{5,3}\big) + g_{u,3} + R_{u,7} \\ &= c_1|\mu|^2\mu + c_2|\nu|^2\mu + \big(c_3|\mu|^4 + c_4|\mu|^2|\nu|^2 + c_5|\nu|^4\big)\mu + g_u, \\ \dot{\nu} &= c_6|u_1|^2\nu_1 + c_7|\nu_1|^2\nu_1 + \big(c_8|u|^4 + c_9|u|^2|\nu|^2 + c_{10}|\nu|^4\big)\nu \\ &\quad + g_{\nu,4} - ie^{ie_1t}\big(\phi_1, G_{5,3}\big) + g_{\nu,3} + R_{\nu,7} \\ &= c_6|\mu|^2\nu + c_7|\nu|^2\nu + \big(c_8|\mu|^4 + c_9|\mu|^2|\nu|^2 + c_{10}|\nu|^4\big)\nu + g_\nu,\end{aligned} \tag{3.75}$$

where

$$\begin{aligned} g_u &= g_{u,4} + g_{u,5} + g_{u,3} + R_{u,7} - ie^{ie_0t}\big(\phi_0, G_{5,3}\big), \\ g_\nu &= g_{\nu,4} + g_{\nu,5} + g_{\nu,3} + R_{\nu,7} - ie^{ie_1t}\big(\phi_1, G_{5,3}\big),\end{aligned} \tag{3.76}$$

$$\begin{aligned} g_{u,5} &= c_1\big(|u_1|^2u_1 - |\mu|^2\mu\big) + c_2\big(|\nu_1|^2u_1 - |\nu|^2\mu\big) \\ &\qquad + c_3\big(|u|^4u - |\mu|^4\mu\big) + c_4\big(|u|^2|\nu|^2u - |\mu|^2|\nu|^2\mu\big) + c_5\big(|\nu|^4u - |\nu|^4\mu\big), \\ g_{\nu,5} &= c_6\big(|u_1|^2\nu_1 - |\mu|^2\nu\big) + c_7\big(|\nu_1|^2\nu_1 - |\nu|^2\nu\big) \\ &\qquad + c_8\big(|u|^4\nu - |\mu|^4\nu\big) + c_9\big(|u|^2|\nu|^2\nu - |\mu|^2|\nu|^2\nu\big) + c_{10}\big(|\nu|^4\nu - |\nu|^4\nu\big).\end{aligned} \tag{3.77}$$

By (3.46) and (3.70), we have

$$\big|g_{u,5}\big| + \big|g_{\nu,5}\big| \le Cn^6|x|. \tag{3.78}$$

We also have, for $j = 1, 2$,

$$\big|e^{ie_jt}\big(\phi_j, G_{5,3}\big)\big| \le Cn^2\big\|\xi^{(3)}\big\|_{L^2_{loc}} \overset{(3.10)}{\le} Cn^{6-1/2}|x| + Cn^2\big\|\xi_0\big\|_Y\langle t\rangle^{-3/2}. \tag{3.79}$$

Finally we recall the estimates (3.37) and (3.39) for $R_{u,7}$ and $R_{\nu,7}$. Observe that, in fact, $-ie^{ie_1t}E_1^{(6)}(|y|)y = -ie^{ie_1t}E_1^{(6)}(|y|)\nu + O(n^6|x|)$ is the only term in $R_{\nu,7}$ which does not satisfy the same estimate of $R_{u,7}$. Together with the estimates (3.56), (3.71), (3.78), and (3.79), we conclude estimates (3.33) for g_u and g_ν. The lemma is proved. ■

Note that the error terms g_u and $\tilde{g}_\nu \equiv g_\nu + ie^{ie_1t}E_1^{(6)}(|y|)\nu$ are of the form

$$g_u, \tilde{g}_\nu \sim \big(x^7 + xy^6\big) + n^2\xi^{(3)} + n^4\xi + n\xi^2 + \big(\xi^3\big)_{loc} + \cdots, \tag{3.80}$$

where $(\xi^3)_{loc}$ denotes the L^1_{loc}-norm of ξ^3.

3.3 Estimates for bound states

We conclude the estimates for x and y stated in Proposition 3.1. Recall that under (3.1) and assumption (3.10), we have proved Lemmas 3.2, 3.3, and 3.4 which contain estimates for ξ, g_u and g_v. We now derive some preliminary estimates.

Let $f = 2|\mu|^2$ and $g = |\nu|^2$. We have, by (3.29),

$$\big|f(t) - 2|x|^2\big| \le 5C_6 n^2 |x|^2, \qquad \big|g(t) - |y|^2\big| \le 5C_6 n^4. \tag{3.81}$$

We also have from (3.30) that

$$\dot f = \operatorname{Re} 4\bar\mu\dot\mu = 2\gamma_0 g^2 f + \operatorname{Re} 4\bar\mu g_u, \tag{3.82}$$

$$\dot g = \operatorname{Re} 2\bar\nu\dot\nu = -2\gamma_0 f g^2 + \operatorname{Re} 2\bar\nu g_\nu. \tag{3.83}$$

By (3.32) and (3.33), we have

$$\big|\dot f + \dot g\big| = \big|\operatorname{Re} 4\bar\mu g_u + \operatorname{Re} 4\bar\nu g_\nu\big| \le 4C_6 n^{8-1/2} + 4C_6 n^3 \big\|\xi_0\big\|_Y \langle t\rangle^{-3/2}. \tag{3.84}$$

Recall that $n_1^2 = |x(0)|^2 + (1/2)|y(0)|^2$. By (3.81) we have $|(f+g)(0) - 2n_1^2| \le 10C_6 n^4$. Thus, for $t < C_3 n^{-4-1/4}$,

$$\begin{aligned}\big|(f+g)(t) - 2n_1^2\big| &\le 10C_6 n^4 + \int_0^t 4C_6 n^{8-1/2} + 4C_6 n^3 \big\|\xi_0\big\|_Y \langle s\rangle^{-3/2}\, ds \\ &\le 5C_3 C_6 n^{3+1/4}.\end{aligned} \tag{3.85}$$

We now prove Proposition 3.1 in three steps.

(1) Initial layer regime. In this period the dispersive part disperses away so much that it becomes negligible locally. The time it takes for this to happen is of order $t_0 = \varepsilon_3 n^{-4}$. We first prove that $(1/2)|x_0| \le |x(t)| \le (3/2)|x_0|$ for $t \in [0, t_0]$. The main ingredients of the proof are the normal form equation of x from Section 3.2 and the following observation. The ξ-dependent terms are of the form $n^2\xi$ or of higher orders. Because of our assumption $\|\xi_0\|_Y \le \varepsilon_2 |x_0| n^{-2}$ and the decay of $\xi(t)$, these terms will not change $x(t)$ very much. More precisely, for $t \in [0, t_0]$, $t_0 = \varepsilon_3 n^{-4}$, we have by (3.82), (3.33), (3.81), the assumptions $\|\xi_0\|_Y \le \varepsilon_2 |x_0| n^{-2}$ and (3.10),

$$\begin{aligned}\big|f(t) - f(0)\big| &\le \int_0^t 4\gamma_0 g^2 f(s) + 5|x_0|\big|g_u(s)\big|\, ds \\ &\le \big(8\gamma_0 n^4 + C n^{6-1/2}\big)|x_0|^2 t_0 + 10|x_0| n^2 \big\|\xi_0\big\|_Y \\ &\le \big(10\gamma_0 + 1\big)\varepsilon_2 |x_0|^2 + 10\varepsilon_3 |x_0|^2 \\ &\le \frac{1}{8} f(0),\end{aligned} \tag{3.86}$$

provided that n, ε_2 and ε_3 are sufficiently small. By (3.81), we have $||x(t)|^2 - |x_0|^2| \le (1/4)|x_0|^2$. Hence we have $(1/2)|x_0| \le |x(t)| \le (3/2)|x_0|$ for $t \in [0, t_0]$.

Similarly, we can show that $|g(t) - g(0)| \le ((10\gamma_0 + 1)\varepsilon_2 + 10\varepsilon_3)n^2$. Hence we have $||y(t)| - |y_0|| \le 0.01|y_0|$ for $t \in [0, t_0]$ if ε_2 and ε_3 are small. The smallness of ε_2 and ε_3 can be guaranteed if we define

$$\varepsilon_2 = \frac{1}{2000(\gamma_0 + 1)}, \qquad \varepsilon_3 = \frac{1}{2000}. \tag{3.87}$$

(2) Transition regime (i). In this period most mass of the disperse wave is faraway and has no effect on the local dynamics; the ground state begins to grow exponentially until it has the order $n/100$. The time it takes is of order n^{-4} to $n^{-4-1/4}$.

Define

$$t_1 \equiv \inf_{t \ge t_0} \{t : |x(t)| \ge 0.01n\}. \tag{3.88}$$

We want to show that

$$0 \le t_1 \le t_1', \quad t_1' \equiv t_0 + \frac{1.01}{\gamma_0 n^4} \log\left(\frac{n}{|x_0|}\right), \tag{3.89}$$

Suppose that (3.89) fails, that is, $|x(t)| < 0.01n$ for all $t \le t_1'$. By (3.29) and (3.81), we have $f(t) \le 0.0004n^2$ and $g(t) \ge 0.9995n^2$ for $t \le t_1'$. Hence

$$\dot{f}(t) \ge 2\gamma_0 (0.9995n^2)^2 f + O(n^{11/2}) f \ge \frac{2}{1.01}\gamma_0 n^4 f, \tag{3.90}$$

if n is sufficiently small. Hence

$$f(t) \ge f(0) \exp\left\{\frac{2}{1.01}\gamma_0 n^4 t\right\}, \tag{3.91}$$

for $t \le t_1'$. We have

$$f(t_1') \ge f(0) \exp\left\{\frac{2}{1.01}\gamma_0 n^4 \frac{1.01}{\gamma_0 n^4} \log\left(\frac{n}{|x_0|}\right)\right\} = f(0) n^2 |x_0|^{-2} \ge 0.99n^2, \tag{3.92}$$

which is a contradiction to the assumption that $|x|(t_1') < 0.01n$. This shows that t_1 satisfies (3.89). We also have that (3.91) holds for all $t \le t_1$, and that $f(t_1) \ge 5 \cdot 10^{-5} n^2$. By (3.85), $|g(t_1) - g(0)| \le n^2/1000$.

Note that, if $x(0)$ is already of order n, say $|x(0)| \geq n/200$, we can let $t_1 = t_0$ and the argument goes through.

(3) Transition regime (ii). In this period x keeps growing while y begins to decay until it reaches the order εn. The time it takes is of order $\varepsilon^{-2} n^{-4}$. Recall the definition of ε (3.3). Define

$$t_2 \equiv \inf\left\{t : g(t) \leq (\varepsilon n)^2\right\}. \tag{3.93}$$

We want to show that

$$t_1 \leq t_2 \leq t_2', \quad t_2' \equiv t_1 + 10100\left(\gamma_0 n^4 \varepsilon^2\right)^{-1}. \tag{3.94}$$

Suppose the contrary, then $g(t) \geq (\varepsilon n)^2$ for all $t \leq t_2'$. Then $\dot{f} > 0$ by (3.82) and (3.33), and hence $f(t) \geq 5 \cdot 10^{-5} n^2$ for $t_1 \leq t \leq t_2'$. Hence, by (3.32) and (3.33),

$$\dot{g} \leq -(1.99)\gamma_0 f g^2 + O\left(n^{13/2}\right) \leq -9.95 \cdot 10^{-5} \gamma_0 n^2 g^2. \tag{3.95}$$

Therefore

$$g(t) \leq \left[g\left(t_1\right)^{-1} + 9.95 \cdot 10^{-5} \gamma_0 n^2 \left(t - t_1\right)\right]^{-1}, \quad t_1 \leq t \leq t_2', \tag{3.96}$$

which implies that $g(t_2') < (\varepsilon n)^2$ by the definition of t_2' and is a contradiction. This contradiction shows the existence of t_2 satisfying (3.94). Estimate (3.85) for $f + g$ then shows the estimate for $f(t_2)$. Because $\dot{g} \geq -(2.01) f g^2$ and $|y(t_1)| \geq 2\varepsilon n$, similar argument shows that $t_2 - t_1 \geq C\varepsilon^{-2} n^{-4}$. By (3.81), these estimates of f and g can be translated into estimates of $x(t_2)$ and $y(t_2)$ stated in Proposition 3.1.

We have proved (3.4), (3.5), (3.7), and (3.8) in Proposition 3.1, using (1.18), (3.1), and the assumption (3.10). Since (3.10) holds for $t = 0$, by continuity it holds for all $t \leq t_2$. From Lemmas 3.2, 3.3, and 3.4 and the above estimates, Proposition 3.1 is proved.

4 Stabilization regime

We study the solution $\psi(t)$ in the third time regime, after the solution has become near some nonlinear ground state. In this regime, it is natural to use the decomposition (4.3) for the solution $\psi(t)$ which emphasizes nonlinear ground states. The setup and proof here are similar to those in [16] except a new idea to prove that the main dispersive wave is out-going. We now briefly recall the setup in [16]. We refer the reader to [16] for more details.

For a nonlinear ground state Q_E with frequency E, let $\mathcal{L}_E$ be the linearized operator around Q_E:

$$\mathcal{L}h = -i\{(-\Delta + V - E + 2\lambda Q_E^2)h + \lambda Q_E^2 \bar{h}\}. \tag{4.1}$$

With respect to $\mathcal{L}_E$, we can decompose $L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, as the direct sum of three invariant subspaces:

$$L^2(\mathbb{R}^3, \mathbb{C}) = S(\mathcal{L}_E) \oplus \mathbf{E}_1(\mathcal{L}_E) \oplus \mathbf{H}_c(\mathcal{L}_E), \tag{4.2}$$

where $S(\mathcal{L}_E)$ and $\mathbf{E}_1(\mathcal{L}_E)$ are generalized eigenspaces, obtained from perturbation of ϕ_0 and ϕ_1, respectively, and $\mathbf{H}_c(\mathcal{L}_E)$ corresponds to the continuous spectrum of $\mathcal{L}_E$. Notice that this decomposition is not orthogonal. Also, $S(\mathcal{L}_E) = \operatorname{span}_{\mathbb{R}}(iQ_E, R_E)$, where $R_E = \partial_E Q_E$.

For each ψ sufficiently close to Q_E, we can decompose ψ as

$$\psi = [Q_E + a_E R_E + \zeta_E + \eta_E] e^{i\Theta_E}. \tag{4.3}$$

Here $a_E, \Theta_E \in \mathbb{R}$, $\zeta_E \in \mathbf{E}_1(\mathcal{L})$ and $\eta_E \in \mathbf{H}_c(\mathcal{L})$. The direction iQ_E is implicitly given in $Q_E(e^{i\Theta} - 1)$. Moreover, for this ψ there is a unique frequency E' such that in the decomposition (4.3) with respect to E' the coefficient a vanishes. In some sense it means that $Q_{E'}$ is the closest nonlinear ground state to ψ.

The main result in [16, Theorem 3], asserts the asymptotic stability of the nonlinear bound states under the following conditions: suppose that the initial data $\psi(t_2)$ is close to $Q_{E'}$ with E' so chosen that the coefficient a in (4.3) with respect to E' vanishes. For all E close to E' with $|E - E'| \le \|\zeta_{E'}\|^2$, the excited state component satisfies

$$\|\zeta_E\| \le \varepsilon_0 n, \tag{4.4}$$

and the dispersive part satisfies

$$\|\eta_E(t_2)\|_Y \le C\|\zeta_E\|^2. \tag{4.5}$$

Here the Y-norm is defined in (1.8).

By Proposition 3.1, we can show that $\psi(t_2)$ is close to a nonlinear ground state $Q_{E_0} e^{i\Theta_0}$ in L^2_{loc}-norm, that is, $\|Q_{E_0}\|_{L^2} = n_{t_2} \le n_0$, $\|\psi(t_2) - Q_{E_0} e^{i\Theta_0}\|_{L^2_{loc}} \le \varepsilon_0 n_{t_2}$, where $n_{t_2} \sim n_1 \sim n$ with n_1 defined in (1.19) and n defined in (1.18). Thus the condition (4.4) is satisfied. The dispersive part, however, is no longer localized and there is no way to satisfy (4.5). In fact, even its L^2-norm is not small enough.

The condition (4.5), however, is used in [16] only to guarantee the decay estimates (4.6) and (4.7) stated in the following Theorem 4.1. These bounds are used to estimate the contribution of $e^{s\mathcal{L}}\eta_E(t_2)$ in the proofs for the L^4 and L^2_{loc} bounds for $\eta_E(t)$ in [16, Lemmas 5.2 and 5.3]. In other words, we can view (4.6) and (4.7) as a measure of an outgoing norm of the dispersive wave. We now state this stronger version in the following Theorem 4.1.

Theorem 4.1. There are small constants $n_0, \varepsilon_0 > 0$ such that the following hold. Suppose that $\psi(t_2)$ is near a nonlinear ground state $Q_{E_0}e^{i\Theta_0}$ in L^2_{loc}-norm such that $\|Q_{E_0}\|_{L^2} = n_{t_2} \sim n \le n_0$ and, in the decomposition (4.3) with $E = E_0$, $|a| + \|\eta_{E_0}\|_{L^2_{loc}} \le \varepsilon^2 n^2$ and $\|\zeta_{E_0}\| \le \varepsilon n$ for some $\varepsilon \in (0, \varepsilon_0)$.

If for all E close to E_0 with $|E - E_0| \le \varepsilon_0^2 n^2$, the dispersive part $\eta_E(t_2)$ in the decomposition (4.3) satisfies

$$\left\|e^{s\mathcal{L}}\eta_E(t_2)\right\|_{L^4} \ll \{s\}^{-3/4+\sigma}, \quad \left\|e^{s\mathcal{L}}\eta_E(t_2)\right\|_{L^2_{loc}} \ll n^{-\sigma}\{s\}^{-1}, \quad s \ge 0, \tag{4.6}$$

$$\left\|e^{s\mathcal{L}}\eta_E(t_2)\right\|_{L^2_{loc}} \ll n\{s\}^{-1}, \quad s \ge \delta t, \tag{4.7}$$

where $\sigma = 1/100$, $\delta t = n^{-1}$ and

$$\{s\} = (\varepsilon n)^{-2} + 2\lambda^2\gamma_0 n^2 s, \tag{4.8}$$

then the conclusion of [16, Theorem 1] remains valid. In particular, there is a frequency E_∞ with $|E_\infty - E_0| \le C\varepsilon_0^2 n^2$ and a function $\Theta(t) = -E_\infty t + O(\log(t))$ for $t \in [t_2, \infty)$ such that, for some constant C_2,

$$\left\|\psi(t) - Q_{E_\infty}e^{i\Theta(t)}\right\|_{L^2_{loc}} \le C_2\left((\varepsilon n)^{-2} + \gamma_0 n^2(t - t_2)\right)^{-1/2}. \tag{4.9}$$

Suppose, furthermore, that $\|\zeta_{E_0}(t_2)\| = \varepsilon n$. Then the following lower bound holds:

$$C_1\left((\varepsilon n)^{-2} + \gamma_0 n^2(t - t_2)\right)^{-1/2} \le \left\|\psi(t) - Q_{E_\infty}e^{i\Theta(t)}\right\|_{L^2_{loc}}, \tag{4.10}$$

for some constant $C_1 > 0$. □

As mentioned previously, condition (4.4) follows from Proposition 3.1. Since $t_2 \le C_3 n^{-4-1/4}$, the estimates (4.12) implies (4.6) and (4.7).

Lemma 4.2. (1) The following estimates for $\xi(t_2)$ hold for all $s \geq 0$:

$$\begin{aligned}
&\|\xi(t_2)\| \ll 1,\\
&\|e^{-isH_0}\xi(t_2)\|_{L^4} \leq Cn^3 t_2 (t_2+s)^{-3/4},\\
&\|e^{-isH_0}\xi(t_2)\|_{L^2_{\mathrm{loc}}} \leq Cn^3 \frac{t_2}{t_2+s}(1+s)^{-1/2}.
\end{aligned}\tag{4.11}$$

(2) For all E with $\|Q_E\| \leq n_0$, let $\mathcal{L} = \mathcal{L}_E$ and $\eta_E(t_2)$ be the dispersive component of $\psi(t_2)$ with respect to E in the decomposition (4.3). For all $s \geq 0$,

$$\begin{aligned}
&\|e^{s\mathcal{L}}\eta_E(t_2)\|_{L^4} \leq Cn^3 t_2 (t_2+s)^{-3/4},\\
&\|e^{s\mathcal{L}}\eta_E(t_2)\|_{L^2_{\mathrm{loc}}} \leq Cn^3 \frac{t_2}{t_2+s}(1+s)^{-1/2}.
\end{aligned}\tag{4.12}$$

□

As we will see in the proof, $|x(t_2)| + |y(t_2)| \leq Cn$ and (4.11) are the only information we need to prove (4.12). Therefore we have a theorem similar to Theorem 4.1 in terms of H_0, which we state as a separate theorem for future reference.

Theorem 4.3. There are small constants $n_0, \varepsilon_0 > 0$ such that, if $\psi(t_2) = x(t_2)\phi_0 + Q_1(y(t_2)) + \xi(t_2)$ with

$$|x(t_2)| = n \ll n_0, \qquad |y(t_2)| \leq \varepsilon_0 n, \tag{4.13}$$

and that $\xi(t_2)$ satisfies (4.11) for some $t_2 \in [1, n^{-4-1/4}]$, then the solution $\psi(t)$ of (1.1) converges to some nonlinear ground state with corresponding estimates as $t \to \infty$. □

Proof of Lemma 4.2

Part 1. By ξ of (2.16) we have

$$e^{-isH_0}\xi(t_2) = e^{-i(t_2+s)H_0}\,\xi_0 + \int_0^{t_2} e^{-i(t_2+s-\tau)H_0}\,\mathbf{P}_c G_\xi(\tau)\,d\tau. \tag{4.14}$$

Hence

$$\begin{aligned}
\|e^{-isH_0}\xi(t_2)\|_{L^4} &\leq C(t_2+s)^{-3/4}\|\xi_0\|_Y\\
&\quad + \int_0^{t_2} C|t_2+s-\tau|^{-3/4}\|G_\xi(\tau)\|_{L^{4/3}}\,d\tau.
\end{aligned}\tag{4.15}$$

Since $\|\xi_0\|_Y \leq n$ and $\|G_\xi(\tau)\|_{L^{4/3}} \leq Cn^3$, the L^4 estimate is obtained after we show that

$$\int_0^{t_2} |t_2+s-\tau|^{-3/4}\,d\tau \leq Ct_2(t_2+s)^{-3/4}. \tag{4.16}$$

If $s > t_2$, the integral is bounded by $\int_0^{t_2} |t_2+s|^{-3/4} d\tau = Ct_2(t_2+s)^{-3/4}$. If $s < t_2$, $(t_2+s) \sim t_2$ and the integral is bounded by $\int_0^{t_2} |t_2-\tau|^{-3/4} d\tau = Ct_2^{1/4} \le Ct_2(t_2+s)^{-3/4}$.

We now estimate L^2_{loc} norm. Let $t = t_2 + s$. The L^2_{loc} norm of the integrand of (4.14) can be bounded by the minimum of L^∞-norm and L^4-norm. Hence

$$\left\|e^{-isH_0}\xi(t_2)\right\|_{L^2_{\text{loc}}} \le C\left(t_2+s\right)^{-3/2}\|\xi_0\|_Y + \int_0^{t_2} C\min\left\{(t-\tau)^{-3/2}, (t-\tau)^{-3/4}\right\}\left\|G_\xi(\tau)\right\|_{L^1\cap L^{4/3}} d\tau. \tag{4.17}$$

Since $\|\xi_0\|_Y \le n$ and $\|G_\xi(\tau)\|_{L^1\cap L^{4/3}} \le Cn^3$, the L^2_{loc} estimate is obtained after we show that

$$\int_0^{t_2} \min\left\{(t-\tau)^{-3/2}, (t-\tau)^{-3/4}\right\} d\tau \le C\frac{t_2}{t}\left\langle t-t_2\right\rangle^{-1/2}. \tag{4.18}$$

If $t \le t_2 + 1$, the left side is bounded by a constant, and hence bounded by the right side. If $t \ge t_2 + 1$, that is, $s \ge 1$, the integral is bounded by

$$\begin{aligned}\int_0^{t_2} (t-\tau)^{-3/2} d\tau &= 2\left(t-t_2\right)^{-1/2} - 2t^{-1/2} \\ &= 2\left[\left(t-t_2\right)^{-1/2} + t^{-1/2}\right]^{-1}\left[\left(t-t_2\right)^{-1} - t^{-1}\right] \\ &\le 2\left(t-t_2\right)^{1/2}\left[\left(t-t_2\right)^{-1} t^{-1} t_2\right].\end{aligned} \tag{4.19}$$

Part 2. Since E will be fixed for the rest of this proof, we shall drop most subscripts E. We have two decompositions of $\psi(t)$:

$$\psi(t) = x(t)\phi_0 + Q_1\left(y(t)\right) + \xi(t) = \left[Q_E + a(t)R_E + \zeta(t) + \eta(t)\right]e^{i\Theta(t)}. \tag{4.20}$$

Hence

$$\zeta(t) + \eta(t) = \left[x(t)\phi_0 e^{-i\Theta(t)} - Q_E\right] + \left[Q_1\left(y(t)\right) + \xi(t)\right]e^{-i\Theta(t)} - a(t)R_E. \tag{4.21}$$

Thus, at $t = t_2$, we have

$$\begin{aligned}\eta(t_2) &= \mathbf{P}_c^{\mathcal{L}}\left\{\left[x(t_2)\phi_0 e^{-i\Theta(t_2)} - Q_E\right] + \left[Q_1\left(y(t_2)\right) + \xi(t_2)\right]e^{-i\Theta(t_2)}\right\} \\ &= \eta_{0,1} + \eta_{0,2},\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}\eta_{0,1} &= \mathbf{P}_c^{\mathcal{L}}\{\xi(t_2)e^{-i\Theta(t_2)}\},\\ \eta_{0,2} &= \mathbf{P}_c^{\mathcal{L}}\{[x(t_2)\phi_0 e^{-i\Theta(t_2)} - Q_E] + Q_1(y(t_2))e^{-i\Theta(t_2)}\}.\end{aligned} \tag{4.23}$$

Note that, since $|x(t_2)| + |y(t_2)| \le Cn$, $\eta_{0,2}$ is a local H^1 function of order n^3, that is, $\|\eta_{0,2}\|_Y \le Cn^3$. Therefore we have

$$\|e^{s\mathcal{L}}\eta_{0,2}\|_{L^4} \le Cn^3 s^{-3/4}, \qquad \|e^{s\mathcal{L}}\eta_{0,2}\|_{L^2_{\mathrm{loc}}} \le Cn^3\langle s\rangle^{-3/2}. \tag{4.24}$$

Hence $e^{s\mathcal{L}}\eta_{0,2}$ satisfies (4.12).

We now focus on the nonlocal term $\eta_{0,1} = \mathbf{P}_c^{\mathcal{L}}\{\xi(t_2)e^{-i\Theta(t_2)}\}$. Note that

$$\|e^{s\mathcal{L}}\eta_{0,1}\|_{L^2} \le C\|\eta_{0,1}\|_{L^2} \le C\|\xi(t_2)\|_{L^2} \ll 1, \tag{4.25}$$

by [16, Lemmas 2.6 and 2.9] and by the first inequality of (4.11). For convenience of notation, we write

$$\mathcal{L} = -iH_0 + W, \tag{4.26}$$

where $Wu = -i\lambda Q_E^2(2u+\bar{u})$ is a local operator of order n^2. By Duhamel's principle,

$$\begin{aligned} e^{s\mathcal{L}}\eta_{0,1} &= \mathbf{P}_c^{\mathcal{L}} e^{s\mathcal{L}}\{\xi(t_2)e^{-i\Theta(t_2)}\} = \Omega_1 + \Omega_2,\\ \Omega_1 &= \mathbf{P}_c^{\mathcal{L}} e^{-isH_0}\xi(t_2)e^{-i\Theta(t_2)},\\ \Omega_2 &= \int_0^s e^{(s-\tau)\mathcal{L}}\,\mathbf{P}_c^{\mathcal{L}} We^{-i\tau H_0}\xi(t_2)e^{-i\Theta(t_2)}\,d\tau.\end{aligned} \tag{4.27}$$

For Ω_1, by (4.11) we have

$$\|\Omega_1\|_{L^4} \le Cn^3 t_2(t_2+s)^{-3/4}, \qquad \|\Omega_1\|_{L^2_{\mathrm{loc}}} \le Cn^3\frac{t_2}{t_2+s}(1+s)^{-1/2}. \tag{4.28}$$

For Ω_2, since W is a local operator of order n^2, by (4.11),

$$\begin{aligned}\|\Omega_2\|_{L^4} &\le C\int_0^s (s-\tau)^{-3/4} n^2\|e^{-i\tau H_0}\xi(t_2)\|_{L^2_{\mathrm{loc}}}\,d\tau\\ &\le Cn^5\int_0^s (s-\tau)^{-3/4}\frac{t_2}{t_2+\tau}(1+\tau)^{-1/2}\,d\tau.\end{aligned} \tag{4.29}$$

If $s \le t_2$, the integral is bounded by

$$\int_0^s (s-\tau)^{-3/4}\tau^{-1/2}\,d\tau = Cs^{-1/4} \le C\left(\frac{t_2}{t_2+s}\right)^{1/2} s^{-1/4}. \tag{4.30}$$

The equality is by scaling. If $s \geq t_2$, the integral is bounded by

$$\int_0^{t_2/2} s^{-3/4}\tau^{-1/2}\,d\tau + \int_{t_2/2}^{t/2} s^{-3/4} t_2 \tau^{-3/2}\,d\tau + \int_{t/2}^{t} (s-\tau)^{-3/4} t_2 \tau^{-3/2}\,d\tau. \tag{4.31}$$

They are bounded by $s^{-3/4}t_2^{1/2} + s^{-3/4}t_2 t_2^{-1/2} + t_2 s^{-5/4} \leq C t_2^{1/2} s^{-3/4} \leq (t_2/(t_2+s))^{1/2} s^{-1/4}$. Combining both cases, we have

$$\|\Omega_2\|_{L^4} \leq C n^5 \left(\frac{t_2}{t_2+s}\right)^{1/2} s^{-1/4}. \tag{4.32}$$

The L^2_{loc}-norm of the integrand of Ω_2 can be bounded by either its L^∞-norm (for τ small) or its L^4-norm (for τ near s). Thus we have,

$$\begin{aligned} \|\Omega_2\|_{L^2_{\mathrm{loc}}} &\leq \int_0^s C \min\left\{|s-\tau|^{-3/2}, |s-\tau|^{-3/4}\right\} n^2 \left\|e^{-i\tau H_0}\xi(t_2)\right\|_{L^2_{\mathrm{loc}}}\,d\tau \\ &\leq C n^5 \int_0^s \min\left\{|s-\tau|^{-3/2}, |s-\tau|^{-3/4}\right\} \frac{t_2}{t_2+\tau}(1+\tau)^{-1/2}\,d\tau. \end{aligned} \tag{4.33}$$

If $s \leq t_2$, the integral is bounded by

$$C\int_0^s \langle s-\tau\rangle^{-3/2}\tau^{-1/2}\,d\tau + C s^{-1/2} \leq C s^{-1/2} \leq C\frac{t_2}{t_2+s} s^{-1/2}. \tag{4.34}$$

If $s \geq t_2$, the integral is bounded by

$$\int_0^s \langle s-\tau\rangle^{-3/2} t_2 \langle\tau\rangle^{-3/2}\,d\tau + C t_2\langle s\rangle^{-3/2} \leq C t_2 \langle s\rangle^{-3/2} \leq C\frac{t_2}{t_2+s} s^{-1/2}. \tag{4.35}$$

Combining both cases, we have

$$\|\Omega_2\|_{L^2_{\mathrm{loc}}} \leq C n^5 \frac{t_2}{t_2+s} s^{-1/2}. \tag{4.36}$$

Since $\|e^{s\mathcal{L}}\eta_{0,1}\| \leq \|\Omega_1\| + \|\Omega_2\|$, we have proven Lemma 4.2. ∎

Appendix

In this appendix we prove Theorem 4.1. The proof is similar to that in [16] and we remark only the main difference. In this proof, we shall set $t_2 = 0$.

Step 1. For each time $T > 0$, choose $E(T)$ so that $Q_{E(T)}$ is the best approximation of $\psi(T)$. With respect to $E = E(T)$, we write the solution $\psi(t)$ as

$$\psi(t) = \left[Q_T + a(t)R_T + \zeta(t) + \eta(t)\right] e^{-iE_T t + i\theta(t)}, \tag{A.1}$$

where $R_T = \partial_E Q_E|_{E=E(T)}$, $\zeta(t) \in \mathbf{E}_1(\mathcal{L})$ is the excited state component, $\eta(t) \in \mathbf{H}_c(\mathcal{L})$ is the dispersive component. We can write

$$\zeta(t) = z(t)u_+ + \bar{z}(t)u_-, \quad z(t) = e^{-i\kappa t}p(t), \tag{A.2}$$

where $u_+ = O(1)$ and $u_- = O(n^2)$ are local functions. We derive the following system for p, a, and η:

$$\begin{aligned} &ie^{-i\kappa t}\dot{p} = (u_+, F) + (u_-, \overline{F}) + \left[(u_+, h) + (u_-, \overline{h}) + (u, R)a\right]\dot{\theta}, \\ &\dot{a} = \left(c_1 Q, \operatorname{Im}\left(F + \dot{\theta}h\right)\right), \quad c_1 = (Q, R)^{-1}, \\ &\partial_t \eta = \mathcal{L}\eta + \mathbf{P}_c^{\mathcal{L}} i^{-1}\left(F + \dot{\theta}(aR + h)\right), \end{aligned} \tag{A.3}$$

where $u = u_+ + u_-$, $h = \zeta + \eta$,

$$\begin{aligned} &F = \lambda Q\left(2|h|^2 + h^2\right) + 2\lambda QRa\left(2h + \bar{h}\right) + 3\lambda QR^2a^2 + \lambda(aR + h)^2\left(aR + \bar{h}\right), \\ &\dot{\theta} = -\left[a + (c_1R, \operatorname{Re} F)\right] \cdot \left[1 + a(c_1R, R) + (c_1R, \operatorname{Re} h)\right]^{-1}. \end{aligned} \tag{A.4}$$

Note that θ enters (A.3) only via $\dot{\theta}$. We have $z \sim (|z(0)|^{-2} + n^2t)^{-1/2}$ and

$$Q \sim n, \quad R \sim n^{-1}, \quad \zeta \sim z, \quad a \lesssim z^2, \quad \eta \lesssim nz^2 + e^{t\mathcal{L}}\eta(0). \tag{A.5}$$

Step 2. We can decompose F and $\dot{\theta}$ according to their order in z, as in [16]. We also decompose a and η

$$a = a_{20}\left(z^2 + \bar{z}^2\right) + b, \qquad \eta = \eta^{(2)} + \eta^{(3)}, \tag{A.6}$$

where $a_{20} = O(n^2)$, $b = O(z^2)$, $\eta^{(2)} = O(nz^2)$, and $\eta^{(3)}$ is smaller than $\eta^{(2)}$ for t large. They are defined in [16] and the decomposition $\eta = \eta^{(2)} + \eta^{(3)}$ is similar to that of $\xi = \xi^{(2)} + \xi^{(3)}$. Moreover, b satisfies

$$\dot{b} = \left(c_1 Q, \operatorname{Im}\left[F - F^{(2)} + \dot{\theta}h\right]\right) - 4a_{20} \operatorname{Re} e^{-2i\kappa t}p\dot{p}. \tag{A.7}$$

We have the following normal form for z and a.

Lemma A.1. Assume that $0 < n \le n_0$ and

$$|z| \ll n, \quad |b| \le C|z|^2, \quad \|\eta\|_{L^4} \le Cn^{7/4}, \quad \|\eta\|_{L^2_{\mathrm{loc}}} \le Cn^3. \tag{A.8}$$

Then there are perturbations $q(t)$, $\widetilde{b}(t)$ of $p(t)$ and $b(t)$ satisfying the estimates

$$\left|q(t) - p(t)\right| \leq Cn\left|z(t)\right|^2 + C\left|b(t)\right|\left|z(t)\right|, \tag{A.9}$$

$$\left|\widetilde{b}(t) - b(t)\right| \leq Cn\left|z(t)\right|^3 + Cn\left|b(t)\right|\left|z(t)\right|, \tag{A.10}$$

and the normal form equations

$$\dot{q} = \delta_{21}|q|^2 q + d_1 b q + f(t) q + g(t), \tag{A.11}$$

$$\frac{d}{dt}\widetilde{b} = B_{22}|z|^4 + B_5. \tag{A.12}$$

Here δ_{21} and d_1 are order one constant, $\operatorname{Re} d_1 = 0$, $\operatorname{Re}\delta_{21} \sim -2\lambda^2\gamma_0 n^2$, $f(t) \leq n^2$ is a purely imaginary function and $B_{22} \sim \lambda^2\gamma_0 n^2/(Q, R)$. The error terms $g(t)$ and $B_5(t)$ are bounded by

$$|g| \lesssim n|z|^4 + n^4|z|^3 + n|z|\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + \left\{|z|^2 + n\|\eta\|_{L^2_{\mathrm{loc}}} + \|\eta\|^2_{L^4}\right\}\|\eta\|_{L^2_{\mathrm{loc}}}, \tag{A.13}$$

$$\left|B_5\right| \lesssim n|z|^5 + n^4|z|^4 + n|z|^2\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + n^2\|\eta\|^2_{L^2_{\mathrm{loc}}} + n\|\eta\|^2_{L^4}\|\eta\|_{L^2_{\mathrm{loc}}}. \tag{A.14}$$

□

Note that $(\operatorname{Re}\delta_{21})|q|^2 q$ is the main damping term in (A.11). Although this normal form is valid for all $t \geq 0$, $g(t)$ may not be sufficiently small, that is, it is not an error, for $t \in [0, \delta t]$.

Proof. The equations for $\dot{p}$ and $\dot{b}$ are of the form

$$\begin{aligned}
\dot{p} &= \cancel{nz^2} + \cancel{z^3} + \cancel{bz} + \cancel{\frac{z^4}{n}} + \cancel{\frac{z^5}{n^2}} + \cancel{\frac{z^6}{n^3}} + \cdots \\
&\quad + \cancel{n^2 z^3} + nz^4 + n^4 z^3 + nz\eta^{(3)} + z^2\eta + n\eta^2 + \left(\eta^3\right)_{\mathrm{loc}} + \cdots, \\
\dot{b} &= n\left\{\cancel{z^3} + \cancel{bz} + \left(\cancel{\frac{z^4}{n}} + \cancel{\frac{z^5}{n^2}} + \cdots\right) + \cancel{nz}\eta + z^2\cancel{\eta^{(2)}} + z^2\eta^{(3)} + n\eta^2 + \left(\eta^3\right)_{\mathrm{loc}} + \cdots\right\} \\
&\quad + n^2 z\left\{\cancel{nz^2} + \cancel{z^3} + \cancel{bz} + O\left(\frac{z^4}{n}\right) + nz\eta + z^2\eta + n\eta^2 + \left(\eta^3\right)_{\mathrm{loc}} + \cdots\right\}.
\end{aligned} \tag{A.15}$$

Here $(\eta^3)_{\mathrm{loc}}$ means terms with same bound as $\|\eta^3\|_{L^1_{\mathrm{loc}}}$. We shall calculate the normal form for p and b by integrating by parts those terms with orders which were crossed out. Notice that there are resonant terms with crossed-out orders, for example, terms of the form $(|z|^2 + b + n^2|z|^2)z$ in the $\dot{p}$ equation. These terms cannot be integrated by parts and will remain on the right-hand side. The final normal form equations are of the form

(A.11) and (A.12). This procedure is routine and was carried out in details in [16] and Section 3 of this article. We shall not repeat it in details but point out a few key steps.

We used in [16] the normal form to remove terms of the form

$$nz^2,\ z^3,\ bz,\ n^2z^3,\ nz\eta^{(3)}_{1-2}, \tag{A.16}$$

where $\eta^{(3)}_{1-2}$ is part of $\eta^{(3)}$. Here we will not remove $nz\eta^{(3)}_{1-2}$, but the terms of the orders $z^4/n+z^5/n^2+z^6/n^3+\cdots$. If these terms are treated as error terms, we need $n^2z^3 \gg z^4/n$. Hence we need $|z| \ll n^3$ which is stronger than the assumptions in Lemma A.1. The term $nz\eta^{(3)}_{1-2}$ will not be treated by normal form. It will be handled by a robust initial layer argument in Step 3. We now identify terms of orders $z^4/n+z^5/n^2+z^6/n^3+\cdots$.

Recall the first equation for $\dot p$ in (A.3). Since $u_- = O(n^2)$ is small, $(u_-, \bar F)$ does not contain such terms. In (u_+, F) we have (u_+, F_1), with

$$F_1 = 3\lambda QR^2a^2 + \lambda(aR+\zeta)^2\big(aR+\bar\zeta\big) - \lambda\zeta^2\bar\zeta. \tag{A.17}$$

Since $a = O(z^2)$ and $u_+ = O(1)$, (u_+, F_1) is of order $z^4/n+z^5/n^2+z^6/n^3$. Also, the main term in $\dot\theta$ is

$$\begin{aligned}\dot\theta &= -\big[a + \big(c_1R, \operatorname{Re} F\big)\big]\cdot\big[1 + a\big(c_1R, R\big) + \big(c_1R, \operatorname{Re} h\big)\big]^{-1}\\ &= -\big[a + \big(c_1R, \operatorname{Re} F_2\big)\big]\cdot\big[1 + b\big(c_1R, R\big) + \big(c_1R, u_+\big)\operatorname{Re} z\big]^{-1} + (\text{error}),\end{aligned} \tag{A.18}$$

where F_2 is the part of F without η:

$$F_2 = \lambda Q\big(2|\zeta|^2+\zeta^2\big) + 2\lambda QRa\big(2\zeta+\bar\zeta\big) + 3\lambda QR^2a^2 + \lambda(aR+\zeta)^2\big(aR+\bar\zeta\big). \tag{A.19}$$

Since $h = \zeta+\eta = zu_+ + \eta + O(n^2z)$ and $a = b + O(n^2z^2)$, the last part of $\dot p$-equation

$$\begin{aligned}&\big[\big(u_+, h\big) + \big(u_-, \bar h\big) + (u, R)a\big]\dot\theta\\ &\quad = \big[\big(u_+, \zeta\big) + \big(u_-, \bar z u_+\big) + (u, R)b\big]\dot\theta + (\text{error}) = P + (\text{error}),\end{aligned} \tag{A.20}$$

where

$$P = \big[\big(u_+, \zeta\big) + \big(u_-, \bar z u_+\big) + (u, R)b\big]\cdot\frac{-\big[a + \big(c_1R, \operatorname{Re} F_2\big)\big]}{1 + b\big(c_1R, R\big) + \frac{1}{2}\big(c_1R, u_+\big)\big(z+\bar z\big)}. \tag{A.21}$$

Hence in the equation of $\dot p$, the terms of order $z^4/n+z^5/n^2+\cdots$ are collected in

$$-ie^{i\kappa t}\big[\big(u_+, F_1\big) + P\big]. \tag{A.22}$$

Observe that there is no η in the above expression. Focus on $-ie^{i\kappa t}P$. The denominator of P is of the form $1 + c_1 b/n^2 + c_2 z/n + c_2\bar{z}/n$ with real coefficients c_1 and c_2 of order one. The numerator can be written as $z^3 f_1 + z^2\bar{z} f_2 + z\bar{z}^2 f_3 + \bar{z}^3 f_4 + bzf_5 + b\bar{z}f_6 + n^{-1}b^2 f_7$, where f_j, $j = 1, \ldots, 7$, are polynomials of z/n, $\bar{z}/n$ and b/n^2 with real coefficients of order one. We can now rewrite $-ie^{i\kappa t}[(u_+, F_1) + P]$ as a series,

$$
\begin{aligned}
-ie^{i\kappa t}\big[(u_+, F_1) + P\big] &= \sum_{\substack{k,l,j=0\\ k+l+2j\ge 3}}^{\infty} c_{k,l,j} e^{i\kappa t} z^k \bar{z}^l b^j \\
&= \sum_{\substack{k,l,j=0\\ k+l+2j\ge 3}}^{\infty} c_{k,l,j} e^{i(1-k+l)\kappa t} p^k \bar{p}^l b^j,
\end{aligned} \tag{A.23}
$$

with purely imaginary $c_{k,l,j}$ satisfying $|c_{k,l,j}| \le Cn^{3-k-l-2j}$. Those terms with $1-k+l \ne 0$ are nonresonant and we can integrate them:

$$
\begin{aligned}
e^{i(1-k+l)\kappa t} p^k \bar{p}^l b^j &= \frac{d}{dt}\left(\frac{1}{i(1-k+l)\kappa} e^{i(1-k+l)\kappa t} p^k \bar{p}^l b^j\right) \\
&\quad - \frac{1}{i(1-k+l)\kappa} e^{i(1-k+l)\kappa t} \frac{d}{dt}\big(p^k \bar{p}^l b^j\big).
\end{aligned} \tag{A.24}
$$

The error term $(d/dt)(p^k \bar{p}^l b^j)$ increases an order $nz + nX/z^2$ where $X = nz\eta + z^2\eta + n\eta^2 + (\eta^3)_{\mathrm{loc}}$. Hence

$$
-ie^{i\kappa t}\big[(u_+, F_1) + P\big] = fp + \frac{d}{dt}p_+ + g_+, \tag{A.25}
$$

with

$$
\begin{aligned}
f &= \sum_{l,j=0,\ldots,\infty} c_{l,l+1,j} |p|^{2l} b^j, \\
p_+ &= \sum_{1-k+l\ne 0, j=0,\ldots,\infty} \frac{c_{k,l,j}}{i(1-k+l)\kappa} e^{i(1-k+l)\kappa t} p^k \bar{p}^l b^j, \\
g_+ &= \sum_{1-k+l\ne 0, j=0,\ldots,\infty} \frac{c_{k,l,j}}{i(1-k+l)\kappa} e^{i(1-k+l)\kappa t} \frac{d}{dt}\big(p^k \bar{p}^l b^j\big).
\end{aligned} \tag{A.26}
$$

This series converge since $|c_{k,l,j}| \le Cn^{3-k-l-2j}$. Moreover, $f = f(t)$ is purely imaginary. This shows that all the terms contributing to $z^4/n + z^5/n^2 + z^6/n^3 + \cdots$ can either be integrated by parts or are of the correct forms.

Finally, we collect all the integrations by parts and obtain the normal form. We can check that q is of the form

$$
q = p + nz^2 + z^3 + bz + n^2 z^3 + \frac{z^4}{n} + \frac{z^5}{n^2} + \frac{z^6}{n^3} + \cdots \tag{A.27}
$$

and thus satisfies the estimate (A.9). Similarly, we can obtain the normal form of b and the estimate for $\widetilde{b}$. The only new terms in $\dot{b}$ are of order $n(z^4/n + z^5/n^2 + \cdots)$, which can be integrated as those $O(z^4/n)$ terms in $\dot{p}$ equation. Also, terms of the form $nz^2\eta^{(3)}$ in the $\dot{b}$ equation are not integrated (but will be treated in Step 3). We conclude the lemma. ∎

Step 3 (initial layer). We first recall the continuity argument in [16]: with respect to a fixed E, define

$$M(T) := \sup_{0\le t\le T} \Big\{ \{t\}^{1/2}|z(t)| + \{t\}^{3/4-\sigma}\|\eta(t)\|_{L^4} + n^{\sigma}\{t\}\|\eta(t)\|_{L^2_{loc}} + \varepsilon^{\sigma}(\varepsilon n)^{-3/4}\{t\}^{9/8}\|\eta^{(3)}_{3-5}(t)\|_{L^2_{loc}} \Big\}, \tag{A.28}$$

where $\sigma = 1/100$, $\{t\}$ is defined in (4.8) and $\eta^{(3)}_{3-5}$ is defined in [16, (4.8)]. Since this term is of lower order, it can be estimated in a rather simple way as in [16] and we shall not repeat it here.

Let $D = 4(Q,R)^{-1}$. We shall assume that $a(T) = 0$ (i.e., we fix $E = E(T)$) and

$$M(T) \le 2, \quad |a(t)| \le D\{t\}^{-1}, \ t \le T. \tag{A.29}$$

Under this assumption, we shall derive in steps 3 and 4 that

$$M(T) \le \frac{3}{2}, \quad |a(t)| \le \frac{D}{2}\{t\}^{-1}, \ t \le T. \tag{A.30}$$

From the standard continuity argument, we then conclude that (A.29) holds for all time provided that it holds for $T = 0$. At $T = 0$, (A.29) is just the assumption of Theorem 4.1. Notice that the main term in $M(T)$ is $\sup\{t\}^{1/2}|z(t)| \sim 1$. The others are of order $o(1)$.

We first show that, for $t < \delta t$, the sizes of $z(t)$ and $a(t)$ do not change much so that (A.30) still holds. Under the assumption (A.30), we can check that the assumption for Lemma A.1 holds and thus we can use its conclusions. Using (A.13) and (A.29), we have for all $t \ge 0$,

$$|g| \lesssim (\varepsilon^{1-\sigma} + n^2)n^2\{t\}^{-3/2} + n\{t\}^{-1/2}\Big(\|\eta^{(3)}_1\|_{L^2_{loc}} + \|\eta^{(3)}_2\|_{L^2_{loc}} + \|\eta^{(3)}_{3-5}(t)\|_{L^2_{loc}}\Big). \tag{A.31}$$

By the explicit form of $\eta^{(3)}_2$ (see [16, pages 174, 181]), we have

$$\|\eta^{(3)}_2\|_{L^2_{loc}} \le C\varepsilon^2 n^3(1+t)^{-9/8}. \tag{A.32}$$

For $t \in [0, \delta t]$ with $\delta t = n^{-1}$, $\{t\}^{-1/2} \le C\varepsilon n$ and $\|\eta_1^{(3)}\|_{L^2_{loc}} \le Cn^{-\sigma}(\varepsilon^2 n^2)$ by (4.6). Together with (A.31) and (A.32),

$$\left|g(t)\right| \lesssim \left(\varepsilon^{1-\sigma} + n^2\right)\varepsilon^3 n^5 + \varepsilon n^2\left[n^{-\sigma}\left(\varepsilon^2 n^2\right) + \varepsilon^2 n^3 (1+t)^{-9/8}\right] \lesssim \varepsilon^3 n^{4-\sigma}. \tag{A.33}$$

Since

$$\frac{d}{dt}\left|q(t)\right| = \mathrm{Re}\left(\frac{\dot{q}\bar{q}}{|q|}\right) = \left(\mathrm{Re}\,\delta_{21}\right)q^3 + \mathrm{Re}\left(\frac{g\bar{q}}{|q|}\right) \tag{A.34}$$

with $\mathrm{Re}\,\delta_{21} = O(n^2)$, we have

$$\left|\left|q(t)\right| - \left|q(0)\right|\right| \le \int_0^{\delta t} C\varepsilon^3 n^{4-\sigma}\, d\tau \le C\varepsilon^3 n^{4-\sigma} n^{-1} \ll \varepsilon n. \tag{A.35}$$

Hence $||z(t)| - |z(0)|| \ll \varepsilon n \sim |z(0)|$. Similarly,

$$\begin{aligned}\left|\left|\widetilde{b}(t)\right| - \left|\widetilde{b}(0)\right|\right| &\le C\int_0^{\delta t} \left|B_{22}\right||z|^4 + \left|B_5\right| d\tau \\ &\le C\int_0^{\delta t} n^2(\varepsilon n)^4 + (\varepsilon n)^2\left[n^{-\sigma}\left(\varepsilon^2 n^2\right) + \varepsilon^2 n^3(1+\tau)^{-9/8}\right] d\tau \\ &\le C\varepsilon^4 n^6 n^{-1} + C\varepsilon^4 n^{4-\sigma} n^{-1} + C\varepsilon^4 n^5 \ll \varepsilon^2 n^2.\end{aligned} \tag{A.36}$$

Hence $||a(t)| - |a(0)|| \ll \varepsilon^2 n^2$. Therefore (A.30) holds for $t \le \delta t$.

Step 4 (after initial layer). For $t > \delta t = n^{-1}$, both (4.6) and (4.7) hold. We also have $\|\eta_2^{(3)}\|_{L^2_{loc}} \le C\varepsilon^2 n^3 (1+t)^{-9/8} \ll n\{t\}^{-1}$. Let $\rho(t) = \{t\}^{-1/2}$. By (A.31),

$$\left|g(t)\right| \ll n^2\rho^3(t), \quad (t > \delta t). \tag{A.37}$$

Thus we have $|q(\delta t)| \le \rho(\delta t)$ and

$$\frac{d}{dt}|q| \le \left(\mathrm{Re}\,\delta_{21}\right)|q|^3 + |g|, \quad \frac{d}{dt}\rho \ge \left(\mathrm{Re}\,\delta_{21}\right)|\rho|^3 + |g|, \quad t > \delta t. \tag{A.38}$$

By comparison principle, we have $|q(t)| \le \rho(t)$. Hence (A.30) holds.

We have handled the term $nz\eta_{1-2}^{(3)}$ in the $\dot{p}$ equation very differently from [16]. Since η always has contribution from the bound states, this term is comparable to the resonance term $n^2|z|^2 z$. Our key observation is that the decay property (4.6) and an initial layer argument control the dynamics in this initial layer. After the initial layer, $\eta_{1-2}^{(3)}$ becomes negligibly small.

Acknowledgments

The work of T.-P. Tsai is partially supported by the National Science Foundation (NSF) grant DMS-9729992. The work of H.-T. Yau is partially supported by NSF grant DMS-0072098.

References

[1] V. S. Buslaev and G. S. Perel′man, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear Evolution Equations, American Mathematical Society, Rhode Island, 1995, pp. 75–98.

[2] V. S. Buslaev and C. Sulem, *On the asymptotic stability of solitary waves of nonlinear Schrödinger equations*, preprint, 2002.

[3] S. Cuccagna, *On asymptotic stability of ground states of NLS*, preprint, 2001.

[4] ———, *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **54** (2001), no. 9, 1110–1145.

[5] M. Grillakis, *Linearized instability for nonlinear Schrödinger and Klein-Gordon equations*, Comm. Pure Appl. Math. **41** (1988), no. 6, 747–774.

[6] ———, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system*, Comm. Pure Appl. Math. **43** (1990), 299–333.

[7] J.-L. Journé, A. Soffer, and C. D. Sogge, *Decay estimates for Schrödinger operators*, Comm. Pure Appl. Math. **44** (1991), no. 5, 573–604.

[8] C.-A. Pillet and C. E. Wayne, *Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations*, J. Differential Equations **141** (1997), no. 2, 310–326.

[9] H. A. Rose and M. I. Weinstein, *On the bound states of the nonlinear Schrödinger equation with a linear potential*, Phys. D **30** (1988), 207–218.

[10] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), no. 2, 173–190.

[11] ———, *Spectral condition for instability*, Nonlinear PDE′s, Dynamics and Continuum Physics (South Hadley, Mass, 1998), Contemp. Math., vol. 255, American Mathematical Society, Rhode Island, 2000, pp. 189–198.

[12] I. M. Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions*, Comm. Math. Phys. **153** (1993), 297–320.

[13] A. Soffer and M. I. Weinstein, *Multichannel nonlinear scattering theory for nonintegrable equations I*, Comm. Math. Phys. **133** (1990), 119–146.

[14] ———, *Multichannel nonlinear scattering theory for nonintegrable equations II*, J. Differential Equations **98** (1992), 376–390.

[15] ———, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math. **136** (1999), 9–74.

[16] T.-P. Tsai and H.-T. Yau, *Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions*, Comm. Pure Appl. Math. **55** (2002), 153–216.

[17] ———, *Stable directions for excited states of nonlinear Schrödinger equations*, to appear in Comm. Partial Differential Equations.

[18] K. Yajima, *The* $W^{k,p}$*-continuity of wave operators for Schrödinger operators*, J. Math. Soc. Japan **47** (1995), no. 3, 551–581.

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