The Hopf Fibration

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by

Santanil Jana

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Dr. Somnath Basu

to the

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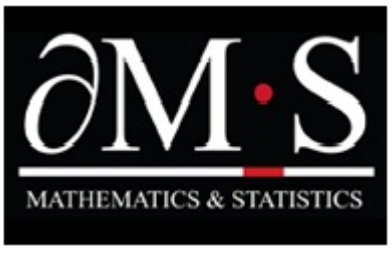

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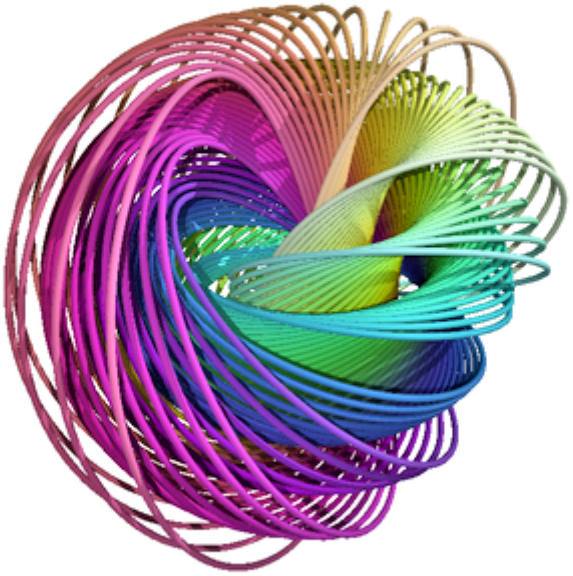

"A Picture is Worth a Thousand Words"

Declaration

I hereby declare that this thesis is my own work and to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Kolkata or any other educational institution, except where due acknowledgement is made in the thesis.

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Abstract

In this thesis, we will give a brief introduction to homotopy groups and fibrations and discuss the influence of the Hopf map in the development of higher homotopy theory. We will also introduce the notion of Hopf invariant to get a better understanding of the elements of the third homotopy group of the 2-sphere. A detailed discussion about the geometry of the Hopf fibration is also presented. In the course of studying Hopf fibration, we will discuss many important concepts of Algebraic and Differential Topology, namely the Cup products, Orientability, de Rham Cohomology, the degree of smooth maps and so on. Finally, we will establish the notion of linking number in terms of degree and illustrate the fact that the fibers of the Hopf fibration are linked once. We will calculate their linking number rigorously as well as try to give a pictorial argument in favor our calculated result.

Keywords: Hopf Fibration, Hopf Invariant, Homotopy Groups, Degree of a Smooth Map, Linking Number.

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Preface

The sole purpose of this thesis is to study the Hopf map and the related notions. The Hopf map was first introduced by Heinz Hopf in his paper [1]. Historically the Hopf map is quite remarkable as it was the first example of a map from a higher dimensional sphere to a lower dimensional sphere, which is not null-homotopic. At the time of the discovery of Hopf map very little was known about the higher homotopy groups of the sphere as it was extremely difficult to calculate them. Even it was not known whether they are trivial or not. The Hopf map, defined from $\mathbb{S}^3$ to $\mathbb{S}^2$, showed that $\pi_3(\mathbb{S}^2)$ is non-trivial. This result really kick-started the development of modern homotopy theory, which revolves mostly around the calculation of higher homotopy groups.

The Hopf map has a plethora of physical applications, especially in the fields of rigid body mechanics [9], quantum information theory [11] and magnetic monopoles [4]. The Hopf map is often called the Hopf fibration. The reason it is called a fibration will be clear in Chapter-1 when we will show that the Hopf map is a fiber bundle over $\mathbb{S}^2$ with fiber $\mathbb{S}^1$. There is a much more general fiber bundle $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$, with fiber $\mathbb{S}^1$. The $n = 1$ case is the one that was introduced by Heinz Hopf in 1931. In the first chapter, We will mostly discuss these topics and the importance of Hopf map in homotopy theory.

In the second chapter, we will review some basic materials from singular cohomology and use cup products to define Hopf invariant. The Hopf invariant will give us an one to one correspondence between the elements of $\pi_3(\mathbb{S}^2)$ and the integers. Later we will see another interpretation of Hopf invariant using linking numbers in the fourth chapter.

Next, we will uncover the geometric significance of the Hopf fibration. Apart from its application in homotopy theory, the Hopf fibration also gives us a way of viewing the 3-sphere as a collection of circles arranged in a special way and parametrized by points on a 2-sphere. In the last two chapters, we will mostly

try to illustrate this fact. We will try to see the spatial arrangement of the fibers of the Hopf map inside the 3-sphere, which will be portrayed as $\mathbb{R}^3 \cup \{\infty\}$ to make our visualization easier.

The third chapter will be mostly devoted to the development of the theory needed to study the fibers of the Hopf map. This mostly is related to defining the notion of the degree of a smooth map. In the fourth chapter, we will introduce the notion of linking number using degree, which will be key in our study of the geometry of the Hopf fibration.

We will assume that the reader has some exposure to basic Algebraic and Differential Topology. An ideal prerequisite for fully understanding the materials presented in this thesis would be the first two chapters of [2] and chapter 1 and 4 of [16]. We will try to give most of the proofs of the results that we state in this thesis, but sometimes due to some technical difficulties, we will skip some proofs and give appropriate reference for the readers.

Chapter 1

Hopf Map in Homotopy Theory

The Hopf map is what showed that the homotopy groups can be interesting and non-trivial. The fundamentals of homotopy theory lie in the computation of homotopy groups $\pi_k(\mathbb{S}^n)$, $k \geqslant n$. Calculating the higher homotopy groups is not at all easy and there is no universal method for calculating these groups. In his paper [1], Hopf showed that there is a continuous surjective map, called the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$, which is not null-homotopic. As a consequence, one has $\pi_3(\mathbb{S}^2) \neq 0$. Later it has been proved using fibrations that $\pi_3(\mathbb{S}^2)$ is infinite cyclic, generated by the Hopf map. The existence of this map first showed that $\pi_k(\mathbb{S}^n)$ can be non-trivial for $k > n$, which is completely opposite to the case of homology groups, where we have $H_k(\mathbb{S}^n) = 0$ for $k > n$. In this chapter, we will discuss homotopy theory and the influence of Hopf map in developing this intriguing theory.

1.1 Higher Homotopy Groups

A basic course in Algebraic Topology typically begins with the definition of Fundamental Group. However, after the basic definitions, examples, and theorems (e.g. Van Kampen and covering space) the focus diverges towards Homology theory. The natural generalization of the Fundamental Group, obtained by replacing $\mathbb{S}^1$ by $\mathbb{S}^n$ in the definition, is rarely discussed despite the fact that it has motivated a large part of modern Algebraic Topology. The reason behind this can be the very complex nature of these groups. To make our job a little easy,

we will mostly structure our development of the theory around the motivating example of the homotopy groups of the n-spheres.

Definition 1.1. (Definition of $\pi_n(X)$) For a space X with basepoint $x_0 \in X$, the n-th homotopy group of X based at x_0, denoted as $\pi_n(X)$, is the set of all homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$ such that the homotopies are required to satisfy $f_t(\partial I^n) = x_0$ for all $t \in I$.

There is an extension of the definition to the case $n = 0$, where $\pi_0(X)$ is defined to be the set of all homotopy classes of all maps from I^0 (which is a single point) to X, which is just the set of path components of X. Although, we are calling this a group, we have not yet defined an operation in $\pi_n(X)$. Let us define the operation $'+'$ for maps $f, g : (I^n, \partial I^n) \to (X, x_0)$ for $n \geqslant 2$.

$$(f + g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \ldots, s_n) & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

The reason we are denoting the operation as $'+'$ will be clear in a while. Note that it is clear that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for $\pi_1(X, x_0)$ show that $'+'$ is an well defined operation on $\pi_n(X, x_0)$ generalizing the concatenation in $\pi_1(X, x_0)$. Moreover $\pi_n(X, x_0)$ is a group with identity element the constant map c_{x_0} sending I^n to x_0 and with inverses given by $-f(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n)$. This can be proved using the same techniques used in case of $\pi_1(X, x_0)$.
The additive notion for the group operation is used because unlike $\pi_1(X, x_0)$, $\pi_n(X, x_0)$ is abelian for $n \geqslant 2$.

Proposition 1.1. *The group $\pi_n(X, x_0)$ is abelian for $n \geqslant 2$.*

Proof. Let $[f], [g] \in \pi_n(X, x_0)$. our aim is to show that $f+g \simeq g+f$. One can write an explicit homotopy between the maps to verify the claim. Instead, we give a pictorial argument to prove this. First, we restrict ourselves to two dimensions. The homotopy begins by shrinking the domains of f and g to smaller sub-cubes of I^2, with the boundary of these sub-cubes mapping to the base-point. Now we have the room to slide the two sub-cubes around as long as they remain disjoint. So, they can be slid past each other to interchange their positions. Then again f and g can be enlarged to their original size. So, a homotopy between $f + g$ and

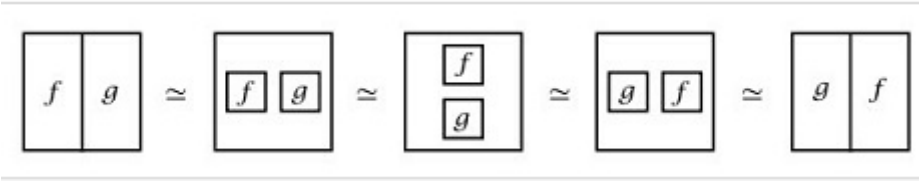

FIGURE 1.1: Homotopy between $f + g$ and $g + f$
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$g + f$ can be pictorially represented as in Figure 1.1

■

Recall that, there is an alternative description of $\pi_1(X, x_0)$ as the set of all homotopy classes of maps $f : (\mathbb{S}^1, 1) \to (X, x_0)$, where $f_t(1) = x_0$ for all $t \in I$. The concatenation operation is defined by,

$$f * g : \mathbb{S}^1 \xrightarrow{c} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{f \vee g} X$$

where the first map c is the "pinch" obtained by collapsing two antipodal points of $\mathbb{S}^1$ together, and the second map is given by taking f on the first factor and g on the second factor (it is well-defined because the point in common of the two $\mathbb{S}^1$ factors is the basepoint where f and g take the same value).
This has a generalization for $n \geqslant 2$, giving us an alternative definition of $\pi_n(X, x_0)$. Let $p \in \mathbb{S}^n$ be the north-pole.

Definition 1.2. $\pi_n(X, x_0)$ is defined to be the set of all homotopy classes of maps $f : (\mathbb{S}^n, p) \to (X, x_0)$, where $f_t(p) = x_0$ for all $t \in I$. The sum operation is defined by the composition,

$$f + g : \mathbb{S}^n \xrightarrow{c} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{f \vee g} X$$

where the first map c is the "pinch" obtained by collapsing the equator($\mathbb{S}^{n-1}$) of $\mathbb{S}^n$ to a point, and the second map is given by taking f on the first factor and g on the second factor.

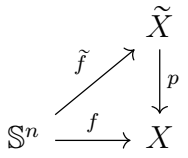
Note that as $I^n/\partial I^n \cong \mathbb{S}^n$, we see that both definitions are equivalent (we choose base-point $p = \partial I^n/\partial I^n$).
In all of the discussions above we have fixed a chosen base-point for X. We have seen that changing the base-point within same component yields isomorphic fundamental groups. For higher homotopy groups, the choice of base-point within a connected component is also irrelevant so that we very well omit the

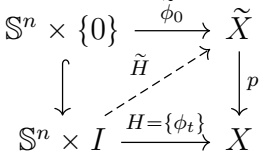
base-point. We will not prove this here. Interested readers can find a detailed discussion about this topic in Chapter-4 of [2]. The proof is a simple generalization of the proof for the case of $\pi_1(X, x_0)$.

Next we observe the fact that π_n is a functor. Given a map $\varphi : (X, x_0) \to (Y, y_0)$, there is a well-defined induced map $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$, given by $\varphi_*[f] = [\varphi \circ f]$. It is immediate that φ_* is a homomorphism for $n \geqslant 1$ and $(\varphi\psi)_* = \varphi_*\psi_*$ and $id_* = id$. Also if $\varphi : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then using the homotopy inverse ψ, we see that $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism.
The spaces which have a contractible universal covering space have trivial higher homotopy groups. The next proposition illustrates this fact.

Proposition 1.2. *A covering space projection $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces an isomorphism $p_* : \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0)$ for $n \geqslant 2$.*

Proof. We will show that p_* is a bijection when $n \geqslant 2$.
CLAIM-1: p_* is surjective.
proof. (CLAIM-1) Using the lifting criterion of covering space and the fact that $\mathbb{S}^n$ is simply-connected for $n \geqslant 2$, we get a lift $\widetilde{f} : \mathbb{S}^n \to \widetilde{X}$ for every map $f : \mathbb{S}^n \to X$. So, given $[f] \in \pi_n(X, x_0)$ we have $p_*[\widetilde{f}] = [p \circ \widetilde{f}] = [f]$(By definition of lift). This shows that p_* is surjective when $n \geqslant 2$.





CLAIM-2: p_* is injective.
proof. (CLAIM-2) Let $[f] \in ker(p_*) \subset \pi_n(\widetilde{X}, \widetilde{x}_0)$, i.e. $[p \circ \widetilde{f}] = [c_{x_0}]$. So, $p \circ \widetilde{f} \simeq c_{x_0}$. By general homotopy lifting property, $\widetilde{f} \simeq c_{\widetilde{x}_0}$.(See the figure above). Hence p_* is injective.
Therefore, by CLAIM-1 and CLAIM-2, p_* is a bijection and hence an isomorphism. ■

Example 1.1. *Let us look at some immediate applications of the above proposition.*

1. *$\pi_n(X, x_0) = 0$ for $n \geqslant 2$ whenever X has a universal cover. In covering space theory, we have seen that $exp : \mathbb{R} \to \mathbb{S}^1$ is a universal covering space. Hence by Proposition 1.2, $\pi_n(\mathbb{S}^1, 1) = 0$ for $k \geqslant 2$.*

2. *$\mathbb{R}^n$ is a universal cover of the n-torus $\mathbb{T}^n$. So, $\pi_k(\mathbb{T}^n) = 0$ for $n \geqslant 2$.*

3. *Let S_g be the surface of genus $g > 1$. From The Uniformization theorem of Koebe and Poincaré of [6] we know that the universal covering space of S_g is the upper half plane $H \subset \mathbb{C}$. Hence by Proposition 1.2, $\pi_n(S_g, x_0) = 0$ for $n \geqslant 2$.*

Proposition 1.3. *$\pi_n(X_1 \times X_2, (x_1, x_2)) \cong \pi_n(X_1, x_1) \times \pi_n(X_2, x_2)$ for all $n \geqslant 1$.*

Proof. The key fact is to observe that every map $\varphi : (\mathbb{S}^n, p) \to (X_1 \times X_2, (x_1, x_2))$ is given by $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i : (\mathbb{S}^n, p) \to (X_i, x_i)$ for $i = 1, 2$. Then $[\varphi] \mapsto ([\varphi_1], [\varphi_2])$ is an isomorphism. The map is clearly surjective. Note that if $\varphi = (\varphi_1, \varphi_2) \simeq \psi = (\psi_1, \psi_2)$ in $X \times Y$ via a homotopy $H = (H_1, H_2) : \mathbb{S}^n \times I \to X_1 \times X_2$, then H_i is a homotopy between φ_i and ψ_i in X_i for $i = 1, 2$. Hence, the map is well-defined and one-one. ■

From elementary algebraic topology and Proposition 1.2, we know all homotopy groups of $\mathbb{S}^1$. Although all homotopy groups of higher dimensional spheres are not known, it is not too difficult to show $\pi_k(\mathbb{S}^n) = 0$ for $k < n$. The essential idea is to make use of the CW complex structure of $\mathbb{S}^n$ (Example A.1(1)).

Definition 1.3. (Cellular Map) A map $f : X \to Y$ between CW complexes is said to be cellular if $f(X^k) \subset Y^k$ for all $k \geqslant 0$.

There is a similarity between cellular maps and linear maps in that they do not increase the dimension. In case of linear maps, the dimension of the range is always less than or equal to that of the domain. Similarly, cellular maps always map the k-skeleton to the k-skeleton. This is a strong form of not increasing dimension as the image of k-skeleton do not even touch the higher dimensional cells.
There are plenty of maps which are non-cellular. The cellular approximation theorem ensures that they are not too far from a cellular map.

Theorem 1.1. *(Cellular Approximation Theorem) Every map $f : X \to Y$ between CW complexes is homotopic to a cellular map. If f is already cellular in a subcomplex $A \subset X$, then the homotopy can be taken to be stationary on A.*

The proof is quite technical and we will not present it here. The interested reader can find the details in Chapter-4 of [2]. Assuming this result, we then immediately have:

Corollary 1.1. $\pi_k(\mathbb{S}^n, p) = 0$ *if* $k < n$.

Proof. Let $[f] \in \pi_k(\mathbb{S}^n, p)$, i.e. $f : \mathbb{S}^k \to \mathbb{S}^n$. By Cellular Approximation Theorem, f is homotopic to a cellular map g. As g is cellular, by definition $g((\mathbb{S}^k)^k) \subset (\mathbb{S}^n)^k$. From the CW structure of $\mathbb{S}^n$ (See Example A.1(1)), k-skeleton of $\mathbb{S}^k$ is $\mathbb{S}^k$ and k-skeleton of $\mathbb{S}^n$ is $\{p\}$ (Assume that 0-cell of $\mathbb{S}^n$ is $\{p\}$). Hence $g(\mathbb{S}^k) = \{p\}$ and $g = c_p$. This shows that $[f] = [c_p]$ and hence $\pi_k(\mathbb{S}^n, p)$ is trivial. ∎

Corollary 1.2. *Let* X *be a CW complex. Then* $\pi_n(X) = \pi_n(X^{n+1})$

Proof. By Cellular Approximation Theorem, any map $f : \mathbb{S}^n \to X$ is homotopic to a cellular map. Let $f \simeq g$ where $g(\mathbb{S}^n) \subset X^n \subset X^{n+1}$. So, $\pi_n(X)$ is the set of maps $\mathbb{S}^n \to X^{n+1}$ modulo homotopies through maps $\mathbb{S}^n \to X$.
Now consider a homotopy $H : \mathbb{S}^n \times I \to X$. This is homotopic, by a second application of Cellular Approximation, to a map which takes values in X^{n+1} (since $\mathbb{S}^n \times I$ is naturally an $(n+1)$-dimensional CW complex). Therefore $\pi_n(X)$ is equal to the set of maps $\mathbb{S}^n \to X^{n+1}$ modulo homotopies through maps $\mathbb{S}^n \to X^{n+1}$, which is just $\pi_n(X^{n+1})$. ∎

1.2 Fibrations and Long Exact Sequence of Homotopy Groups

In this section we will introduce two important classes of maps, namely the *Hurewicz* fibration and *Serre* fibration. Moreover, associated to a Serre fibration we will obtain a long exact sequence of homotopy groups of the fibre, total space and the base space.

Definition 1.4. (Right Lifting Property) A map $p : E \to B$ of spaces is said to have the right lifting property (RLP) with respect to a map $i : A \to X$ if for any two maps $f : A \to E$ and $g : X \to B$ with $p \circ f = g \circ i$, there exists a map $h : X \to E$ with $p \circ h = g$ and $h \circ i = f$:

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array}$$

(So, h extends f and lifts g at the same time)

Definition 1.5. (Serre & Hurewicz Fibration) A map $p : E \to B$ of spaces is called a Serre Fibration if it has the RLP with respect to all inclusions of the form $i : I^n \times \{0\} \hookrightarrow I^n \times I$, $n \geqslant 0$ and a Hurewicz fibration if it has the RLP with respect to all maps of the form $i : A \times \{0\} \hookrightarrow A \times I$, $n \geqslant 0$ for any space A. (So, every Hurewicz fibration is a Serre fibration)

Definition 1.6. (Fiber) If $p : E \to B$ is a map of spaces and $b \in B$, then $p^{-1}(b) \subset E$ is called the fiber of p over b.

Thus, Hurewicz fibrations are those maps $p : E \to B$ which has the **homotopy lifting property** with respect to all spaces: given a homotopy $H : A \times I \to B$ of maps with target B (Let's say $H = \{\varphi_t\}$) and a lift $\widetilde{\varphi}_0 : A \times \{0\} \to E$ of $\varphi_0 = H(-, 0) : A \to B$ there is a lift $\widetilde{H}$ of the entire homotopy H which satisfies $p \circ \widetilde{H} = H$ and $\widetilde{H} \circ i = \widetilde{\varphi}_0$.

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\widetilde{\varphi}_0} & E \\ \downarrow{\scriptstyle i} & \overset{\widetilde{H}}{\nearrow} & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow{H=\{\varphi_t\}} & B \end{array}$$

Proposition 1.4. *For a Hurewicz fibration $p : E \to B$, the fibers are homotopy equivalent if the base B is path-connected.*

Proof. Let γ be a path in B. Then define $G : F_{\gamma(0)} \times I \to B$ by $G(x, t) = \gamma(t)$. Then the inclusion $i_1 : F_{\gamma(0)} \hookrightarrow E$ provides a lift $\widetilde{G}_0$, so by homotopy lifting property we have a lift $\widetilde{G} : F_{\gamma(0)} \times I \to E$ of the homotopy G with $p \circ \widetilde{G} = G$. So, $\widetilde{G}(x, 1) \in p^{-1}(G(x, 1)) = p^{-1}(\gamma(1)) = F_{\gamma(1)}$. Let us define $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ by setting $L_\gamma(x) = \widetilde{G}(x, 1)$.

$$\begin{array}{ccc} F_{\gamma(0)} \times \{0\} & \xrightarrow{i_1} & E \\ \downarrow{\scriptstyle i} & \overset{\widetilde{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{\gamma(0)} \times I & \xrightarrow{G} & B \end{array}$$

This map has certain properties:

1. If $\gamma \simeq \gamma'$ rel ∂I, then $L_\gamma \simeq L_{\gamma'}$.
   proof. Let H be a homotopy between γ and γ'. Let us set $G(x,s) = \gamma(s)$ and $G'(x,s) = \gamma'(s)$ to be the defining maps for L_γ and $L_{\gamma'}$ respectively. Define $h : F_{\gamma(0)} \times I \times I \to B$ by $h(x,s,t) := H(s,t)$. Then $h(x,s,0) = H(x,0) = \gamma(s) = G(x,s)$ and $h(x,s,1) = H(x,1) = \gamma'(s) = G'(x,s)$. We want to use the lifting property with respect to space suitable for our purpose and which will also be homeomorphic to $F_{\gamma(0)} \times \{0\} \times I$. Let us define $J = F_{\gamma(0)} \times (\{0\} \times I \cup I \times \partial I)$. Since $(\{0\} \times I \cup I \times \partial I)$ is homeomorphic to $\{0\} \times I$, the same is true after taking product with $F_{\gamma(0)}$. As, p has the RLP with respect to $i : F_{\gamma(0)} \times \{0\} \times I \to F_{\gamma(0)} \times I \times I$, it also has the RLP with respect to the homeomorphic space J. Also let us define $j : J \to E$ by $j(x,s,0) = G(x,s)$; $j(x,s,1) = G'(x,s)$ and $j(x,0,t) = e_1$ for $e_1 \in F_{\gamma(1)}$.

$$\begin{array}{ccc} J & \xrightarrow{j} & E \\ \downarrow{\scriptstyle i'} & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{\gamma(0)} \times I \times I & \xrightarrow{h} & B \end{array}$$

   ($i' = i \circ$(homeomorphism between J & $F_{\gamma(0)} \times \{0\} \times I$))
   By RLP, h has a lift $\tilde{h}$ which extends j. So, $p \circ \tilde{h}(x,1,t) = h(x,1,t) = H(1,t) = \gamma(1)$ (As H is a homotopy rel ∂I). So, $\tilde{h}(x,1,t) \in p^{-1}(\gamma(1)) = F_{\gamma(1)}$. Define $\widetilde{H} : F_{\gamma(0)} \times I \to F_{\gamma(1)}$ by $\widetilde{H}(x,t) = \tilde{h}(x,1,t)$. Note that $\widetilde{H}(x,0) = \tilde{h}(x,1,0) = \widetilde{G}(x,1) = L_\gamma(x)$ (As $\tilde{h}$ extends j). Similarly, $\widetilde{H}(x,1) = \tilde{h}(x,1,1) = \widetilde{G'}(x,1) = L_{\gamma'}(x)$. Hence $\widetilde{H}$ is a homotopy between L_γ and $L_{\gamma'}$.

2. For a concatenation of paths $\gamma * \gamma'$, $L_{\gamma * \gamma'}$ is homotopic to $L_{\gamma'} \circ L_\gamma$.
   proof. This is true since for Lifts $\widetilde{G}$ and $\widetilde{G'}$ defining L_γ and $L_{\gamma'}$ we obtain a lift defining $L_{\gamma * \gamma'}$ by taking $\widetilde{G}(x,2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\widetilde{G'}(L_\gamma(x), 2t-1)$ for $\frac{1}{2} \leqslant t \leqslant 1$.

Let $b_0, b_1 \in B$ be arbitrary. Let γ be a path joining b_0 and b_1. Let $\overline{\gamma}$ be the inverse path. Note that $\gamma * \overline{\gamma} \simeq c_{b_0}$ rel ∂I. So, by (1) and (2) $L_\gamma \circ L_{\overline{\gamma}} \simeq L_{\overline{\gamma} * \gamma} \simeq L_{c_{b_0}} = id_{F_{b_0}}$. Similarly, $L_{\overline{\gamma}} \circ L_\gamma \simeq id_{F_{b_1}}$. ∎

From now on we will be concerned with Serre fibrations. By Proposition 1.4, we can talk about $\pi_n(F, e_0)$ without any ambiguity as long as B is path-connected

as we know that fibers of different points are homotopy equivalent and hence same at the level of π_n. Let us move on to the main theorem of this section.

Theorem 1.2. *(The long exact sequence of a Serre fibration) Let $p : (E, e_0) \to (B, b_0)$ be a map of pointed spaces with B path-connected and $(F, e_0) \hookrightarrow (E, e_0)$ being the fiber. Suppose that p is a Serre fibration. Then there is a long exact sequence of the form:*

$$\cdots \to \pi_{n+1}(B, b_0) \xrightarrow{\delta} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\delta} \pi_{n-1}(F, e_0) \xrightarrow{i_*} \cdots \xrightarrow{\delta} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0).$$

Here F, E, B are all pointed sets, hence their π_0 is also a pointed set. The definition of $kernel$ here is the pre-image of the chosen base-point and the definition of exactness is the same: $ker(p_*) = im(i_*)$.

Before going to the proof of the theorem, let us define a space $J^n \subset I^{n+1}$ by,

$$J^n = (I^n \times \{0\}) \cup (\partial I \times I) \subset \partial I^{n+1} \subset I^n$$

By flattening the sides of the cube, one can construct a homeomorphism of pairs

$$(I^{n+1}, J^n) \xrightarrow{\cong} (I^{n+1}, I^n \times \{0\})$$

Thus, any Serre fibration also has the Right Lifting Property(RLP) with respect to the inclusion $J^n \subset I^{n+1}$. We will use this fact quite often in the proof of Theorem 1.2.

Proof. (Theorem 1.2)
The proof of the theorem will mostly rely on repeated use of the homotopy lifting property. The essential step in the proof is to construct the map δ. Let $\alpha : (I^n, \partial I^n) \to (B, b_0)$ represent an element of $\pi_n(B, b_0)$. Let $c_{e_0} : J^{n-1} \to E$ be the constant map with value e_0. Then the square

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{c_{e_0}} & E \\ \downarrow{\scriptstyle i} & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle p} \\ I^n & \xrightarrow{\alpha} & B \end{array}$$

commutes. Hence by the definition of Serre fibration there is a $\beta : I^n \to E$ such that $p \circ \beta = \alpha$ and $\beta(J^{n-1}) = e_0$. Then define $\delta[\alpha]$ to be the element of $\pi_{n-1}(F, e_0)$ represented by the map

$$\beta(-, 1) : I^{n-1} \to F, t \mapsto \beta(t, 1)$$

This δ is often called the connecting homomorphism. Note that $\beta(-, 1)$ indeed represents an element of $\pi_{n-1}(F, e_0)$, because the boundary of $I^{n-1} \times \{1\}$ is contained in J^{n-1}, so $\beta(\partial(I^{n-1} \times \{1\})) = e_0$ and $\beta(I^{n-1} \times \{1\}) \in p^{-1}(\alpha(I^{n-1} \times \{1\})) = p^{-1}(b_0) = F$ (As $\alpha(\partial I^n) = b_0$). Before we proceed further into the proof, we need to verify that δ is well defined.

Lemma 1.1. *δ is well defined on homotopy classes.*

Proof. Let $[\alpha_0] = [\alpha_1]$ with $h : I^n \times I \to B$ being a homotopy between α_0 and α_1. Suppose also that we have chosen lifts β_0 and β_1 of α_0 and α_1 as above. Let $\widetilde{J^n}$ be the union of all faces of I^{n+1} except $\{t_n = 1\}$, i.e. it is same as J^n except the role of t_n and t_{n+1} is interchanged. Define $k : \widetilde{J^n} \to E$ by setting it β_0 and β_1 on $I^n \times \{0\}$ and $I^n \times \{1\}$ respectively. On the other faces set k to be constant e_0.

$$\begin{array}{ccc} \widetilde{J^n} & \xrightarrow{k} & E \\ \downarrow{\scriptstyle i} & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow{h} & B \end{array}$$

Now by homotopy lifting property, there is a diagonal $l : I^n \times I \to E$, which lifts h and extends k. Now consider $l' = l|_{I^{n-1} \times \{1\} \times I}$. then $p \circ l'(s, 1, t) = p \circ l(s, 1, t) = h(s, 1, t) = b_0$, because h is a homotopy relative to ∂I^n. So, image of l' lies entirely on $p^{-1}(b_0) = F$ and l' gives a homotopy between $\beta_0(-, 1)$ and $\beta_1(-, 1)$. This proves the lemma. ■

Exactness at $\pi_n(\mathbf{E}, \mathbf{e_0})$. $p \circ i : F \to B$ is constant b_0. So, $p_* \circ i_* = 0$ and hence $im(i_*) \subset ker(p_*)$. Suppose $\alpha : I^n \to E$ represents an element $\pi_n(E, e_0)$ such that $p_*[\alpha] = [p \circ \alpha] = 0$. Let $h : I^n \times I \to B$ be a homotopy relative to ∂I^n from $p \circ \alpha$ to the constant map c_{b_0}. Define $k : J^n \to E$ by $k|_{I^n \times \{0\}} = \alpha$ and k is constant e_0 on other faces.

$$\begin{array}{ccc} J^n & \xrightarrow{k} & E \\ \downarrow{\scriptstyle i} & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow{h} & B \end{array}$$

By homotopy lifting property, there is a lift l. Set $l' = l|_{I^n \times \{1\}}$. Note that $p \circ l'(s,1) = p \circ l(s,1) = h(s,1) = b_0$, because h is a homotopy between $p \circ \alpha$ to c_{b_0}. So, image of l' lies in $p^{-1}(b_0) = F$ and $l'(s,1) = e_0$ for $s \in \partial I^n$. We have $[l'] \in \pi_n(F, e_0)$ with $i_*[l'] = [i \circ l'] = [\alpha]$ by the homotopy l, which implies $ker(p_*) \subset im(i_*)$ and hence proves the exactness at $\pi_n(E, e_0)$.

Exactness at $\pi_n(\mathbf{B}, \mathbf{b_0})$. If $\beta : I^n \to E$ represents an element of $\pi_n(B, b_0)$ then for $\alpha = p \circ \beta$ we can take the same β as the lift in the construction of $\delta[\alpha]$. So, $\delta \circ p_*[\beta] = [\beta(-, 1)]$. But as $I^{n-1} \times \{1\} \subset \partial I^n$, $\beta(-, 1)$ is the constant map c_{e_0}. So, $\delta \circ p_* = 0$ and hence $im(p_*) \subset ker(\delta)$. For the reverse inclusion, suppose $\alpha : I^n \to B$ represent an element of $\pi_n(B, b_0)$ with $\delta[\alpha] = 0$. Then for a lift β as in

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{c_{e_0}} & E \\ \downarrow{\scriptstyle i} & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle p} \\ I^n & \xrightarrow{\alpha} & B \end{array}$$

we have that $\beta(-, 1)$ is homotopic to the constant map by a homotopy h relative to ∂I^{n-1} which maps into the fiber F. Now define a map $\gamma : I^n \to E$ by

$$\gamma(s,t) = \begin{cases} \beta(s, 2t) & t \in [0, \frac{1}{2}] \\ h(s, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

for $s \in I^{n-1}, t \in I$. Clearly γ represents an element of $\pi_n(E, e_0)$. Note that as $p \circ h$ is constant, $p \circ \gamma$ is homotopic to $p \circ \beta = \alpha$. So, we have $[\alpha] = p_*[\gamma]$ and hence $ker(\delta) \subset im(p_*)$, which shows the exactness at $\pi_n(B, b_0)$.

Exactness at $\pi_{\mathbf{n-1}}(\mathbf{F}, \mathbf{e_0})$. Let $\alpha : I^n \to B$ represent an element of $\pi_n(B, b_0)$. The map β in the definition of $\delta[\alpha] = [\beta(-, 1)]$ shows that $c_{e_0} \simeq \beta(-, 0) \simeq \beta(-, 1)$ in E. So, we have $i_* \circ \delta[\alpha] = 0$ and hence $im(\delta) \subset ker(i_*)$. To see the other inclusion, let $\gamma : I^{n-1} \to F$ represent an element of $ker(i_*) \subset \pi_{n-1}(F, e_0)$. So, we have $i_*[\gamma] = 0$ and suppose $h : I^{n-1} \times I \to E$ be a homotopy between $i \circ \gamma$ and c_{e_0} relative to ∂I^{n-1}. Then $\alpha = p \circ h$ represents an element of $\pi_n(B, b_0)$, and in the definition of $\delta[\alpha]$ we can choose the diagonal lift β to be h, in which case

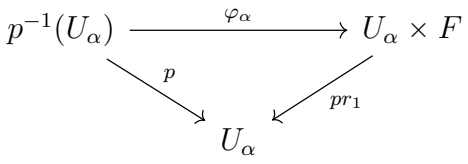
$\delta[\alpha]$ is represented by $[\gamma]$. This shows that $ker(i_*) \subset im(\delta)$, and this completes the proof of the theorem. ■

There are plenty of examples of fibrations. For example any projection $X \times F \to X$, the evaluation at 1 map $ev_1 : P(X) \to X$ from the path space $\mathcal{P}(X)$ to X are Hurewicz fibrations. It is also easy to see that any covering space is a Hurewicz fibration with discrete fiber. One of the important classes of fibrations are fiber bundles and we will be mostly concerned with fiber bundles rather than general fibrations. Fiber bundles show up in homotopy theory quite often and we will see later in the chapter that the Hopf map is a fiber bundle.

1.3 Fiber Bundles

A fiber bundle E over a base B with fiber F is nothing but a geometric way of expressing E in terms of B and F. In algebra whenever we have a short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups, we can express B in terms of A and C, namely $B \cong A \bigoplus C$. Likewise one can expect to have something similar in topology, namely $E \cong B \times F$. But unfortunately, that is not always true. For a fiber bundle, we can only express E as a product of B and F locally.

Definition 1.7. (Fiber Bundle)
A map $p : E \to B$ between topological spaces is a fiber bundle with fiber F if there is a covering $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of B such that for each α there is a $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ which is a homeomorphism and the following diagram commutes:



This property is known as the local trivialization. Before going to the examples of fiber bundles let us look at the relationship between fiber bundles and Serre fibrations.

Proposition 1.5. *A fiber bundle is a Serre fibration.*

Proof. Let $p : E \to B$ be a fiber bundle with fiber F and we have a diagram

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{\widetilde{\varphi}_0} & E \\ \downarrow{\scriptstyle i} & \overset{\widetilde{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow{h=\{\varphi_t\}} & B \end{array}$$

where h and $\widetilde{\varphi}_0$ is given. Our aim is to construct $\widetilde{h}$ such that $p \circ \widetilde{h} = h$ and $\widetilde{h}|_0 = \widetilde{\varphi}_0$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a open covering of B such that each U_α has the local trivialization property. Since I^n and I are compact, we can divide I^n into finitely many cubes $C_1, C_2, \ldots, C_k$ and I into finitely many intervals $J_1, J_2, \ldots, J_l$ such that each $C_i \times J_j \subset h^{-1}(U_\alpha)$ for some $\alpha \in \Lambda$.
We proceed by induction on n. The base case is just path lifting. So, we assume that $\widetilde{h}$ is defined over $\partial C_i \times I$ for each of the subcubes C_i. We will extend this $\widetilde{h}$ over $C_i \times I$ by extending it along I using the sub-intervals J_j. Let us choose cube that contains origin and the sub-interval that contains 0, let's call them C and J. We already know that $\widetilde{h}$ is defined to be $\widetilde{\varphi}_0$ on $C \times \{0\}$. Also by induction hypothesis $\widetilde{h}$ is also defined on $\partial C \times J$. So, $\widetilde{h}$ is define on $C \times \{0\} \cup \partial C \times J$. As $\widetilde{h}$ is a lift of h, we have

$$p \circ \widetilde{h}(C \times \{0\} \cup \partial C \times J) = h(C \times \{0\} \cup \partial C \times J) \subset h(C \times J) \subset U_\alpha$$

Hence $\widetilde{h}(C \times \{0\} \cup \partial C \times J) \subset p^{-1}(U_\alpha) \cong U_\alpha \times F$.
Let $\psi = pr_1 \circ \varphi_\alpha \circ \widetilde{h}|_{C \times \{0\} \cup \partial C \times J} : C \times \{0\} \cup \partial C \times J \to F$. Let us define a map $\beta : C \times J \to U_\alpha \times F$, $\beta = (\beta_1, \beta_2)$. Set $\beta_1 = h$ on $C \times J$. Note that, $C \times \{0\} \cup \partial C \times J$ is a retract of $C \times J$. Let r be a retraction. Set $\beta_2 = \psi \circ r$, i.e.

$$\beta_2 : C \times J \xrightarrow{r} C \times \{0\} \cup \partial C \times J \xrightarrow{\psi} F$$

Finally we set $\widetilde{h} = \varphi_\alpha^{-1} \circ \beta$ on $C \times J$. $\widetilde{h}$ is indeed a lift as $p \circ \widetilde{h} = p \circ \varphi_\alpha^{-1} \circ \beta = pr_1 \circ \beta = \beta_1 = h$. Having defined it on $C \times J$, we will do the same construction along I except we will take $\widetilde{h}|_{C \times \{t\}}$ (end point of J is t) instead of the map $\widetilde{\varphi}_0$ in the induction step. Also having defined $\widetilde{h}$ for $C \times J$, we will consider this $\widetilde{h}$ in the induction step of adjacent cells so that $\widetilde{h}$ is continuous on $I^n \times I$. ∎

Now let us look at some examples of fiber bundles. We will use the notation $F \hookrightarrow E \to B$ for a fiber bundle E over B with fiber F. Some of the examples

will be useful later in other contexts. By Proposition 1.5 and Theorem 1.2, there is a long exact sequence associated to every such fiber bundle.

Example 1.2. *1. A covering space $p : E \to B$, where B is connected, is a fiber bundle with discrete fiber. This readily follows from the definition as $p^{-1}(U)$ is a disjoint union of open sets, each homeomorphic to U for evenly covered neighborhoods. The resulting long exact sequence yields $p_* : \pi_n(E) \to \pi_n(B)$ is an isomorphism for $n \geqslant 2$ as $\pi_k(F) = 0$ for $n \geqslant 1$. We also have a short exact sequence $0 \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to 0$, which implies $p_* : \pi_1(E) \to \pi_1(B)$ is injective. Note that these results are indeed consistent with the results from covering space theory.*

2. *One of the simplest non-trivial fiber bundles is the Möbius band, which is a fiber bundle over $\mathbb{S}^1$ with fiber an interval. Define the Möbius band $\mathcal{M}$ to be the quotient of $I \times [-1, 1]$ under the identifications $(0, t) \sim (1, -t)$, with $p : \mathcal{M} \to \mathbb{S}^1$ induced by the projection $I \times [-1, 1] \to I$, so the fiber is $[-1, 1]$. If we attach two copies of $\mathcal{M}$ along their boundaries via the identity map, then we get Klein bottle, a bundle over $\mathbb{S}^1$ with fiber $\mathbb{S}^1$.*

3. *The next example involves projective spaces and is more of our concern. In the real setting we have the covering space $\mathbb{S}^0 \hookrightarrow \mathbb{S}^n \to \mathbb{RP}^n$. Over the complex numbers we have the much more interesting fiber bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. Here $\mathbb{S}^{2n+1}$ is seen as the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{CP}^n$ is seen as the quotient space of $\mathbb{S}^{2n+1}$ under the equivalence relation $(z_0, z_1, \ldots, z_n) \sim \lambda(z_0, z_1, \ldots, z_n)$ for $\lambda \in \mathbb{S}^1$, the unit circle in $\mathbb{C}$. We will come back to this example later in more details when we will talk about the Hopf bundle.*

Lets us now introduce the concept of degree in the continuous setting. We will discuss about degree of a map in the smooth setting in Chapter-3 in detail. We will also state (without a proof) the Freudenthal Suspension Theorem, which will be important in the calculation of the groups $\pi_n(\mathbb{S}^n)$.

1.4 Suspension and Degree

We will define degree of a map $f : \mathbb{S}^n \to \mathbb{S}^n$ using homology. Our definition of degree will use the fact that $H_n(\mathbb{S}^n) \cong \mathbb{Z}$ (see [8] for a proof).

Definition 1.8. (Degree of a map) Let $f : \mathbb{S}^n \to \mathbb{S}^n$. Then, $f_* : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$ is a homomorphism. So, f_* is multiplication buy some integer d as f is a homomorphism $\mathbb{Z} \to \mathbb{Z}$. This d is called the degree of f and denoted as $\deg(f)$.

Once we choose a generator 1 of $H_n(\mathbb{S}^n)$, then $f_*(1) = \deg(f)$ and if we choose -1 as our generator, then $f_*(-1) = -\deg(f)$ by $\mathbb{Z}$-linearity. Thus $\deg(f)$ is well defined and equal to $f_*(1)$. There is a nice relationship between homotopic maps and their degrees.

Proposition 1.6. *If f and g are homotopic maps from $\mathbb{S}^n$ to $\mathbb{S}^n$, then* $\deg(f) = \deg(g)$.

Proof. Let f and g be homotopic maps. Then they induce the same map in homology, i.e. $f_* = g_*$ (see Theorem-2.10 of [2]). So, if f_* is multiplication my d, then so is g_*. Hence, $\deg(f) = \deg(g)$. ■

The converse of the above proposition is also true, i.e. if two maps have same degree, then they are homotopic. This result is known as the Hopf degree theorem (see [10] for a detailed discussion). On $\mathbb{S}^1 \subset \mathbb{C}$ the map $f(z) = z^k$ has degree k. So, we can construct maps of arbitrary degree on $\mathbb{S}^1$. We can also construct maps of any degree on $\mathbb{S}^n$ for $n \geqslant 2$. To do that we need to introduce the concept of suspension.

Definition 1.9. (Cone and Suspension) Let X be a topological space. The cone of X, denoted as CX, is defined to be the quotient space of $X \times I$ with the identification $(x, 1) \sim (x', 1)$ for all $x, x' \in X$. The suspension of X, denoted as SX, is defined to be the quotient space of $X \times [-1, 1]$ with the identification $(x, 1) \sim (x', 1)$ and $(x, -1) \sim (x', -1)$ for all $x, x' \in X$.

Although, suspension of a space does not look like a nice space due to the quotients involved, sometimes it does produce some nice spaces. As an example, suspension of a sphere is again a sphere in dimension one more than the previous one. This will be very useful to us, so let us give a proof of this fact.

Proposition 1.7. $S\mathbb{S}^n \cong \mathbb{S}^{n+1}$

Proof. $S\mathbb{S}^n = (\mathbb{S}^n \times [-1, 1])/\sim$, where the relation '$\sim$' is defined as $(x, 1) \sim (y, 1)$ and $(x, -1) \sim (y, -1)$ for all $x, y \in \mathbb{S}^n$.

Define $f : \mathbb{S}^n \times [-1,1] \to \mathbb{S}^{n+1}$ by

$$((x_0,\ldots,x_n),t) \mapsto (x'_0,\ldots,x'_n,t)$$

where $x'_i = \sqrt{1-t^2}\cdot x_i$. Note that f is well defined as

$$\sum_i x'^2_i + t^2 = (1-t^2)\sum_i x_i^2 + t^2 = 1.$$

f induces a map $\widetilde{f} : S\mathbb{S}^n \to \mathbb{S}^{n+1}$.

$$\begin{array}{ccc} \mathbb{S}^n \times [-1,1] & \xrightarrow{f} & \mathbb{S}^{n+1} \\ \downarrow q & \nearrow \tilde{f} & \\ S\mathbb{S}^n & & \end{array}$$

As each x'_i is a continuous map of x_i and t, we have f is continuous. By property of quotient topology, $\widetilde{f}$ is also continuous. Let $(y_0,\ldots,y_{n+1}) \in \mathbb{S}^{n+1}$ be given. Set $t = y_{n+1}$, $x_i = \frac{y_i}{\sqrt{1-t^2}}$ if $y_{n+1} = t \neq 1,-1$ and $x_i = 0$ if $y_{n+1} = t = 1$ or -1. Then $f((x_0,\ldots,x_n),t) = (\sqrt{1-t^2}\cdot x_0,\ldots,\sqrt{1-t^2}\cdot x_n,t) = (y_0,\ldots,y_{n+1})$. This shows that f is surjective. As, q is surjective, we have $\widetilde{f} = f \circ q$ is also surjective.
Let $\widetilde{f}[(x_0,\ldots,x_n),t] = \widetilde{f}[(x'_0,\ldots,x'_n),t']$. So, $t = t'$ and $\sqrt{1-t^2}\cdot x_i = \sqrt{1-t'^2}\cdot y_i$. Hence $x_i = x'_i$ whenever $t = t' \neq 1,-1$. All points $(x,\pm 1) \in \mathbb{S}^n \times \{\pm 1\}$ will map to $(0,0,\ldots,0,\pm 1)$. then by definition of '$\sim$', all points $(x,\pm 1)$ are identified. Hence the map $\widetilde{f}$ is injective.
$\mathbb{S}^n \times [-1,1]$ is compact by Tychonoff theorem and $S\mathbb{S}^n$ being the image of $\mathbb{S}^n \times [-1,1]$ under the continuous map q, is also compact. As $\widetilde{f}$ is a continuous bijection from a compact space $S\mathbb{S}^n$ to a hausdorff space $\mathbb{S}^{n+1}$, $\widetilde{f}$ is a homeomorphism. ■

Definition 1.10. A space X is n-connected if $\pi_i(X) = 0$ for all $i \leqslant n$.

Note that 0-connected means path-connected and 1-connected means simply connected. Next, we give equivalent conditions of being n-connected.

Proposition 1.8. *The following are equivalent:*
(1) Every map $\mathbb{S}^i \to X$ is homotopic to a constant map.
(2) Every map $\mathbb{S}^i \to X$ extends to a map $\mathbb{D}^{i+1} \to X$
(3) $\pi_i(X) = 0$.

Proof. $(1) \Rightarrow (2)$: Suppose first that $f : \mathbb{S}^i \to X$ is homotopic to a constant map, so there is a homotopy $H : \mathbb{S}^i \times I \to X$ such that $H(y, 1) = f(y)$ and $H(y, 0) = x_0$ where x_0 is the basepoint of X. Let us take polar coordinates $(\phi_1, \ldots, \phi_i, r)$ on $\mathbb{D}^{i+1}$. Define $\overline{f} : \mathbb{D}^{i+1} \to X$ by,

$$\overline{f}(\phi_1, \ldots, \phi_i, r) = H((\phi_1, \ldots, \phi_i), r)$$

where $(\phi_1, \ldots, \phi_i)$ is the polar coordinate representation of a point in $\mathbb{S}^i$ and $r \in I$. The map is well-defined as $H((\phi_1, \ldots, \phi_i), 0) = x_0$ for all $(\phi_1, \ldots, \phi_i) \in \mathbb{S}^i$ and is an extension of of f as $\overline{f}(\phi_1, \ldots, \phi_i, 1) = H(\phi_1, \ldots, \phi_i, r) = f(\phi_1, \ldots, \phi_i)$ for all $(\phi_1, \ldots, \phi_i) \in \mathbb{S}^i$.
$(2) \Rightarrow (3)$: Suppose that there is a map $\overline{f} : \mathbb{D}^{i+1} \to X$ extending a given $f : \mathbb{S}^i \to X$ with $f(p) = x_0$ ($p \in \mathbb{S}^i$ be the basepoint). Define $H : \mathbb{S}^i \times I \to X$ by,

$$H(y, t) = \overline{f}(ty + (1 - t)p)$$

This is well-defined because $\mathbb{D}^{i+1}$ is convex, so that $ty + (1 - t)p \in \mathbb{D}^{i+1}$ for all $y \in \mathbb{S}^I \subset \mathbb{D}^{i+1}$. We have $H(y, 0) = \overline{f}(p) = x_0$ and $H(y, 1) = \overline{f}(y) = f(y)$ since $\overline{f}$ extends f. Furthermore $H(p, t) = \overline{f}(p) = f(p) = x_0$ for all t, so that H is a basepoint preserving homotopy between f and the constant map c_{x_0}. Hence $[f] = 0$ in $\pi_i(X, x_0)$ and $\pi_i(X, x_0) = 0$.
$(3) \Rightarrow (1)$: Follows from the definition. ∎

As a consequence of the above proposition, we can say that a space X is n-connected if one of the three conditions in Proposition 1.8 holds for all $i \leqslant n$. For a map $f : X \to Y$, there is a map $Sf : SX \to SY$, called the suspension of f, given by $Sf[(x, t)] = [(f(x), t)]$. It is clear that if $f \simeq g$, then $Sf \simeq Sg$. So, for a map $f : \mathbb{S}^i \to \mathbb{S}^n$, there is a map $Sf : S\mathbb{S}^i = \mathbb{S}^{i+1} \to \mathbb{S}^{n+1} = S\mathbb{S}^n$. So, we get a well-defined map $S : \pi_i(\mathbb{S}^n) \to \pi_{i+1}(\mathbb{S}^{n+1})$, given by $f \mapsto Sf$. We call this the suspension map. Now the map Sf has the property that $\deg(Sf) = \deg(f)$. We omit the proof of this fact as it involves some tools from homology (e.g. Mayer-Vietrois sequence) that we have not properly introduced in this thesis. The upshot is that we can construct maps of degree k from $\mathbb{S}^n$ to $\mathbb{S}^n$ by taking suspension repeatedly (n times to be precise) of a degree k map on $\mathbb{S}^1$.

Theorem 1.3. *(Freudenthal Suspension Theorem)*
The suspension map $\pi_i(\mathbb{S}^n) \to \pi_{i+1}(\mathbb{S}^{n+1})$ is an isomorphism for $i < 2n - 1$ an a

surjection for $i = 2n-1$. More generally, this holds for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ whenever X is an $(n-1)$-connected CW complex.

The proof is rather technical and tedious. So, we omit the proof. A detailed proof can be found in Chapter-4 of [2]. Instead we look at the important consequence of the theorem.

Corollary 1.3. *The group $\pi_n(\mathbb{S}^n)$ is isomorphic to $\mathbb{Z}$, generated by the identity map, for all $n \geqslant 1$.*

Proof. From the Freudenthal Suspension Theorem, we know that in the suspension sequence

$$\pi_1(\mathbb{S}^1) \to \pi_2(\mathbb{S}^2) \to \pi_3(\mathbb{S}^3) \to \pi_4(\mathbb{S}^4) \to \dots$$

the first map is surjective and the subsequent maps are all isomorphisms. We know from basic algebraic topology that $\pi_1(\mathbb{S}^1)$ is $\mathbb{Z}$ generated by the identity map. If we can prove that $\pi_2(\mathbb{S}^2)$ is $\mathbb{Z}$, then we are done as all consequent maps are isomorphisms. Applying Theorem 1.2 and Corollary 1.1 on the Hopf bundle (we will describe later) $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ gives us the desired result. ■

1.5 The Hopf Bundle

We will introduce the Hopf bundle in this section and see one of its most important implications. Recall that in Example 1.2, we said that $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle. Let us first prove this fact.

Proposition 1.9. *$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle, where the map $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is given by, $(z_0, z_1, \dots, z_n) \mapsto [z_0, z_1, \dots, z_n]$*

Proof. Regard $\mathbb{S}^{2n+1}$ as the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{CP}^n$ as the quotient space of $\mathbb{S}^{2n+1}$ under the equivalence relation $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$ for any $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$.
Let $U_i = \{[z_0, \dots, z_n] : z_i \neq 0\}$. U_i is an open set in $\mathbb{CP}^n$. Note that

$$p^{-1}(U_i) = \{(z_0, \dots, z_n : z_i \neq 0\}.$$

Define $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{S}^1$ by

$$(z_0, \ldots, z_n) \mapsto ([z_0, \ldots, z_n], z_i/|z_i|)$$

The map φ is continuous as $z_i \neq 0$ on U_i.
CLAIM-1: φ_i is surjective.
Proof. Let $[z_0, \ldots, z_n] \in U_i$ and $\xi \in \mathbb{S}^1$. We know that $z_i \neq 0$. Let $\lambda = \xi/z_i$ and $\lambda' = \lambda/|\lambda|$. Then

$$\varphi([\lambda'(z_0, \ldots, z_n)]) = ([z_0, \ldots, z_n], \lambda' z_i/|\lambda' z_i|) = ([z_0, \ldots, z_n], \xi).$$

which shows that φ_i is surjective.
CLAIM-2: φ is injective.
Proof. Let $([z_0, \ldots, z_n], z_i/|z_i|) = ([w_0, \ldots, w_n], w_i/|w_i|)$. Then we have

$$[z_0, \ldots, z_n] = [w_0, \ldots, w_n] \quad \text{and} \quad \frac{z_i}{|z_i|} = \frac{w_i}{|w_i|}.$$

Hence $(z_0, \ldots, z_n) = \lambda(w_0, \ldots, w_n)$ for some $\lambda \in \mathbb{S}^1$ and $z_i = \frac{|z_i|}{|w_i|} w_i$. So, we must have $\lambda = \frac{|z_i|}{|w_i|}$, i.e. λ is real and positive with $|\lambda| = 1$. The only possible value for λ is 1. Hence $(z_0, \ldots, z_n) = (w_0, \ldots, w_n)$, which shows that φ_i is injective.
So, φ_i is a continuous bijection. To show that φ_i is a homeomorphism, we need to show that φ_i is an open map.
CLAIM-3: φ_i is an open map.
Proof. Let $V \subseteq \mathbb{S}^{2n+1}$ be open. Note that

$$\varphi_i(V) = \{([z_0, \ldots, z_n], z_i/|z_i|) : (z_0, \ldots, z_n) \in V\}.$$

Write $\varphi_i = (\varphi_i^1, \varphi_i^2)$, where $\varphi_i^1(z_0, \ldots, z_n) = [z_0, \ldots, z_n]$ and $\varphi_i^2(z_0, \ldots, z_n) = z_i/|z_i|$. As φ_i^2 is a projection map, it is open. Now φ_i^1 is also open as

$$\begin{aligned} p^{-1}(\varphi_i^1(V) &= \{(z_0, \ldots, z_n) \in \mathbb{S}^{2n+1} : p(z_0, \ldots, z_n) \in \varphi_i^1(V)\} \\ &= \{(z_0, \ldots, z_n) \in \mathbb{S}^{2n+1} : [z_0, \ldots, z_n] \in \varphi_i^1(V)\} \\ &= \{(z_0, \ldots, z_n) \in \mathbb{S}^{2n+1} : [z_0, \ldots, z_n] \in p(V)\} \\ &= \cup_{\xi \in \mathbb{S}^1} \xi V \end{aligned}$$

is open. As both φ_i^1 and φ_i^2 are open maps, φ_i is also an open map.

So, this shows that φ_i is a homeomorphism. Also $pr_1 \circ \varphi_i = p$. So $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is fiber bundle with fiber $\mathbb{S}^1$. ■

The $n = 1$ case is special in the above fiber bundle. For $n = 1$, the fiber bundle is $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{CP}^1$. By Proposition A.1, $\mathbb{CP}^1 \cong \mathbb{S}^2$ and hence the above fiber bundle becomes $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$.

Definition 1.11. (Hopf Bundle and Hopf Map)
The fiber bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ is called the Hopf bundle and the map $\mathfrak{h} : \mathbb{S}^3 \to \mathbb{S}^2$ is called the Hopf map. The Hopf map $\mathfrak{h}$ is the map p for the $n = 1$ case in Proposition 1.9 along with the identification $\mathbb{CP}^1 \cong \mathbb{S}^2$ (see Proposition A.1).

We will give more descriptions of the Hopf map as we go along. Now, the next obvious step is to apply Theorem 1.2 to the Hopf bundle to see what information we get about the homotopy groups of $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$. So, the long exact sequence of homotopy groups for this fiber bundle is,

$$\cdots \to \pi_{k+1}(\mathbb{S}^2) \xrightarrow{\delta} \pi_k(\mathbb{S}^1) \xrightarrow{i_*} \pi_k(\mathbb{S}^3) \xrightarrow{p_*} \pi_k(\mathbb{S}^2) \xrightarrow{\delta} \pi_{k-1}(\mathbb{S}^1) \xrightarrow{i_*} \pi_{k-1}(\mathbb{S}^3) \xrightarrow{p_*} \cdots$$

For $k > 1$, $\pi_k(\mathbb{S}^1) = 0$. So, $\pi_k(\mathbb{S}^3) \xrightarrow{p_*} \pi_k(\mathbb{S}^2)$ is an isomorphism for $k - 1 > 1$ or $k > 2$. In particular for $k = 3$, $\mathfrak{h}_* : \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2)$ is an isomorphism. From Corollary 1.3, $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$, generated by the identity map $\mathbb{1} : \mathbb{S}^3 \to \mathbb{S}^3$. Hence, we see that $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, generated by the Hopf map $\mathfrak{h}$.

Theorem 1.4. *The group $\pi_3(\mathbb{S}^2)$ is isomorphic to $\mathbb{Z}$ and generated by the Hopf map.*

Proof. Follows from the above discussion. ■

Note that Theorem 1.4 gives us some description about the first group of the type $\pi_k(\mathbb{S}^n), k > n$. We will investigate more about this group and its elements.

Remark 1.1. Replacing the field $\mathbb{C}$ by the field of quaternions $\mathbb{H}$, the same constructions yields a fiber bundle $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{HP}^n$ over the quaternionic projective spaces $\mathbb{HP}^n$. Taking $n = 1$ gives a second Hopf bundle $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$. There is another Hopf bundle $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8$, whose definition uses the non-associative 8-dimensional algebra $\mathbb{O}$ of Cayley octonions.

Next we will study the elements of the group $\pi_3(\mathbb{S}^2)$. We only know that the Hopf map $\mathfrak{h}$ generates this group. Let we have two maps $f : \mathbb{S}^3 \to \mathbb{S}^3$ and

$g : \mathbb{S}^2 \to \mathbb{S}^2$. Then the maps $f \circ \mathfrak{h}$ and $\mathfrak{h} \circ g$ represent some elements of $\pi_3(\mathbb{S}^2)$. What are those elements and can we identify them using the notions that we have already established? To answer these we will need some more tools, which we will now develop in the next chapter.

Chapter 2

The Hopf Invariant

Given a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, one can assign a number to it, which we will call Hopf Invariant. This "Hopf Invariant" will help us to answer our question that we asked at the end of the last chapter. The Hopf Invariant is somewhat similar to that of the notion of degree. A fundamental theorem by Adams states that a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ of Hopf Invariant 1 exist only when $n = 2, 4, 8$. A very interesting consequence of this theorem is that $\mathbb{R}^n$ is a division algebra only when $n = 1, 2, 4, 8$. The Hopf invariant has many definitions. We will, for now, define it in terms of cohomology. There are other definitions of Hopf invariant using K-theory and linking number. We will talk about linking number in chapter-4 and interested readers can find details about the k-theory approach in [23]. So, before defining the Hopf invariant, let us review the notion of cup products from basic cohomology.

2.1 Cup Products

We want to define a product $H^k(X; R) \times H^l(X; R) \to H^{k+l}(X; R)$. To define cup product we consider cohomology with coefficients in a ring R. We will first define the cup product in the cochain level and then we will hopefully be able to pass it to the quotient and get a well-defined product in cohomology.

Definition 2.1. (Cup Product)
For cochains $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$, the cup product $\varphi \smile \psi \in C^{k+l}(X; R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \to X$ is given by the

formula

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+l}]})$$

where the right hand side is the product in R.

To see that the cup product of cochains induces a cup product of cohomology classes we need a formula relating it to the coboundary map.

Lemma 2.1. *For $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$*

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

Proof. Let $\sigma : \Delta^{k+l+l} \to X$ be a singular simplex. Then we have

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$(-1)^k(\varphi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,\widehat{v_i},\dots,v_{k+l+1}]})$$

Note that the last term $((-1)^{k+1}\varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}))$ of the first expression is the negative of the first term $((-1)^k\varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}))$ of the second expression. Hence when we add the above two expressions, these two terms gets cancelled and the remaining terms are exactly $(\varphi \smile \psi)(\delta\sigma)$ as $\delta\sigma = \sum_{i=0}^{k+l+1}(-1)^i\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{k+l+1}]}$. Also by definition $(\varphi \smile \psi)(\delta\sigma) = \delta(\varphi \smile \psi)(\sigma)$, which proves the lemma. ∎

From the formula it is apparent that the cup product of two cocycles is again a cocycle. Also the cup product of a cocycle and a coboundary , in either order is a coboundary since $\varphi \smile \delta\psi = \pm\delta(\varphi \smile \psi)$ if $\delta\varphi = 0$ and $\delta\varphi \smile \psi = \delta(\varphi \smile \psi)$ if $\delta\psi = 0$. It follows that there is an induced cup product $H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R)$.

Proposition 2.1. *For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.*

Proof. Regard α and β as cochains representing their cohomology class. Then the formula comes from the cochain formula $f^{\#}(\alpha \smile \beta) = f^{\#}(\alpha) \smile f^{\#}(\beta)$:

$$\begin{aligned} f^{\#}(\alpha) \smile f^{\#}(\beta) &= f^{\#}(\alpha)(\sigma|_{[v_0,\dots,v_k]}) f^{\#}(\beta)(\sigma|_{[v_k,\dots,v_{k+l}]}) \\ &= (\alpha)(f\sigma|_{[v_0,\dots,v_k]})(\beta)(f\sigma|_{[v_k,\dots,v_{k+l}]}) \\ &= (\alpha \smile \beta)(f\sigma) \\ &= f^{\#}(\alpha \smile \beta) \end{aligned}$$

■

We will now define the cross product or external cup product. The maps

$$H^k(X;R) \times H^l(Y;R) \xrightarrow{\times} H^{k+l}(X \times Y;R)$$

given by $a \times b = p_1^*(a) \smile p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y. The relative forms of the cup product and the cross product are similarly defined.

The product in R is associative and distributive and hence so is the cup product. So, it is natural to ask whether we can make this cup product a multiplication in a ring structure on the cohomology groups of a space X. The answer is actually Yes!!

To do this we simply define $H^*(X;R)$ to be the direct sum of the groups $H^n(X;R)$. Elements of $H^*(X;R)$ are the finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X;R)$, and the product of two such terms is defined to be $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \smile \beta_j$. It is routine to check that this makes $H^*(X;R)$ into a ring with identity if R has an identity (because we will set $\imath \in H^0(X;R)$, which is given by the cocyle that takes every zero simplex to $1 \in R$).

This kind of construction of a ring is called a graded ring: a ring A that is decomposed as a sum $\bigoplus_{k \geqslant 0} A_k$ of additive subgroups A_k such that the multiplication takes $A_k \times A_l$ to A_{k+l}. To dictate that an element $a \in A$ lies in A_k, we will $|a| = k$. This applies in particular to elements of $H^k(X;R)$, and we will call $|a|$ to be the dimension of the element a.

$H^*(X;R)$ often has a more compact description than the sequence of groups $H^n(X;R)$, so it is beneficial for us to work with the single object $H^*(X;R)$ rather than regarding all groups $H^n(X;R)$. We will now see some examples of cohomology rings that will be helpful to us later.

Proposition 2.2. *The cohomology ring of the n-sphere is given by*

$$H^*(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2),$$

where $\alpha \in H^n(\mathbb{S}^n;\mathbb{Z})$ is a generator.

Proof. From cellular cohomology, we have

$$H^k(\mathbb{S}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & otherwise \end{cases}$$

Let 1 be the generator of $H^0(\mathbb{S}^n;\mathbb{Z})$. Then the only possible cup products are

$$\alpha \smile 1 = \alpha, \quad 1 \smile \alpha = \alpha \quad \text{and} \quad \alpha \smile \alpha = 0$$

as $H^{2n}(\mathbb{S}^n;\mathbb{Z}) = 0$. Hence we have $H^*(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$. ■

Proposition 2.3. *The cohomology rings of the projective spaces are given as follows:*

$$H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1}),$$

where $\alpha \in H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ and

$$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1}),$$

where $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$

This is quite an important result and has many implications. We will not give a proof of this here as it will involve lots of concepts from cohomology theory that we have not mentioned. Interested readers can look at [13] for a short and elegant proof. For a detailed proof using basic cohomology and diagram chasing see Theorem-3.12 of [2].

2.2 The Hopf Invariant

Let us prove a result first, which will be useful in defining Hopf invariant.

Proposition 2.4. *If (X_1, A) is a CW pair and we have attaching maps $f, g : A \to X_0$ that are homotopic, then $X_0 \amalg_f X_1 \simeq X_0 \amalg_g X_1$ rel X_0*

Here we define $W \simeq Z \text{rel} Y$ for pairs (W, Y) and (Z, Y) by requiring the existence of maps $\varphi : W \to Z$ and $\psi : Z \to W$ such that they restrict to the identity on Y and $\psi \circ \varphi \simeq \mathbb{1}$ and $\varphi \circ \psi \simeq \mathbb{1}$ via homotopies that restrict to the identity on Y all times.

Proof. Let us take $F : A \times I \to X_0$ is a homotopy from f to g. Consider the space $X_0 \amalg_F (X_1 \times I)$. This contains both $X_0 \amalg_f X_1$ and $X_0 \amalg_g X_1$ as subspaces. $X_0 \amalg_F (X_1 \times I)$ deformation retracts onto both $X_0 \amalg_f X_1$ and $X_0 \amalg_g X_1$. Also both these deformation retractions restrict to the identity on X_0. Hence we have the homotopy equivalence $X_0 \amalg_f X_1 \simeq X_0 \amalg_g X_1$ rel X_0. ∎

For a map $f : \mathbb{S}^m \to \mathbb{S}^n$ with $m \geqslant n$, we can form a CW complex C_f to be the quotient space of $\mathbb{S}^n \amalg e^{m+1}$ with the identification $x \sim f(x)$, for $x \in \partial e^{m+1} = \mathbb{S}^m$. The homotopy type of C_f only depends on the homotopy class of f, by Proposition 2.4. Now, if $m = n$ and f has degree d, then from cellular chain complex of C_f we see that $H_n(C_f) = \mathbb{Z}_{|d|}$ and $H_k(C_f) = 0$ for $k > 0 (\neq n)$. Similarly in cohomology we have $H^n(C_f) = \mathbb{Z}_{|d|}$ and $H^k(C_f) = 0$ for $k > 0 (\neq n)$. So, the ring structure of $H^*(C_f)$ in this case is trivial. But cup products may have a chance of being nontrivial in $H^*(C_f)$ when $m = 2n-1$. From the cellular cochain complex of C_f, the cohomology of C_f is

$$H^k(C_f; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, n, 2n \\ 0 & otherwise \end{cases}$$

From now on we will consider cohomology with integer coefficients unless otherwise stated. Let us take generators $\alpha \in H^n(C_f)$ and $\beta \in H^{2n}(C_f)$. The cup product of α with itself will land in $H^{2n}(C_f)$ and hence will be of the form $\alpha \smile \alpha = H(f)\beta$, where $H(f)$ is an integer.

Definition 2.2. (Hopf Invariant) The integer $H(f)$ in the above discussion is called the Hopf invariant.

Note that $H(f)$ depends on the choice of the generator β, but this can be specified by requiring β to correspond to a fixed generator of $H^{2n}(\mathbb{D}^{2n}, \partial\mathbb{D}^{2n})$ under the

map $H^{2n}(C_f) \cong H^{2n}(C_f, \mathbb{S}^n) \to H^{2n}(\mathbb{D}^{2n}, \partial\mathbb{D}^{2n})$ induced by the characteristic map of the cell e^{2n}, which is determined by f. We can then change the sign of $H(f)$ by composing f with a reflection of $\mathbb{S}^{2n-1}$, of degree -1. If $f \simeq g$, then by Proposition 2.4, the homotopy equivalence $C_f \simeq C_g$ gives us that the chosen generators β_f for $H^{2n}(C_f)$ and β_g for $H^{2n}(C_g)$ correspond, so $H(f)$ depends only on the homotopy class of f. If f is a constant map then $C_f = \mathbb{S}^n \vee \mathbb{S}^{2n}$ and $H(f) = 0$ since C_f retracts onto $\mathbb{S}^n$.
The unshot is that we have a well-defined map $H : \pi_{2n-1}(\mathbb{S}^n) \to \mathbb{Z}$, given by $[f] \mapsto H(f)$. Let us now look at some properties of this Hopf invariant.

Proposition 2.5. *The Hopf invariant has the following properties:*
1. *The Hopf invariant $H : \pi_{2n-1}(\mathbb{S}^n) \to \mathbb{Z}$ is a homomorphism.*
2. *The Hopf invariant of a composition $\mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n \xrightarrow{g} \mathbb{S}^n$ is given by*

$$H(g \circ f) = (\deg(g))^2 H(f)$$

3. *The Hopf invariant of a composition $\mathbb{S}^{2n-1} \xrightarrow{g} \mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n$ is given by*

$$H(f \circ g) = (\deg(g)) H(f)$$

Proof. (1)
Let $f, g : \mathbb{S}^{2n-1} \to \mathbb{S}^n$. We have to show that $H(f+g) = H(f) + H(g)$.

$$f + g : \mathbb{S}^{2n-1} \xrightarrow{c} \mathbb{S}^{2n-1} \vee \mathbb{S}^{2n-1} \xrightarrow{f \vee g} \mathbb{S}^n$$

Let $C_{f\vee g}$ be the quotient space obtained from collapsing the equatorial disk of the $2n$-cell of C_{f+g} to a point. Hence $C_{f\vee g}$ is the space obtained from $\mathbb{S}^n$ by attaching two $2n$-cells via f and g. Let $q : C_{f+g} \to C_{f\vee g}$ be the quotient map. Let e^{2n}_{f+g} be the generator of $H^{2n}(C_{f+g}) = \mathbb{Z}$ and e^{2n}_f and e^{2n}_g are the generators of $H^{2n}(C_{f\vee g}) = \mathbb{Z} \oplus \mathbb{Z}$. The induced cellular map in homology q_* sends e^{2n}_{f+g} to $e^{2n}_f + e^{2n}_g$. As the dual of the map $1 \mapsto (1,1)$ is given by $(1,0) \mapsto 1$ and $(0,1) \mapsto 1$, the induced map in cohomology q^* is given by $q^*(\beta_f) = q^*(\beta_g) = \beta_{f+g}$ where β_f, β_g, β_{f+g} are the cohomology classes dual to the $2n$-cells. Letting α_{f+g} and $\alpha_{f\vee g}$ be the cohomology classes corresponding to the n-cells, we have $q^*(\alpha_{f\vee g}) = \alpha_{f+g}$ since q is a homeomorphism on the n-cells. Now consider the inclusion maps

$$i_f : C_f \hookrightarrow C_{f\vee g}, \quad i_g : C_g \hookrightarrow C_{f\vee g}$$

Note that $\alpha_{f\vee g} \smile \alpha_{f\vee g} \in H^{2n}(C_{f\vee g}) \cong \mathbb{Z}\oplus\mathbb{Z}$ and β_f and β_g generate $H^{2n}(C_{f\vee g})$. Hence,

$$\alpha_{f\vee g} \smile \alpha_{f\vee g} = n_f\beta_f + n_g\beta_g \tag{2.1}$$

As, i_f^* induces isomorphism on H^n, we have

$$i_f^*(\alpha_{f\vee g}) = \alpha_f, \quad i_g^*(\alpha_{f\vee g}) = \alpha_g$$
$$i_f^*(\beta_f) = \beta_f, \quad i_f^*(\beta_g) = 0$$
$$i_g^*(\beta_g) = \beta_g, \quad i_g^*(\beta_f) = 0$$

Applying, i_f^* on both sides of Eq. 2.1, we get $\alpha_f \smile \alpha_f = n_f\beta_f$. Hence $n_f = H(f)$. Similarly applying i_g^* in Eq. 2.1 we get $n_g = H(g)$. Hence,

$$\alpha_{f+g}^2 = q^*(\alpha_{f\vee g}^2) = q^*(H(f)\beta_f + H(g)\beta_g) = (H(f) + H(g))\beta_{f+g}$$

as $q^*(\beta_f) = q^*(\beta_g) = \beta_{f+g}$. So, by definition of Hopf invariant of the map $f + g$, we get $H(f + g) = H(f) + H(g)$. ■

Proof. (2)
We have $g \circ f : \mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n \xrightarrow{g} \mathbb{S}^n$. Let us define $G : \mathbb{S}^n \amalg e^{2n} \to \mathbb{S}^n \amalg e^{2n}$ by setting $G = id$ on e^{2n} and $G = g$ on $\mathbb{S}^n$.

$$\begin{array}{ccc} \mathbb{S}^n \amalg e^{2n} & \xrightarrow{G} & \mathbb{S}^n \amalg e^{2n} \\ \downarrow{\scriptstyle q_f} & & \downarrow{\scriptstyle q_{g\circ f}} \\ C_f & \xrightarrow{F} & C_{g\circ f} \end{array}$$

(F is such that the diagram commutes)
In cohomology of degree n, the diagram gives

$$\begin{array}{ccc} H^n(C_{g\circ f}) & \xrightarrow{F^*} & H^n(C_f) \\ \downarrow{\scriptstyle q_{g\circ f}^*} & & \downarrow{\scriptstyle q_f^*} \\ H^n(\mathbb{S}^n) \oplus H^n(e^{2n}) & \xrightarrow{G^*} & H^n(\mathbb{S}^n) \oplus H^n(e^{2n}) \end{array}$$

As $H^n(e^{2n}) = 0$, all the groups in the above diagram are $\mathbb{Z}$. Note that the map G is given by g on $\mathbb{S}^n$. So, the map G^* is multiplication by $\deg(g)$ by definition of

degree. We also have

$$\mathbb{S}^n \xhookrightarrow{i} \mathbb{S}^n \amalg e^{2n} \xrightarrow{q_f} C_f$$

where the composition is an isomorphism on H^n by the long exact sequence of cohomology for pairs (as $H^k(C_f, \mathbb{S}^n) = 0$ for $k \neq 2n$). Hence q_f^* and $q_{g\circ f}^*$ are isomorphisms. From the diagram we therefore get F^* is also multiplication by $\deg(g)$. Let α_f and $\alpha_{g\circ f}$ be generators of $H^n(C_f)$ and $H^n(C_{g\circ f})$ respectively. As G is identity on e^{2n}, F^* takes the generator $\beta_{g\circ f}$ of $H^{2n}(C_{g\circ f})$ to a generator β_f of $H^{2n}(C_f)$. Then we have

$$F^*(\alpha_{g\circ f}) = \deg(g)\alpha_f, \quad F^*(\beta_{g\circ f}) = \beta_f \tag{2.2}$$

Hence by Eq. 2.2 and Proposition 2.1,

$$H(g\circ f)\beta_f = F^*(H(g\circ f)\beta_{g\circ f}) = F^*(\alpha_{g\circ f}^2) = (\deg(g))^2\alpha_f^2 = (\deg(g))^2 H(f)\beta_f$$

which implies $H(g\circ f) = (\deg(g))^2 H(f)$. ■

Proof. (3)
We have $f\circ g : \mathbb{S}^{2n-1} \xrightarrow{g} \mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n$. Let us define $G : \mathbb{S}^n \amalg e^{2n} \to \mathbb{S}^n \amalg e^{2n}$ by setting $G = Cg$ on e^{2n} and $G = id$ on $\mathbb{S}^n$. Note that $Cg : C\mathbb{S}^{2n-1} \to C\mathbb{S}^{2n-1}$ is the cone of g. As, $C\mathbb{S}^{2n-1} \cong e^{2n}$, we can take Cg to be a map on e^{2n}.

$$\begin{array}{ccc} \mathbb{S}^n \amalg e^{2n} & \xrightarrow{G} & \mathbb{S}^n \amalg e^{2n} \\ \downarrow{\scriptstyle q_{f\circ g}} & & \downarrow{\scriptstyle q_f} \\ C_{f\circ g} & \xrightarrow{F} & C_f \end{array}$$

(F is such that the diagram commutes)
Now let us consider another diagram as follows :

$$\begin{array}{ccc} C_{f\circ g} & \xrightarrow{F} & C_f \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ C_{f\circ g}/(C_{f\circ g}\backslash e^{2n}) & \xrightarrow{\widetilde{F}} & C_f/(C_f\backslash e^{2n}) \end{array}$$

where q is the quotient map and $\widetilde{F}$ is the induced map such that the diagram commutes. Note that as in the construction of F we have taken the map to G to

be Cg on e^{2n}, we have

$$\widetilde{F} = Sg : C_{f\circ g}/(C_{f\circ g}\backslash e^{2n}) \cong \mathbb{S}^{2n} \to C_f/(C_f\backslash e^{2n}) \cong \mathbb{S}^{2n}$$

where Sg is the suspension map. Hence we finally have a diagram as follows:

$$\begin{array}{ccc} C_{f\circ g} & \xrightarrow{F} & C_f \\ \downarrow q & & \downarrow q \\ \mathbb{S}^{2n} & \xrightarrow{Sg} & \mathbb{S}^{2n} \end{array}$$

In cohomology of degree $2n$, the diagram gives

$$\begin{array}{ccc} H^{2n}(\mathbb{S}^{2n}) & \xrightarrow{Sg^*} & H^{2n}(\mathbb{S}^{2n}) \\ \downarrow q^* & & \downarrow q^* \\ H^{2n}(C_f) & \xrightarrow{F^*} & H^{2n}(C_{g\circ f}) \end{array}$$

All the groups in the above diagram are $\mathbb{Z}$ and the map Sg^* is multiplication by $\deg(Sg) = \deg(g)$. We also have by the long exact sequence of cohomology for pairs that q^* is an isomorphism (as $H^k(C_f, C_f/(C_f\backslash e^{2n})) = 0$ for $k \neq 2n$). From the diagram we therefore get F^* is also multiplication by $\deg(g)$. Let α_f and $\alpha_{f\circ g}$ be generators of $H^n(C_f)$ and $H^n(C_{f\circ g})$ respectively. As F is identity on $\mathbb{S}^n$, F^* takes α_f to $\alpha_{f\circ g}$. Also let β_f and $\beta_{f\circ g}$ be generators of $H^{2n}(C_f)$ and $H^{2n}(C_{f\circ g})$ respectively. Then we have

$$F^*(\beta_f) = \deg(g)\beta_{f\circ g}, \quad F^*(\alpha_f) = \alpha_{f\circ g} \tag{2.3}$$

Hence by Eq. 2.3 and Proposition 2.1,

$$H(f\circ g)\beta_{f\circ g} = \alpha^2_{f\circ g} = F^*(\alpha^2_f) = F^*(H(f)\beta_f) = (\deg(g))H(f)\beta^2_{f\circ g}$$

which implies $H(f\circ g) = (\deg(g))H(f)$. ■

Now, let us calculate the Hopf invariant of the Hopf map $\mathfrak{h} : \mathbb{S}^3 \to \mathbb{S}^2$.

Proposition 2.6. *The Hopf invariant of the Hopf map is* 1.

Proof. We have $\mathfrak{h} : \mathbb{S}^3 \to \mathbb{S}^2$ and $C_\mathfrak{h}$ is by definition given by the quotient space of $\mathbb{S}^2 \amalg e^4$ with the identification $x \sim \mathfrak{h}(x)$ for all $x \in \partial e^4 = \mathbb{S}^3$. The cohomology of $C_\mathfrak{h}$ is given by

$$H^k(C_\mathfrak{h}) = \begin{cases} \mathbb{Z} & k = 0, 2, 4 \\ 0 & otherwise \end{cases}$$

The CW structure of $\mathbb{CP}^2$ (see Example A.1(2)) gives $\mathbb{CP}^2 = (\mathbb{CP}^1 \amalg e^4)/\sim$, with the identification '$\sim$' given by $(z_0, z_1) \sim [z_0, z_1]$ for all $(z_0, z_1) \in \partial e^4 = \mathbb{S}^3$. Note that the Hopf map $\mathfrak{h}$ is also defined by $(z_0, z_1) \mapsto [z_0, z_1]$. Hence using the homeomorphism $\mathbb{CP}^1 \cong \mathbb{S}^2$ (see Proposition A.1), we see that $C_\mathfrak{h}$ and $\mathbb{CP}^2$ are the quotient space of the same space $\mathbb{S}^2 \amalg e^4$ and the identification in the quotient are also same. Hence $\mathbb{CP}^2$ and $C_\mathfrak{h}$ are homeomorphic and $H^*(C_\mathfrak{h}) = H^*(\mathbb{CP}^2)$. Now from the cohomology ring structure of $\mathbb{CP}^2$ (see Proposition 2.3), we have $\alpha \smile \alpha = \beta$, where α and β are the generators of $H^2(\mathbb{CP}^2)$ and $H^4(\mathbb{CP}^2)$. Hence we must have $\alpha_\mathfrak{h}^2 = \beta_\mathfrak{h}$ and $H(\mathfrak{h}) = 1$. ∎

From Proposition 2.5(1), we know that the Hopf invariant $H : \pi_3(\mathbb{S}^2) \to \mathbb{Z}$ is a homomorphism. From Proposition 2.6, H is surjective. It can be also shown that $H : \pi_3(\mathbb{S}^2) \to \mathbb{Z}$ is injective. Hence H gives us an isomorphism from $\pi_3(\mathbb{S}^2)$ to $\mathbb{Z}$. From Proposition 2.5, we have a better understanding of the elements of $\pi_3(\mathbb{S}^2)$ using Hopf invariant. By Proposition 2.5(1), we have that the element of $\pi_3(\mathbb{S}^2)$ that corresponds to the element $k \in \mathbb{Z}$ is the homotopy class of the map $\mathfrak{h} \circ g$, where $g : \mathbb{S}^3 \to \mathbb{S}^3$ is a map of degree k. As there are maps of arbitrary degree on $\mathbb{S}^3$, we have an idea of every element of $\pi_3(\mathbb{S}^2)$. Also from Proposition 2.5(2), we know that the element $k^2 \in \mathbb{Z}$ corresponds to the homotopy class of the map $f \circ \mathfrak{h}$, where $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a map of degree k.
Let $g : \mathbb{S}^3 \to \mathbb{S}^3$ be a map of degree k^2 and $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a map of degree k. By Proposition 2.5, both $\mathfrak{h} \circ g$ and $f \circ \mathfrak{h}$ corresponds to the element of $k^2 \in \mathbb{Z}$ via the isomorphism H. Hence we must have $[\mathfrak{h} \circ g] = [f \circ \mathfrak{h}]$. Using the properties of Hopf invariant, we concluded that two maps are homotopic with only knowing their degrees!!!
The existence of Hopf invariant 1 is not at all a common phenomenon, an element of Hopf invariant 1 in $\pi_{2n-1}(\mathbb{S}^n$ exists if and only if $n = 2, 4$ or 8. This result was proved by J.F. Adams in 1960 (see [21]) using secondary cohomology operations.

Chapter 3

Degree of a Smooth Map

We will now introduce an important concept in Differential Topology, called the degree of a map. We have already seen one definition of degree of a continuous map $\mathbb{S}^n \to \mathbb{S}^n$ using homology in Chapter-1. Here we will be mostly dealing with smooth manifolds and develop the notion of degree of a smooth map between smooth manifolds of same dimension. We will use de Rham cohomology groups to define degree instead of the homology groups as we have done before. One of the key concept that will be important in our discussion is the integration on a manifold. We will briefly introduce orientability and integration and then go to the theory of degree.

3.1 Orientations on a Manifold

On a vector space an orientation is specified by a choice of an ordered basis. We say that two different orientations are equivalent if the determinant of the change of basis matrix is positive. So, it is quite clear that there can only be two different orientations of a vector space. We will define the orientation of a manifold in a similar fashion. To give a manifold an orientation, we orient the tangent spaces at each point of the manifold in a "coherent" way so that it does not change abruptly anywhere.

Definition 3.1. An n-manifold is said to be orientable if it has an atlas

$$\{(U_\alpha, \varphi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha)\}$$

such that

$$det\Big(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\Big) > 0$$

on the intersection $U_\alpha \cap U_\beta$.

Although we have defined what orientation on a manifold is, we would like to give some motivation behind such a definition. Let us recall the change of variables formula from calculus:

$$\int_{\mathbb{R}^n} f(y_1, \dots, y_n) dy_1 \cdots dy_n = \int_{\mathbb{R}^n} f(y_1(x), \dots, y_n(x)) |det(\partial y_i / \partial x_j)| dx_1 \cdots dx_n.$$

Now using the change of coordinate formula for an n-form ω, we have:

$$\begin{aligned} \int_{\mathbb{R}^n} \omega &= \int_{\mathbb{R}^n} f(y_1, \dots, y_n) dy_1 \cdots dy_n \\ &= \int_{\mathbb{R}^n} f(y_1(x), \dots, y_n(x)) det(\partial y_i / \partial x_j) dx_1 \cdots dx_n. \end{aligned}$$

The only difference in the above two expressions is the absolute value. It is quite clear from Definition 3.1 that for an orientable manifold, the above two expression agree. So, we have a consistent sign of the above integral over all coordinate chats. This enables us to assert a coordinate-independent value to the integral of an n-form over an orientable manifold. Next we give an important characterization of orientability in terms of differential forms.

Proposition 3.1. *A manifold M of dimension n is orientable if and only if there exists a nowhere vanishing n-form on M.*

Proof. Let ω be a nowhere vanishing n-form on M, and consider an atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that U_α's are connected. Our goal is to construct a new atlas where the change of coordinate has positive determinant. Let,

$$\omega|_{U_\alpha} = f_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha,$$

where the function $f \neq 0$ at all points of U_α. As U_α is connected, f_α has a fixed sign. If f_α is positive we keep the corresponding chart. If f_α is negative, then we change the chart to a new one by composing φ_α with the change of coordinate map $(x_1, \dots, x_n) \mapsto (-x_1, \dots, x_n)$. Clearly, in the new coordinate system, f_α is positive. We repeat this process for each coordinate neighbourhoods and obtain

a new atlas which we also denote as $\{(U_\alpha, \varphi_\alpha)\}$, where the coefficient functions f_α are all positive. Moreover whenever $U_\alpha \cap U_\beta \neq \varnothing$, we have

$$\begin{aligned}\omega|_{U_\alpha \cap U_\beta} &= f_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \\ &= f_\alpha det(\partial x_i^\alpha / \partial x_j^\beta) dx_1^\beta \wedge \cdots \wedge dx_n^\beta \\ &= f_\beta dx_1^\beta \wedge \cdots \wedge dx_n^\beta\end{aligned}$$

Since $f_\alpha > 0$ and $f_\beta > 0$, we must have $det(\partial x_i^\alpha / \partial x_j^\beta) > 0$.
Conversely, if M is orientable, we have an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which the change of coordinates have positive determinant. Take a partition of unity $\{\rho_\alpha\}$ subordinate to this cover and put

$$\omega = \sum_\alpha \rho_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha. \tag{3.1}$$

Then on a coordinate neighbourhood $\{(U_\beta, \varphi_\beta)\}$ we have

$$\omega|_{U_\beta} = \sum_\alpha \rho_\alpha det(\partial x_i^\alpha / \partial x_j^\beta) dx_1^\beta \wedge \cdots \wedge dx_n^\beta.$$

Since $\rho_\alpha \geqslant 0$, $det(\partial x_i^\alpha / \partial x_j^\beta) > 0$ and $\rho_\beta \neq 0$ on U_β, $\omega|_{U_\beta}$ is non-vanishing for each U_β. Hence ω in 3.1 is non-vanishing on M. ■

Now this characterization helps us to quickly see some examples of orientable and non-orientable manifolds.

Example 3.1. *1. The most trivial example to start with is $\mathbb{R}^n$. Clearly, it has a nowhere vanishing n-form which is $dx_1 \wedge \cdots \wedge dx_n$. Hence by Proposition 3.1 it is orientable.*

2. *The next class of manifolds that comes to our mind is the n-sphere. Consider $\mathbb{S}^n$ as a sub-manifold of $\mathbb{R}^{n+1}$. Note that*

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}$$

is a non-vanishing n-form on $\mathbb{S}^n$. By Proposition 3.1 $\mathbb{S}^n$ is orientable. The n-form ω is also a volume form.

3. *Consider the real projective space $\mathbb{RP}^n$ and the map $p : \mathbb{S}^n \to \mathbb{RP}^n$ which maps a unit vector in $\mathbb{R}^{n+1}$ to the one dimensional subspace it spans. Note that the map p is smooth and locally invertible as only the antipodal points map to the same point in $\mathbb{RP}^n$. Let $-\mathbb{1} : \mathbb{S}^n \to \mathbb{S}^n$ be the diffeomorphism $x \mapsto -x$, called the antipodal map. Then*

$$\begin{aligned}(-\mathbb{1})^*\omega &= \sum_{i=1}^{n+1}(-1)^{i-1}(-x_i)d(-x_1)\wedge\cdots\wedge\widehat{d(-x_i)}\wedge\cdots\wedge d(-x_{n+1}) \\ &= (-1)^{n+1}\omega.\end{aligned}$$

Suppose $\mathbb{RP}^n$ is orientable. Then by Proposition 3.1 it has a non-vanishing n-form τ. Since the map p is smooth and locally invertible, its derivative is invertable at all points. So, $p^*\tau$ is a non-vanishing n-form on $\mathbb{S}^n$ and so $p^*\tau = f\omega$ for some non-vanishing smooth function f. As $p \circ (-\mathbb{1}) = p$, we have

$$f\omega = p^*\tau = (p\circ(-\mathbb{1}))^*\tau = (-\mathbb{1})^*f\omega = (f\circ(-\mathbb{1})) = (-1)^{n+1}\omega.$$

Thus if n is even, $f \circ (-\mathbb{1}) = -f$. So, if $f(x) > 0$, $f(-x) = -f(x) < 0$. Hence f must vanish somewhere, which is a contradiction. So $\mathbb{RP}^n$ is a non-orientable manifold when n is even.

3.2 Integration on Manifold

In this section, we give a brief discussion about integration on manifolds. The contents of this section are mostly taken from [3]. Let M be an orientable manifold of dimension n with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$. Suppose ω is an n-form with compact support on U, where $\{(U, \phi)\}$ is a chart in the given oriented atlas. Note that $(\phi^{-1})^*\omega$ is an n-form with compact support on the open set $\phi(U) \subset \mathbb{R}^n$. We define

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^*\omega \tag{3.2}$$

If (U, ψ) is another chart in the oriented atlas with the same U, then $\phi \circ \psi^{-1} : \psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism as the chart is oriented, and so

$$\int_{\phi(U)} (\phi^{-1})^*\omega = \int_{\psi(U)} (\phi\circ\psi^{-1})^*(\phi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega$$

Now let ω be an n-form with compact support. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha\omega$ are identically zero and $\omega = \sum_\alpha \rho_\alpha\omega$ is a finite sum. Since $supp(\rho_\alpha\omega) \subset supp(\rho_\alpha) \cap supp(\omega)$, $supp(\rho_\alpha\omega)$ is a closed subset of the compact set $supp(\omega)$. Since $\rho_\alpha\omega$ is a compact form on U_α, $\int_{U_\alpha} \rho_\alpha\omega$ is well defined. So, we define the integral of ω over M to be

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha\omega \tag{3.3}$$

In the definition of the integral we have a choice of partition of unity. So, to say the integral is well defined, we must show that it is independent of the choice of partition of unity. Let $\{V_\beta, \psi_\beta\}$ be another oriented atlas of M and $\{\chi_\beta\}$ be a partition of unity subordinate to $\{V_\beta\}$. Then $\{U_\alpha \cap V_\beta, \phi_\alpha|_{U_\alpha\cap V_\beta}\}$ and $\{U_\alpha \cap V_\beta, \psi_\alpha|_{U_\alpha\cap V_\beta}\}$ are two new atlases of M, specifying the orientation of M, and

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta\omega \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha\chi_\beta\omega \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha\cap V_\beta} \rho_\alpha\chi_\beta\omega
\end{aligned}$$

where we can interchange the sum and integral as all are finite sums. The last line follows from the fact that $supp(\rho_\alpha\chi_\beta) \subset supp(\rho_\alpha) \cap supp(\chi_\beta) \subset U_\alpha \cap V_\beta$. Similarly, the other integral $\sum_\beta \int_{V_\beta} \chi_\beta\omega$ is also equal to $\sum_{\alpha,\beta} \int_{U_\alpha\cap V_\beta} \rho_\alpha\chi_\beta\omega$. Hence,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega = \sum_\beta \int_{V_\beta} \chi_\beta\omega$$

showing that Eq. 3.3 is well defined.

3.3 The de Rham Cohomology

In this section we will introduce the de Rham cohomology of a manifold and compute a few examples. One important question to answer when a differential form is exact. As, $d^2 = 0$, a necessary condition is that the forms is closed. It was proved by Poincare that every k-form on $\mathbb{R}^n$ is exact if and only if it is exact

for $k = 1, 2, 3$. Later a more general version was proved, called the Poincare Lemma. For a general manifold, whether every closed form is exact depends on the topology of the manifold. For example on $\mathbb{R}^n$ every closed form is exact, but on $\mathbb{S}^n$ there are closed forms which are not exact. The de Rham cohomology measures the extent to which closed forms are not exact.

Definition 3.2. Let $Z^k(M)$ be the vector space of all closed k-forms and $B^k(M)$ be the vector space of all exact k-forms. As $d^2 = 0$, all exact forms are closed and $B^k(M) \subset Z^k(M)$. We define the k-th de Rham cohomology vector space of M to be the quotient

$$H^k_{\mathbb{R}}(M) = Z^k(M)/B^k(M)$$

The equivalence class of a closed form is called the cohomology class and two closed forms ω and ω' are called cohomologous if $\omega = \omega' + d\tau$.

Proposition 3.2. *Let M be a connected manifold. Then $H^0_{\mathbb{R}}(M) = \mathbb{R}$.*

Proof. Note that there is no exact 0-form. So, $H^0_{\mathbb{R}}(M) = Z^0(M)$. Let us take a 0-form, i.e. a C^∞-function f such that $df = 0$. On a coordinate chart $(U, x_1, \ldots, x_n)$, we can write $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. As $df = 0$, we have all partial derivatives of f are zero. Hence f is locally constant function on U. We see that the closed 0-forms are identified with the constant value it takes. So, $H^0_{\mathbb{R}}(M) = \mathbb{R}$. ∎

Proposition 3.3. *Let M be a manifold of dimension n. Then $H^k_{\mathbb{R}}(M) = 0$ for $k > n$.*

Proof. For $p \in M$, T_pM is a vector space of dimension n. For a k-form ω on M, $\omega_p \in A_k(T_pM)$, the vector space of all k-covectors. But as $k > n$, $A_k(T_pM) = 0$. So, ω is the zero form and $H^k_{\mathbb{R}}(M) = 0$ for $k > n$. ∎

Example 3.2.

1. *(de Rham Cohomology of $\mathbb{R}$) Since $\mathbb{R}$ is connected, by Proposition 3.2 $H^0_{\mathbb{R}}(\mathbb{R}) \cong \mathbb{R}$. On $\mathbb{R}$ there are no non zero 2-forms. So, every form is closed. Let $f dx$ be a 1-form on $\mathbb{R}$. Define the function g on $\mathbb{R}$ by setting $g(x) := \int_0^x f(t)dt$. Then by fundamental theorem of calculus, $g'(x) = f(x)$ and $dg = g'(x)dx = f(x)dx$. This proves every 1-form on $\mathbb{R}$ is exact. Using this fact and Proposition 3.3, we have $H^k_{\mathbb{R}}(\mathbb{R}) = 0$ for $k > 0$.*

2. *Let U be a disjoint union of m open intervals in $\mathbb{R}$. Then $H^0_{\mathbb{R}}(U) = \mathbb{R}^m$ and $H^k_{\mathbb{R}}(U) = 0$ for $k > n$.*

3. *In general*

$$H^k_{\mathbb{R}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

This result is known as the Poincaré lemma and will be proved later.

For any smooth map $f : M \to N$, there is a map $f^* : \Omega^*(N) \to \Omega^*(M)$, called the pullback of f.

Lemma 3.1. *The pullback map sends closed forms to closed forms and exact forms to exact forms.*

Proof. Let ω be a closed form, i.e. $d\omega = 0$. As the pullback commutes with the exterior derivative d, we have $df^*\omega = f^*d\omega = f^*0 = 0$. Hence, $f^*\omega$ is closed. Similarly $f^*d\tau = df^*\tau$ is an exact form. ■

It follows that f^* induces a map between the quotients $f^* : H^k_{\mathbb{R}}(N) \to H^k_{\mathbb{R}}(M)$, also denoted as f^*, given by $f^*[\omega] = [f^*\omega]$. The functorial properties of the pullback map on differential forms easily yield the same functorial properties for the induced map in cohomology namely,

1. If $\mathbb{1} : M \to M$ is the identity map, then $\mathbb{1}^* : H^k_{\mathbb{R}}(M) \to H^k_{\mathbb{R}}(M)$ is also the identity map.
2. If $f : M \to N$ and $g : N \to P$ are smooth maps, then $(g \circ f)^* = f^* \circ g^*$.

In case of singular cohomology, we have the cup product which gives the product structure on the singular cohomology ring. Similarly, the wedge product gives a product structure on the vector space $\Omega^*(M)$. This product induces a product on de Rham cohomology: if $[\omega] \in H^k_{\mathbb{R}}(M)$ and $[\tau] \in H^l_{\mathbb{R}}(M)$, define

$$[\omega] \wedge [\tau] := [\omega \wedge \tau] \in H^{k+l}_{\mathbb{R}}(M)$$

To see that the product is well defined we observe the following facts about wedge product.

1. The wedge product $\omega \wedge \tau$ of closed forms is closed.
   This follows from the formula: $d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge (d\tau)$.

2. The class $[\omega \wedge \tau]$ is independent of the choice of representative of $[\omega]$ or $[\tau]$.
   If ω is replaced by $\omega' = \omega + d\alpha$, then we have $\omega' \wedge \tau = \omega \wedge \tau + d\alpha \wedge \tau$. We have to show that $d\alpha \wedge \tau$ is exact. Indeed we have $d(\alpha \wedge \tau) = d\alpha \wedge \tau + (-1)^{k-1}\alpha \wedge d\tau = d\alpha \wedge \tau$ as $d\tau = 0$. Similarly the other one follows.

For an n-manifold we set

$$H^*_{\mathbb{R}}(M) := \bigoplus_{k=1}^{n} H^k_{\mathbb{R}}(M)$$

Every element $\omega \in H^*_{\mathbb{R}}(M)$ can be written as $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$, $\omega_k \in H^k_{\mathbb{R}}(M)$. Elements of $H^*_{\mathbb{R}}(M)$ can be added and multiplied, multiplication here being the wedge product. It can be verified that under this addition and multiplication $H^*_{\mathbb{R}}(M)$ becomes a ring, called the de Rham cohomology ring. The cohomology ring has a natural grading by degree of closed forms. Also, the cohomology ring is an anticommutative graded ring, i.e. $\omega \wedge \tau = (-1)^{kl}\tau \wedge \omega$. Since $H^*_{\mathbb{R}}(M)$ is a real vector space, it is a real graded algebra.
For a C map $f : M \to N$, we have the pull back map $f^* : H^k_{\mathbb{R}}(N) \to H^k_{\mathbb{R}}(M)$. Because one has $f * (\omega \wedge \tau) = f^*\omega \wedge f^*\tau$, $f^* : H^*_{\mathbb{R}}(N) \to H^*_{\mathbb{R}}(N)$ is well defined and becomes a ring homomorphism. So, the de Rham cohomology ring gives us a contravariant functor from the category of smooth manifolds to anticommutative graded ring. If two manifolds M and N are diffeomorphic, then $H^*_{\mathbb{R}}(M)$ and $H^*_{\mathbb{R}}(N)$ are isomorphic anticommutative graded ring. Hence, the de Rham cohomology ring is an important diffeomorphism invariant of smooth manifolds. It is also a homotopy invariant as in the case of singular cohomology ring. We have defined homotopy in the continuous setting. But as our discussion is regarding smooth manifolds and smooth maps, we need to extend the notion of homotopy to the smooth setting.

Definition 3.3. (Smooth Homotopy) Two smooth maps $f, g : M \to N$ between two smooth manifolds M and N are said to be smoothly homotopic if there is a smooth map $F : M \times \mathbb{R} \to N$ such that $F|_{t=0} = f$ and $F|_{t=1} = g$.

The map F is called a homotopy between f and g. The map F can be thought of a smooth family $\{f_t = F(-, t) : t \in \mathbb{R}\}$ of maps varying smoothly in t so that $f_0 = f$ and $f_1 = g$. As an example we can think of the straight-line homotopy $F(x, t) = (1 - t)f(x) + tg(x)$ for maps $f, g : M \to \mathbb{R}^n$. F is smooth on $M \times \mathbb{R}$ and

$F|_{t=0} = f$ and $F|_{t=1} = g$.

Definition 3.4. (Smooth Homotopy Equivalence) A smooth map $f : M \to N$ is said to be a smooth homotopy equivalence if there is a map $g : N \to M$ such that $g \circ f$ is smoothly homotopic to $\mathbb{1}_M$ and $f \circ g$ is smoothly homotopic to $\mathbb{1}_N$.

In this case we say that M is smoothly homotopy equivalent to N, or that M and N have the same smooth homotopy type.

Example 3.3. *1. A diffeomorphism is a smooth homotopy equivalence.*

2. *Let $p \in \mathbb{R}^n$ and $i : \{p\} \hookrightarrow \mathbb{R}^n$ be the inclusion map. Let $r : \mathbb{R}^n \to \{p\}$ be the constant map $x \mapsto p$ for all $x \in \mathbb{R}^n$. Then $r \circ i = \mathbb{1}_{\{p\}}$ and the straight-line homotopy*

$$F(x,t) = (1-t)x + tp$$

gives a smooth homotopy between $\mathbb{1}_{\mathbb{R}^n}$ and $i \circ r$. Hence $\mathbb{R}^n$ and $\{p\}$ have the same homotopy type.

3. *Let $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}\backslash\{0\}$ be the inclusion and $r : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{S}^n$ be the map $x \mapsto \frac{x}{|x|}$. Note that r is a smooth map and $r \circ i = \mathbb{1}_{\mathbb{S}^n}$. Let us define the map $F : \mathbb{R}^{n+1}\backslash\{0\} \times \mathbb{R} \to \mathbb{R}^{n+1}\backslash\{0\}$ by*

$$F(x,t) := (1-t)^2 x + t^2 r(x) = (1-t)^2 x + t^2 \frac{x}{|x|}.$$

Note that F never takes the value 0 as $F(x,t) = 0 \leftrightarrow (1-t)^2 + t^2/|x| \leftrightarrow t = 1 = 0$, which is a contradiction. Also F is smooth and F gives a smooth homotopy between $\mathbb{1}_{\mathbb{R}^{n+1}\backslash\{0\}}$ and $i \circ r$. Hence $\mathbb{S}^n$ and $\mathbb{R}^{n+1}\backslash\{0\}$ are smoothly homotopic.

We will now state the homotopy axiom for de Rham Cohomology without a proof. A proof can be found in Section 28 of [3].

Theorem 3.1. *Smoothly homotopic maps $f_0, f_1 : M \to N$ induce the same map $f_0^* = f_1^* : H^*_{\mathbb{R}}(N) \to H^*_{\mathbb{R}}(M)$ in cohomology.*

Corollary 3.1. *If $f : M \to N$ is a smooth homotopy, then the induced map $f^* : H^*_{\mathbb{R}}(N) \to H^*_{\mathbb{R}}(M)$ is an isomorphism.*

Proof. By Definition 3.4 there is a $g : N \to M$ such that $g \circ f \sim \mathbb{1}_M$ and $f \circ g \sim \mathbb{1}_N$. By Theorem 3.1 and functoriality we have $f^* \circ g^* = (g \circ f)^* = \mathbb{1}_M^*$ and $g^* \circ f^* = (f \circ g) = \mathbb{1}_N^*$. Hence g^* is the inverse of f^*, making both f^* and g^* isomorphisms. ∎

From Example 3.3 we know that $\mathbb{R}^n$ has the same homotopy type as a point. Hence the de Rham Cohomology of $\mathbb{R}^n$ is same as that of a point. This proves the Poincaré Lemma that we mentioned in Example 3.2.
Next we prove a result concerning the de Rham cohomology of a compact connected orientable manifold in the top dimension. We will first prove two lemmas that will be needed for proving the result we are looking for.

Lemma 3.2. *Let f be a smooth function on $\mathbb{R}^n$ with support in the open cube $C^n = (-1,1)^n$ and*

$$\int_{\mathbb{R}^n} f dx_1 \cdots dx_n = 0$$

Then there exist smooth functions $f_1, \ldots, f_n$ on $\mathbb{R}^n$ with support in C^n such that

$$f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

Proof. We will prove the lemma using induction on the dimension n.
For $n = 1$, we are given $\int_{\mathbb{R}} f dx = 0$. Set

$$g(x) := \int_{-1}^{x} f(t)dt$$

Then we have $\frac{\partial g}{\partial x} = f$. Note that $supp(g) = \overline{\{x | \int_{-1}^{x} f(t)dt \neq 0\}}$. As, $supp(f) \subset (-1,1)$, $\int_{-1}^{1} f dx = 0$. Hence $supp(g) \subset (-1,1)$.
Let us assume that the statement is true for all values less than n. We are given a smooth function f on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}} f dx_1 \cdots dx_n = 0$$

Fix $x_n = t$ and define the function g on $\mathbb{R}^{n-1}$ by

$$g(x_1, \ldots, x_{n-1}) := f(x_1, \ldots, x_{n-1}, t)$$

Let σ be a bump function on C^{n-1} such that

$$\int_{C^{n-1}} \sigma dx_1 \cdots dx_{n-1} = 1$$

Set

$$h(t) := \int_{C^{n-1}} f(x_1, \ldots, x_{n-1}, t) dx_1 \cdots dx_{n-1}$$

Now we have

$$g(x_1, \ldots, x_{n-1}) - h(t)\sigma(x_1, \ldots, x_{n-1}) =: h'(x_1, \ldots, x_{n-1}) \tag{3.4}$$

Using the fact that $supp(\sigma) \subset C^{n-1}$ and $supp(f) \subset C^{n-1}$ and integrating 3.4 over $\mathbb{R}^{n-1}$, we get

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} h' dx_1 \cdots dx_{n-1} &= \int_{\mathbb{R}^{n-1}} g dx_1 \cdots dx_{n-1} - \int_{\mathbb{R}^{n-1}} h(t)\sigma dx_1 \cdots dx_{n-1} \\
&= \int_{\mathbb{C}^{n-1}} f(x_1, \ldots, x_{n-1}, t) dx_1 \cdots dx_{n-1} - h(t) \int_{\mathbb{C}^{n-1}} \sigma dx_1 \cdots dx_{n-1} \\
&= h(t) - h(t) = 0
\end{aligned}$$

As h' is a function on $\mathbb{R}^{n-1}$, we have by induction hypothesis

$$h' = \sum_{i=1}^{n-1} \frac{\partial f_i}{\partial x_i} \tag{3.5}$$

Set $f_n(x_1, \ldots, x_n) := (\int_{-1}^{x_n} h(t)dt)\sigma(x_1, \ldots, x_{n-1})$. Then,

$$\frac{\partial f_n}{\partial x_n} = h(x_n)\sigma(x_1, \ldots, x_{n-1}) \tag{3.6}$$

Using Eq. 3.5 & Eq. 3.6 and putting $t = x_n$ in Eq. 3.4 we get,

$$\begin{aligned}
f(x_1, \ldots, x_{n-1}, x_n) &= h'(x_1, \ldots, x_{n-1}) + h(x_n)\sigma(x_1, \ldots, x_{n-1}) \\
&= \sum_{i=1}^{n-1} \frac{\partial f_i}{\partial x_i} + \frac{\partial f_n}{\partial x_n} \\
&= \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}
\end{aligned}$$

By hypothesis, $f_1, \ldots f_{n-1}$ has support in C^{n-1}. Also, note that $h(t)$ has support in $(-1, 1)$. From Eq. 3.4, we see that h' is zero if $t > 1 - \delta$ or $t < -1 + \delta$. So, $supp(f_j) \subset C^{n-1} \times (-1, 1) = C^n$ for $j = 1, \ldots, n-1$.

For f_n, if $t > 1 - \delta$,

$$\begin{aligned}\int_{-1}^{t} h(t)dt &= \int_{-1}^{t} \Big(\int_{C^{n-1}} f(x_1, \ldots, x_{n-1}, t)dx_1 \cdots dx_{n-1} \Big) dt \\ &= \int_{-1}^{1} \Big(\int_{C^{n-1}} f(x_1, \ldots, x_{n-1}, t)dx_1 \cdots dx_{n-1} \Big) dt \text{ (as } f = 0 \text{ for } t > 1 - \delta) \\ &= \int_{C^{n-1}} f(x_1, \ldots, x_n)dx_1 \cdots dx_{n-1} \\ &= 0 \text{ (by hypothesis)}\end{aligned}$$

Thus $supp(f_n) \subset C^n$. ■

Lemma 3.3. *Let ω be an n-form on $\mathbb{R}^n$ with support contained in the open cube C^n such that*

$$\int_{\mathbb{R}^n} \omega = 0$$

Then there exists an $(n-1)$-form η on $\mathbb{R}^n$ with support in C^n such that $d\eta = \omega$.

Proof. Write $\omega = f dx_1 \wedge \cdots \wedge dx_n$ with $supp(f) \subset C^n$. As $\int_{\mathbb{R}^n} \omega = 0$, we have

$$\int_{\mathbb{R}^n} f dx_1 \cdots dx_n = 0$$

By Lemma 3.2,

$$f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

with $supp(f_i) \subset C^n$. Define the $(n-1)$-form η on $\mathbb{R}^{n-1}$ by,

$$\eta = \sum_{i=1}^{n} (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

Then we have

$$d\eta = \Big(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \Big) dx_1 \wedge \cdots \wedge dx_n = f dx_1 \wedge \cdots \wedge dx_n = \omega$$

and $supp(\eta) \subset C^n$ as $supp(f_i) \subset C^n$ for all $i = 1, \ldots, n$. ■

These two lemmas will be helpful in proving our next result regarding the cohomology in top dimension of a compact orientable manifold.

Proposition 3.4. *If M is a compact connected orientable smooth manifold of dimension n, then $H^n_{\mathbb{R}}(M) \cong \mathbb{R}$.*

Proof. By compactness, there is a finite cover $\{U_1, \ldots, U_m\}$ by coordinate nbds diffeomorphic to the open cube C^n. Let ω_0 be a bump n-form with support contained in U_1 and total integral 1. If $\omega_0 = d\eta$, then $\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$, by Stokes Theorem. So, ω_0 is not exact. It is closed as there is no $(n+1)$-form on M. Hence ω_0 defines a non-zero cohomology class in $H^n_{\mathbb{R}}(M)$. We shall show that every n-form on M is cohomologus to a multiple of ω_0. Let ω be a n-form on M. we have to show

$$\omega = c\omega_0 + d\eta \; ; c \in \mathbb{R}, \eta \in \Omega^{n-1}(M). \tag{3.7}$$

Using a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$, we write $\omega = \sum_i \rho_i\omega$, where $\rho_i\omega$ is an n-form with support in U_i. By linearity of d, it is enough to prove Eq. 3.7 for $\rho_i\omega$. So, without loss of any generality, we assume that ω has support in U_k for some $k \in \{1, \ldots, m\}$.

Let $x \in U_1$ and $y \in U_k$ be two disjoint points. As M is connected, there is a path connecting x and y. Using the compactness of the path we find $U_{i_1}, \ldots, U_{i_r}$ such that they cover the path and $U_{i_1} = U_1$, $U_{i_r} = U_k$ and $U_{i_j} \cap U_{i_{j+1}} \neq \varnothing$ for all $j = 1, \ldots, r-1$.

For all $j = 1, \ldots, r-1$, choose n-form ω_j with support in $U_{i_j} \cap U_{i_{j+1}}$ and total integral 1. Now $\omega_0 - \omega_1$ has support in $U_{i_1} = U_1$ and total integral zero. By Lemma 3.3, there exist a $(n-1)$-form η_1 with support in U_1 such that

$$\omega_0 - \omega_1 = d\eta_1.$$

Similarly $\omega_1 - \omega_2$ has support in U_{i_2} and total integral zero. So, $\omega_1 - \omega_2 = d\eta_2$, where η_2 has support in U_{i_2}. Continuing for $j = 1, \ldots, r-1$ we get,

$$\begin{aligned} \omega_0 - \omega_1 &= d\eta_1 \\ \omega_1 - \omega_2 &= d\eta_2 \\ &\vdots \\ \omega_{r-2} - \omega_{r-1} &= d\eta_{r-1} \end{aligned}$$

Adding both sides we get $\omega_0 - \omega_{r-1} = \sum_{j=1}^{r-1} d\eta_j$.

Let $\eta = \sum_{j=1}^{r-1} \eta_j$. By linearity of d,

$$\omega_0 - \omega_{r-1} = d\eta. \tag{3.8}$$

Now, the support of ω and ω_{r-1} is contained in $U_{i_r} = U_k$. Let $\int_M \omega = c$. Then $\int_M(\omega - c\omega_{r-1}) = 0$ and $\omega - c\omega_{r-1}$ has support in U_k. Again by Lemma 3.3,

$$\omega - c\omega_{r-1} = d\zeta \tag{3.9}$$

From Eq. 3.8 and Eq. 3.9,

$$\omega - c(\omega_0 - d\eta) = d\zeta \Rightarrow \omega = c\omega_0 + d(\zeta - c\eta)$$

which proves Eq. 3.7. Hence $H^n_{\mathbb{R}}(M) \cong \mathbb{R}$, generated by the bump n-form ω_0. ■

3.4 Degree of a Smooth Map between Manifolds

In this section we will define the notion of degree of a smooth map between compact connected orientable manifolds of same dimension and see some properties and applications of degree.

Definition 3.5. Let $f : M \to N$ be a smooth map between compact connected orientable smooth manifolds of same dimension n. let ω_M and ω_N be the n-form on M, N respectively, with total integral 1, that generate the cohomology group $H^n_{\mathbb{R}}(M)$ and $H^n_{\mathbb{R}}(N)$. $f^* : H^n_{\mathbb{R}}(N) \to H^n_{\mathbb{R}}(M)$ carries $[\omega_N]$ to a multiple of $[\omega_M]$. This number is called the degree of f, denoted as $\deg(f)$.

The definition is same as our previous definition of degree in Chapter-1. Next we will prove two propositions, which will be useful for calculation purpose.

Proposition 3.5. *Let $f : M \to N$ be smooth. Then for all $\omega \in \Omega^n(M)$,*

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

Proof. If τ is an exact n-form on M, say $\tau = d\eta$, then $\int_M \tau = \int_M d\eta = \int_{\partial M} \eta = 0$ (by Stokes' Theorem).
So, the integral of an exact form is zero. Hence the integral only depends on the cohomology class. Let $[\omega_N]$ be the cohomology class that generates $H^n_{\mathbb{R}}(N)$, i.e. $\int_N \omega_N = 1$. So, $[\omega] = c[\omega_N]$. Also, we have $\int_N \omega = c\int_N \omega_N = c$.

Now, by definition

$$\begin{aligned} f^*[\omega] &= f^*(c[\omega]) \\ &= cf^*[\omega_N] \\ &= c \cdot \deg(f)[\omega_N] \end{aligned}$$

Integrating and using the fact that $\int_M \omega_M = 1$ & $\int_N \omega = c$ we get,

$$\int_M f^*\omega = c \cdot \deg(f) \int_M \omega_M = \deg(f) \int_N \omega$$

■

By our previous definition of degree, it was clear that degree of a map is an integer, but this notion of degree does not immediately ensures that it is an integer. It is clear that degree of a smooth map is a real number. We will show now it is indeed an integer.

Proposition 3.6. *Let $f : M \to N$ be a smooth map. If $q \in N$ is a regular value of f, then*

$$\deg(f) = \sum_{p \in f^{-1}(q)} \text{sign}(det(df|_p))$$

Proof. Since q is a regular value, $df|_p$ is surjective at each $p \in f^{-1}(q)$. As $dim(M) = dim(N)$, $df|_p$ is a bijection. So, by Inverse Function Theorem, f is a local diffeomorphism at each $p \in f^{-1}(q)$. As, M is compact and $f^{-1}(q)$ being closed, $f^{-1}(q)$ is compact. Also, f being a local diffeomorphism at each point $p \in f^{-1}(q)$, $f^{-1}(q)$ is discrete and hence finite (only discrete compact sets are finite sets).
Write $f^{-1}(q) = \{p_1, \ldots, p_k\}$ and choose disjoint nbds U_i of p_i and V_i of q such that $f : U_i \to V_i$ is a diffeomorphism. Set $V = \cap_{i=1}^k V_i$ and $\widetilde{U}_i = U_i \cap f^{-1}(V)$.
Then $f : \widetilde{U}_i \to V$ is a diffeomorphism for all i. Moreover $f(M \setminus \cup_{i=1}^k \widetilde{U}_i)$ is a compact subset of N. So, by further shrinking V, we can assume that $f^{-1}(V) = \cup_{i=1}^k \widetilde{U}_i$.
Let ω be an n-form on N, with total integral 1 and support contained in V. Then $f^*\omega$ is an n-form on M with support in $\cup_{i=1}^k \widetilde{U}_i$.

$$\int_{\widetilde{U}_i} f^*\omega = \text{sign}(det(df|_p)) \int_V \omega = \text{sign}(det(df|_p))$$

Here we are considering the determinant of the jacobian matrix of f at p_i, relative to the orientation preserving local charts around p_i and q. So, sign$(det(df|_{p_i}))$ is $+1$ is $df|_{p_i} : T_{p_i}M \to T_qN$ preserves orientation and -1 if it reverses orientation. Hence,

$$def(f) = \int_M f^*\omega = \sum_{i=1}^{k} \int_{\widetilde{U_i}} f^*\omega = \sum_{i=1}^{k} \text{sign}(det(df|_{p_i})).$$

■

Corollary 3.2. $\deg(f)$ *is an integer.*

Proof. Follows from Proposition 3.6 ■

Before looking at some examples, we discuss some important properties and applications of degree. The following proposition lists some of the important properties of degree.

Proposition 3.7. *Let $f : M \to N$ and $g : N \to S$ be two smooth maps between compact connected orientable smooth manifolds of same dimension n. Then*

1. *Let $N = M$ and $f = \mathbb{1}_M$. Then* $\deg(\mathbb{1}_M) = 1$
2. *If $\{f_t | t \in \mathbb{R}\}$ is smooth homotopy, then* $\deg(f_0) = \deg(f_1)$.
3. *If f is not surjective, then* $\deg(f) = 0$.
4. $\deg(g \circ f) = \deg(f)\deg(g)$.

Proof.

1. Since, $\mathbb{1}_M^* = \mathbb{1}_{H^n_{\mathbb{R}}(M)}$, from definition of degree $\deg(\mathbb{1}_M) = 1$.
2. Note that by Corollary 3.1 we have $f_0 = f_1$. By our definition of degree, we must have $\deg(f_0) = \deg(f_1)$.
3. As M is compact, $f(M)$ is also compact and hence a closed subset of N. Set $N \backslash f(M) =: V$. Note that V is open and we choose a bump n-form ω on N such that $supp(\omega) \subseteq V$. Then $f^*\omega = 0$ and by Proposition 3.5, we have $\deg(f) = 0$.

4. By Proposition 3.5, we have

$$\begin{aligned}\deg(g\circ f)\int_S \omega_S &= \int_M (g\circ f)^*\omega_S \\ &= \int_M f^*(g^*\omega_S) \\ &= \deg(f)\int_N g^*\omega_S \\ &= \deg(f)\deg(g)\int_S \omega_S.\end{aligned}$$

As $\int_S \omega_S \neq 0$, we have $\deg(g\circ f) = \deg(f)\deg(g)$.

■

Lemma 3.4. *The antipodal map $-\mathbb{1} : \mathbb{S}^n \to \mathbb{S}^n$ has degree $(-1)^{n+1}$*

Proof. Let ω be the volume form on $\mathbb{S}^n$ as in Example 3.1 and we have seen that $(-\mathbb{1})^*\omega = (-)^{n+1}\omega$. Hence using Proposition 3.5, we see that $\deg(-\mathbb{1}) = (-1)^{n+1}$. ■

An important application of degree is the following result.

Proposition 3.8. *$\mathbb{S}^n$ has a non-vanishing tangent vector field if and only if n is odd.*

Proof. Let $\mathbb{S}^n$ has a non-vanishing smooth vector field $v : \mathbb{S}^n \to T\mathbb{S}^n$, assigning to each point x the tangent vector $v(x) \in T_x\mathbb{S}^n$. Now, as

$$T_x\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} : \langle u, x\rangle = 0\},$$

we have x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$. Take $u(x) = v(x)/|v(x)|$. Then u is a non-vanishing smooth vector field as $v(x) \neq 0$. Also $|u(x)| = 1$ for all $x \in \mathbb{S}^n$. Now define

$$f_t(x) = (\cos t)x + (\sin t)u(x), \quad t \in \mathbb{R}.$$

Then $f_t(x)$ lies in the unit circle of the plane spanned by x and $u(x)$ and thus gives us a homotopy between $\mathbb{1}_{\mathbb{S}^n}$ ($t = 0$) and the antipodal map $-\mathbb{1}$ ($t = \pi$). By Proposition 3.7 and Lemma 3.4, we have $1 = \deg(\mathbb{1}_{\mathbb{S}^n}) = \deg(-\mathbb{1}) = (-1)^{n+1}$. Hence n must be odd.
Conversely, if $n = 2k-1$ is odd, we define

$$v(x_1, y_1, \ldots, x_k, y_k) = (-y_1, x_1, \ldots, -y_k, x_k).$$

Then for $x = (x_1, y_1, \ldots, x_k, y_k)$ we have

$$\langle x, v(x)\rangle = (-x_1y_1 + y_1x_1) + \cdots + (-x_ky_k + y_kx_k) = 0.$$

Hence x and $v(x)$ is orthogonal and $|v(x)| = 1$ for all $x \in \mathbb{S}^{2k-1}$. So, v is a smooth non-vanishing tangent vector field on $\mathbb{S}^{2k-1}$. ■

Example 3.4. *Let us now calculate the degree of some smooth maps between some known smooth manifolds.*

1. *To start with, let us first consider the example $f : \mathbb{S}^1 \to \mathbb{S}^1$, given by $z \mapsto z^n$. If we parametrize $\mathbb{S}^1$ by $\phi : (0, 2\pi) \to \mathbb{S}^1$, $\phi(t) = (\cos t, \sin t)$, the map f locally looks like $(\cos t, \sin t) \mapsto (\cos nt, \sin nt)$. Take the 1-form $\omega = xdy - ydx$ on $\mathbb{S}^1 \subset \mathbb{R}^2$. Note that*

   $$\phi^*\omega = (x \circ \phi)d(y \circ \phi) - (y \circ \phi)d(x \circ \phi) = \cos t(\cos t dt) - \sin t(-\sin t dt) = dt$$

   and $f \circ \phi(t) = (\cos nt, \sin nt)$. Hence, $(f \circ \phi)^*\omega = ndt$. Now by Proposition 3.5, we have

   $$\begin{aligned} & \int_{\mathbb{S}^1} f^*\omega = def(f) \int_{\mathbb{S}^1} \omega \\ \Rightarrow & \int_0^{2\pi} \phi^* f^*\omega = \deg(f) \int_0^{2\pi} \phi^*\omega \\ \Rightarrow & \int_0^{2\pi} ndt = \deg(f) \int_0^{2\pi} dt \\ \Rightarrow & \quad \deg(f) = n \end{aligned}$$

2. *Consider $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, given by $(z_1, z_2) \mapsto (z_1^{n_1}, z_2^{n_2})$. $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ is the set of all points $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ such that $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$. Consider for $j = 1, 2$ the 1-forms $\omega_j = x_j dy_j - y_j dx_j$ and take the 2-form $\omega_1 \wedge \omega_2$ on $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$. Take ϕ to be the parametrization of $\mathbb{S}^1 \times \mathbb{S}^1$ given by $(t_1, t_2) \mapsto (\cos t_1, \sin t_1, \cos t_2, \sin t_2)$. By similar calculations as in the previous example, we can show that $\phi^*\omega_j = dt_j$ and $(f \circ \phi)^*\omega_j = n_j dt_j$. Hence using the formula in Proposition 3.5 we conclude that $\deg(f) = n_1n_2$.*

3. *Consider $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, given by $(z_1, z_2) \mapsto (z_1^p z_2^r, z_1^q z_2^s)$, where*

   $$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Z}).$$

Using the notations used in previous examples, we have

$$f \circ \phi(t_1, t_2) = (\cos(pt_1 + rt_2), \sin(pt_1 + rt_2), \cos(qt_1 + st_2), \sin(qt_1 + st_2)).$$

It is easy to verify that $(f \circ \phi)^*\omega_1 = pdt_1 + rdt_2$ *and* $(f \circ \phi)^*\omega_2 = qdt_1 + sdt_2$. *Hence we have*

$$\begin{aligned}\int_{\mathbb{S}^1 \times \mathbb{S}^1} f^*(\omega_1 \wedge \omega_2) &= \int_0^{2\pi} \int_0^{2\pi} (pdt_1 + rdt_2) \wedge (qdt_1 + sdt_2) \\ &= \int_0^{2\pi} \int_0^{2\pi} (ps - qr) dt_1 \wedge dt_2 \\ &= det \begin{pmatrix} p & q \\ r & s \end{pmatrix} \int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega_1 \wedge \omega_2.\end{aligned}$$

Using the formula in Proposition 3.5 we conclude that $\deg(f) = det \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

4. *Let* $\mathbb{C} \cup \{\}$ *be the extended complex plane and* $f : \mathbb{C} \cup \{\} \to \mathbb{C} \cup \{\}$ *be the map given by*

$$f(z) = \begin{cases} z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0 & z \neq \\ & z = \end{cases}$$

This is smooth at all points $z \neq$ *as it is a polynomial on* $\mathbb{C}$. *For points near* $z =$, $f(z)$ *is smooth if and only if* $f(1/z)$ *is smooth near* $z = 0$. *Note that on a small disc around* $z = 0$, $f(1/z)$ *is given by*

$$w \mapsto \frac{w^k}{1 + a_{k-1}w + \cdots + a_0 w^k}$$

which is smooth as the denominator never vanish on a small disc around zero. Now let us define for each $t \in \mathbb{R}$

$$f_t(z) = z^k + t(a_{k-1} + \cdots + a_1 z + a_0).$$

This is a smooth map for all $t \in \mathbb{R}$. *By Proposition 3.7(2),*

$$\deg(f_0) = \deg(f_1) = \deg(f)$$

where $f_0(z) = z^k$. *To calculate the degree of this map, we take a 2-form* $\phi(x^2 + y^2)dx \wedge dy$ *with* $supp(\phi)$ *compact. Writing it in polar co-ordinates* r *and* θ, *we*

have

$$\phi(x^2 + y^2)dx \wedge dy = \phi(r)rdr \wedge d\theta.$$

Then the degree is given by

$$\begin{aligned}\deg(f_0)\int_{\mathbb{R}^2} f(r)rdr \wedge d\theta &= \int_{\mathbb{R}^2} f_0^*(f(r)rdr \wedge d\theta) \\ &= \int_{\mathbb{R}^2} f(r^k)(r^k)d(r^k) \wedge d(k\theta) \\ &= k\int_{\mathbb{R}^2} f(r)rdr \wedge d\theta.\end{aligned}$$

Thus $\deg(f) = \deg(f_0) = k$.

This example shows the existence of maps of arbitrary degree on $\mathbb{S}^2$. *Note that in particular if* $k > 0$, *then* f *is surjective and takes the value* 0 *at some point. Therefore every polynomial in* $\mathbb{C}$ *has a root. This is a proof of the* ***fundamental theorem of algebra*** *using degree theory.*

Degree is an important concept in Algebraic and Differential Topology and has many applications. For further reading one can refer to [14] and [15]. The notion of *degree* will be key in our discussion of *Linking Number* in the next chapter. We will define the linking number as a degree of some map from the torus to the sphere.

Chapter 4

Geometry of The Hopf Fibers

In this chapter we will discuss about the geometry of the fibers of the Hopf map. In Proposition 1.9 we have seen that the fibers of the Hopf fibration are circles. The first question that arises after this observation is whether the fibers are linked or not. We will mainly address this question in this chapter and unveil the geometry of the Hopf fibers. This hopefully will give us more insight on the spatial arrangement of the Hopf fibers. Before going to the Hopf map, we introduce the concept of linking number. For a pair of disjoint linked closed curves in $\mathbb{R}^3 \subset \mathbb{S}^3$ we can associate an integer to the pair, known as the linking number, which represents the winding of one curve around the other. In Figure 4.1, we see three links. In the first one, there is no linking. That is why it is known as the Un-link. The second one is linked once and the third one is linked twice. So, we expect that the linking number for the Un-link must be 0, for the Hopf link must be 1 and for the Solomon link must be 2. We will see that the notion of linking number that we will define will be consistent with this observation.

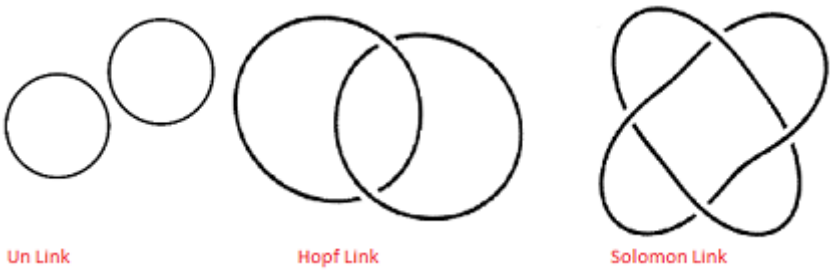


FIGURE 4.1: Various Links
SOURCE: GOOGLE IMAGE

4.1 Linking Number

Carl Friedrich Gauss introduced the concept of linking number in a brief note on his diary in 1833. Although he gave no proof or derivation in that note, it was a cornerstone in the development of the modern theory of linking number, which became fundamental in the field of knot theory and modern topological field theory. Here we will define linking number in terms of degree. There are other interpretations of linking number in terms of signed crossing and intersection number. Interested readers can find detailed discussion about them in [7].
Let C_1 and C_2 be two disjoint, closed, oriented, smooth curves in $\mathbb{R}^3 \subset \mathbb{S}^3$. As we are identifying $\mathbb{R}^3$ with $\mathbb{S}^3 \backslash \{point\}$ via the stereographic projection, we will regard the infinite curves, such as $\{(0,0,z) : z \in \mathbb{R}\}$ as closed curves. So, the only type of curves that are not allowed are the finite non-closed curves and the half-infinite curves, such as $\{(0,0,z) : z \geqslant 0\}$. Let $\gamma_j : [0, 2\pi] \to \mathbb{R}^3$ be parametrizations for C_j. To each pair $(p_1, p_2) \in C_1 \times C_2$ there is a corresponding point (t_1, t_2) on the torus $\mathbb{T}$ such that $\gamma_j(t_j) = p_j$ for $j = 1, 2$.

Definition 4.1. The Gauss map $\psi : \mathbb{T} \to \mathbb{S}^2$ is defined by associating to each point (t_1, t_2) the unit vector

$$n(t_1, t_2) = \frac{\gamma_1(t_1) - \gamma_2(t_2)}{|\gamma_1(t_1) - \gamma_2(t_2)|}.$$

Definition 4.2. The linking number of C_1 and C_2, denoted as $Lk(C_1, C_2)$ is given by

$$Lk(C_1, C_2) := \deg(\psi).$$

In the definition of linking number we have a choice of parametrization of the curves C_1 and C_2. For the definition to be well defined, it should be invariant under change of parametrizations. The next proposition tells us indeed it is.

Proposition 4.1. *Let $\phi_j : [0, 2\pi] \to [0, 2\pi]$ for $j = 1, 2$ be two orientation preserving diffeomorphisms. Let $\psi' : \mathbb{T} \to \mathbb{S}^2$ be the Gauss map associated to the new parametrizations $\gamma_j \circ \phi_j$ for $j = 1, 2$. Then $\deg(\psi') = \deg(\psi)$.*

Proof. Note that ϕ_j is homotopic to $\mathbb{1}_{[0,2\pi]}$ via the linear homotopy. Let H_j be the corresponding homotopies. Then we can define a homotopy F between ψ and ψ' by setting

$$F((t_1, t_2), s) := \frac{\gamma_1(H_1(t_1, s)) - \gamma_2(H_2(t_2, s))}{|\gamma_1(H_1(t_1, s)) - \gamma_2(H_2(t_2, s))|}.$$

By Proposition 3.7(2), we have $\deg(\psi) = \deg(\psi')$. ■

Lemma 4.1. $Lk(C_1, C_2) = Lk(C_2, C_1)$

Proof. The Gauss map associated to $Lk(C_2, C_1)$ is given by $\psi'(t_2, t_1) = -\psi(t_1, t_2)$, where ψ is the Gauss map associated to $Lk(C_1, C_2)$. If $-\mathbb{1}$ is the antipodal map of $\mathbb{S}^2$ and ζ is the map $\mathbb{T} \to \mathbb{T}$ given by $(t_1, t_2) \mapsto (t_2, t_1)$, then $\psi' = (-\mathbb{1}) \circ \psi \circ \zeta$. So,

$$\deg(\psi') = \deg(-\mathbb{1})\deg(\psi)\deg(\zeta) = \deg(\psi)$$

as $\deg(-\mathbb{1}) = \deg(\zeta) = -1$. ■

From Proposition 3.6, we know that

$$\deg(\psi) = \sum_{p \in \psi^{-1}(q)} sgn(det(d\psi|_p)). \tag{4.1}$$

Let $q \in \mathbb{S}^2$ be a regular value of ψ and let $\psi^{-1}(q) = \mathcal{T} = \{(t_{1_1}, t_{2_2}), \ldots, (t_{k_1}, t_{k_2})\}$. By Eq. 4.1 any point of $\mathcal{T}$ contributes ± 1 to the value of $\deg(\psi)$. At any point $\psi(t_{i_1}, t_{i_2})$ $(i = 1, 2, \ldots, k)$ the normal vector $\nu(t_{i_1}, t_{i_2}) = \left(\frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2}\right)_{(t_{i_1}, t_{i_2})}$ gives an orientation to the surface $\psi(\mathbb{T})$ at $\psi(t_{i_1}, t_{i_2})$. Hence the orientation of $\nu(t_{i_1}, t_{i_2})$ (outwards or inwards) determines whether the point (t_{i_1}, t_{i_2}) contributes $+1$ or -1 to the value of degree. Note that $\frac{\partial n}{\partial t_1}$ and $\frac{\partial n}{\partial t_2}$ lives in the tangent space to the sphere at $n(t_1, t_2)$. Hence their cross product must be parallel to to the unit vector $n(t_1, t_2)$. So, the sign is given by the sign of $n(t_{i_1}, t_{i_2}) \cdot \nu(t_{i_1}, t_{i_2})$. Let us denote

$$n(t_{i_1}, t_{i_2}) \cdot \nu(t_{i_1}, t_{i_2}) =: \left(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\right)_{(t_{i_1}, t_{i_2})}; \tag{4.2}$$

and we see that

$$\deg(\psi) = \sum_{(t_{i_1}, t_{i_2}) \in \mathcal{T}} sgn\left(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\right)_{(t_{i_1}, t_{i_2})}. \tag{4.3}$$

Also note that $\left(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\right)_{(t_{i_1}, t_{i_2})} = sgn\left(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\right)_{(t_{i_1}, t_{i_2})} \left|\frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2}\right|_{(t_{i_1}, t_{i_2})}$ as $n(t_{i_1}, t_{i_2})$ and $\nu(t_{i_1}, t_{i_2})$ are parallel. It can be shown that the degree of ψ is given by

$$\deg(\psi) = \frac{1}{4\pi} \int_{\mathbb{T}} \left(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\right) d^2\theta, \tag{4.4}$$

where $d^2\theta$ is the volume form on the torus. For further details and a proof see Proposition-5.6 of [7].

We can also calculate the degree of ψ by calculating the oriented area of $\psi(\mathbb{T})$. Let us sub-divide $\mathbb{T}$ into regions $\mathcal{R}_i$, where $\frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2}$ has a constant sign. The oriented area of $\psi(\mathcal{R}_i)$ is given by

$$\pm \int \int \Big| \frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2} \Big| dt_1 dt_2,$$

where sign depends on the orientation of the surface. The oriented area $\mathcal{A}$ is given by summing up all the positive and negative contributions from the regions $\psi(\mathcal{R}_j)$. Since $\big(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\big)_{(t_{i_1}, t_{i_2})} = \pm \big| \frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2} \big|_{(t_{i_1}, t_{i_2})}$ has the sign of the oriented surface $\psi(\mathbb{T})$, the oriented area is given by

$$\mathcal{A} = \int_0^{2\pi} \int_0^{2\pi} \Big(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2} \Big) dt_1 dt_2. \tag{4.5}$$

Thus from Eq. 4.4 and Eq. 4.5 we have $\deg(\psi) = \mathcal{A}/4\pi$. This oriented area interpretation of degree will be useful for our subsequent calculations. In a non-technical language the degree of ψ is simply given by the number of times $\psi(\mathbb{T})$ covers $\mathbb{S}^2$.

Let us consider the Gauss map associated with the Hopf link. The image $\psi(\mathbb{T})$ is given by Figure 4.2. The lighter region is covered once and the darker region is covered twice, by two opposite orientations. The orientated area of $\psi(\mathbb{T})$ in this case is $\pm 4\pi$ depending on the surface orientation. Hence linking number is 1 in this case.

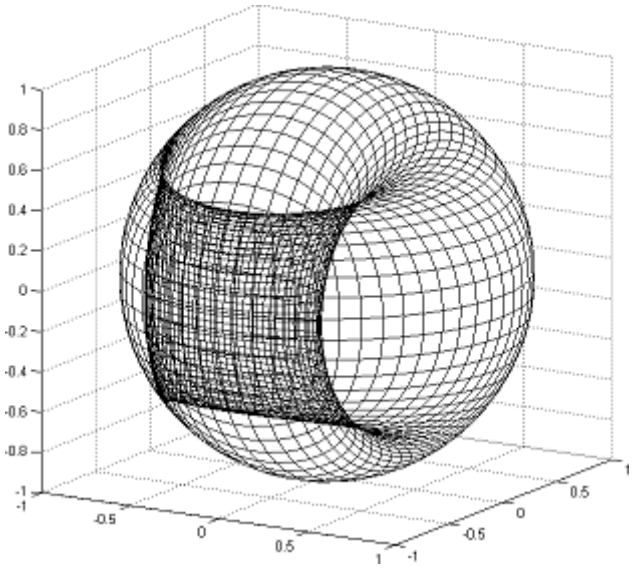


FIGURE 4.2: $\psi(\mathbb{T})$ resulting from the Gauss map for Hopf link
SOURCE: WORLD SCIENTIFIC [7]

4.2 The Hopf Fibers

From now on we will use the name Hopf fibers for referring to the fibers of the Hopf map. In this section we will parametrize the Hopf fibers. In Chapter-1 we have seen that the Hopf fibers are circles. There are infinitely many circles in $\mathbb{S}^3$. Given any point in $\mathbb{S}^2$, we do not know exactly which circle represents the fiber over that point. We will now try to identify the fiber circles and find an explicit parametrization of them. Let us recall the definition of the Hopf map from Chapter-1.

Definition 4.3. (Hopf Map) The Hopf map $\mathfrak{h} : \mathbb{S}^3 \to \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$ is given by

$$\mathfrak{h}(z_1, z_2) = \frac{z_2}{z_1}.$$

Taking the homeomorphism $\mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$ to be inverse steriographic projection and regarding $\mathbb{S}^3 \subset \mathbb{R}^4$, one can write the Hopf map $\mathfrak{h} : \mathbb{S}^3 \to \mathbb{S}^2$ as

$$\mathfrak{h}(x_1, y_1, x_2, y_2) = (2(x_1x_2 + y_1y_2), 2(x_1y_2 - y_1x_2), x_2^2 + y_2^2 - x_1^2 - y_1^2) \tag{4.6}$$

Now let us consider $\mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R}$, i.e. $\mathbb{S}^2 = \{(z, x) : |z|^2 + x^2 = 1\}$. Then the Hopf map $\mathfrak{h}$ is defined as

$$\mathfrak{h}(z_1, z_2) = (2\overline{z_1}z_2, |z_2|^2 - |z_1|^2). \tag{4.7}$$

Now we will try to illustrate the fact that the Hopf fibers are linked with each other. We can calculate their linking number to conclude that, but we would like to give a more visual argument in favor of this.
Let us take $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Then

$$\mathfrak{h}(\lambda z_1, \lambda z_2) = (2\overline{\lambda z_1}\lambda z_2, |\lambda z_2|^2 - |\lambda z_1|^2) = |\lambda|^2(2\overline{z_1}z_2, |z_2|^2 - |z_1|^2) = \mathfrak{h}(z_1, z_2).$$

Also if $\mathfrak{h}(z_1, z_2) = \mathfrak{h}(w_1, w_2)$, then $\frac{z_2}{z_1} = \frac{w_2}{w_1}$. Taking $z_j = r_j e^{i\theta_j}$ and $w_j = s_j e^{i\phi_j}$, we have $\frac{r_2}{r_1} = \frac{s_2}{s_1}$ and $\theta_2 - \theta_1 = \phi_2 - \phi_1 + 2k\pi$. Using the relations $r_1^2 + r_2^2 = s_1^2 + s_2^2 = 1$, we conclude that $r_1 = s_1$ and $r_2 = s_2$. So, we can write

$$\begin{aligned}(s_1e^{i\phi_1}, s_2e^{i\phi_2}) = e^{i(\phi_1-\theta_1)}(r_1e^{i\theta_1}, r_2e^{i(\phi_2-\phi_1+\theta_1)}) &= e^{i(\phi_1-\theta_1)}(r_1e^{i\theta_1}, r_2e^{i(\theta_2-\theta_1-2k\pi+\theta_1)}) \\ &= e^{i(\phi_1-\theta_1)}(r_1e^{i\theta_1}, r_2e^{i\theta_2}).\end{aligned}$$

So, we have $\mathfrak{h}(w_1, w_2) = \mathfrak{h}(z_1, z_2)$ if and only if $(w_1, w_2) = \lambda(z_1, z_2)$ with $|\lambda| = 1$. From this it is clear that the fibers are circles. Now consider the circle

$$C_0 = \{(z_1, z_2) : z_2 = 0\} \subset \mathbb{S}^3.$$

Under the Hopf map C_0 maps to $(0, -1) \in \mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R}$, hence $\mathfrak{h}^{-1}(0, -1) = C_0$. Let us denote the equatorial sphere of $\mathbb{S}^3$ by S_0^2, which is given by

$$S_0^2 = \{(z_1, z_2) : Im(z_2) = 0\}.$$

Lemma 4.2. *Any point $(z_1, z_2) \in \mathbb{S}^3$ can be connected to only one pair of antipodal points on S_0^2 by some Hopf fiber.*

Proof. We have already seen that if (z_1, z_2) belongs to some Hopf fiber then $\lambda(z_1, z_2)$ also belongs to the same fiber for every λ such that $|\lambda| = 1$. Lets us write $z_j = r_j e^{i\theta_j}$ for $j = 1, 2$. Take $\lambda = e^{-i\theta_2}$, then $|\lambda| = 1$ and $Im(\lambda r_2 e^{i\theta_2}) = 0$. Also taking $\lambda = -1$, we see that the antipodal point $(-r_1 e^{i(\theta_1 - \theta_2)}, -r_2)$ of $(r_1 e^{i(\theta_1 - \theta_2)}, r_2)$ also belongs to the same Hopf fiber as (z_1, z_2). If there is any other point (z_1', r) on S_0^2 which belongs to the same fiber, then it must satisfy $(r_1 e^{i(\theta_1 - \theta_2)}, r_2) = \lambda(z_1', r)$, with $|\lambda| = 1$. But as the 2nd coordinate is real, $\lambda = \pm 1$ and hence the pair of antipodal point is unique. ■

Since every Hopf fiber contains a pair of antipodal points of S_0^2 and the circular fiber joins the southern hemisphere of $\mathbb{S}^3$ ("inside" of S_0^2, $Im(z_2) < 0$) with the northern hemisphere of $\mathbb{S}^3$ ("outside" of S_0^2, $Im(z_2) > 0$), every Hopf fiber is linked with C_0, which is the equatorial circle of S_0^2. To see that any two fiber circles are linked, we give an intuitive argument. The 2-sphere can be rotated so that any great circle can be used as an equator or any pair of antipodal points can be used as the poles. Similarly, the 3-sphere can also be rotated so that any circle can be moved to where C_0 is. Hence any two Hopf fiber circles are linked, as one can be moved to be at C_0 and the other fiber circle will be linked to it by our previous discussion. So we have proved the following:

Proposition 4.2. *Any two Hopf fiber circles are linked.*

Although we now know that the Hopf fibers are linked, we do not know their linking number. There are several questions we need to address at this point. The first is whether the linking number of any two fibers are same. If it is, then

we can take any two Hopf fibers and calculate their linking number, which will give us an invariant for the Hopf map.

Proposition 4.3. *Let $p = (e^{i\phi}\sin\theta, \cos\theta) \in \mathbb{S}^2$, where θ and ϕ are the polar and azimuthal angles respectively. Then*

$$\mathfrak{h}^{-1}(p) = \{(\sin(\theta/2)e^{i(t-\phi/2)}, \cos(\theta/2)e^{i(t+\phi/2)}) : t \in [0, 2\pi]\}.$$

Proof. Let $(z_1, z_2) \in \mathfrak{h}^{-1}(p)$ and $z_j = r_j e^{i\xi_j}$ for $j = 1, 2$. Then we have

$$\mathfrak{h}(z_1, z_2) = (2r_1 r_2 e^{i(\xi_2 - \xi_1)}, r_2^2 - r_1^2)$$

and the relations

$$\begin{aligned} \sin\theta\cos\phi &= 2r_1 r_2 \cos(\xi_2 - \xi_1) \\ \sin\theta\sin\phi &= 2r_1 r_2 \sin(\xi_2 - \xi_1) \\ \cos\theta &= r_2^2 - r_1^2 \end{aligned}$$

From the first two relations we see that $\tan\phi = \tan(\xi_2 - \xi_1)$, which immediately tells us that $\phi = \xi_2 - \xi_1$. Again from the third relation,

$$\begin{aligned} & (r_2^2 - r_1^2) + (r_2^2 + r_1^2) = \cos\theta + 1 \\ \Rightarrow \quad & r_2^2 = \frac{1 + \cos\theta}{2} = \cos^2\theta/2 \\ \Rightarrow \quad & r_2 = \cos(\theta/2) \end{aligned}$$

So, $r_1 = \sin(\theta/2)$ and we have

$$\begin{aligned} z_1 &= \sin(\theta/2)e^{i\xi_1} \\ z_2 &= \cos(\theta/2)e^{i\xi_2} \end{aligned}$$

Note that $\lambda(t) = e^{i(t-\xi_2+\phi/2)}$, $t \in [0, 2\pi]$ is a parametrizarion of $\mathbb{S}^1$. Hence

$$(w_1(t), w_2(t)) = (\lambda(t)z_1, \lambda(t)z_2)$$

gives the parametrization for the Hopf fiber $\mathfrak{h}^{-1}(p)$. We have

$$\begin{aligned} w_1(t) &= \sin(\theta/2)e^{i(t-\phi/2)} \\ w_2(t) &= \cos(\theta/2)e^{i(t+\phi/2)} \end{aligned}$$

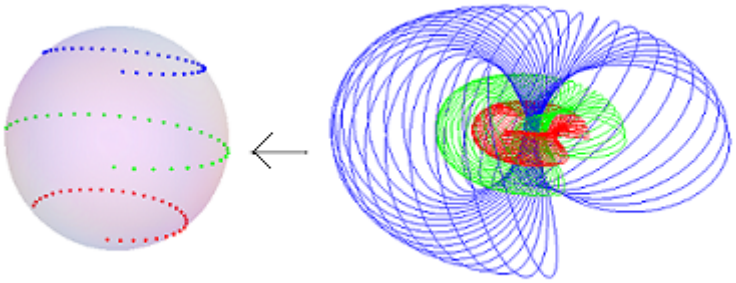


FIGURE 4.3: Stereographic Projection of The Hopf Fibers
SOURCE: MATHEMATICA

which completes the proof. ■

Once we know the parametrization of the Hopf fibers, we can actually plot the steriographic projections of the fibers in $\mathbb{R}^3$ and see how they are oriented in $\mathbb{R}^3$. In figure 4.3, one can see that the fibers are linked once. Hence, one should expect to have the linking number of the Hopf fibers to be ± 1. In the next section we will see this is indeed true by using the notion of linking number.

4.3 Linking Number of the Hopf Fibers

Now we have a parametrization of the Hopf fiber of a given point $(e^{i\phi}\sin\theta, \cos\theta)$ in $\mathbb{S}^2$. We will now show that given any two pair of points in $\mathbb{S}^2$ the linking number of the corresponding Hopf fibers are the same. To prove this we will fix one pair to be $((0,1),(0,-1))$ and choose the other pair (p_1, p_2) ($p_j \neq (0,1)$ or $(0,-1)$ for $j = 1, 2$) in $\mathbb{S}^2$ arbitrarily.

Proposition 4.4. *$Lk(\mathfrak{h}^{-1}(0,1), \mathfrak{h}^{-1}(0,-1)) = Lk(\mathfrak{h}^{-1}(p_1), \mathfrak{h}^{-1}(p_2))$ for any two distinct points p_1 and p_2 in $\mathbb{S}^2$.*

Proof. Let α_1 and α_2 be two paths disjoint from each other such that

$$\alpha_1(0) = (0,1), \quad \alpha_1(1) = p_1$$
$$\alpha_2(0) = (0,-1), \quad \alpha_2(1) = p_2$$

We can find such paths as one can choose α_1 to be the part of the great circle that passes through $(0,1)$ and p_1 and choose α_2 to be in $\mathbb{S}^2 \backslash \alpha_1([0,1])$. Let us write α_1

and α_2 as

$$\alpha_1(s) = (e^{i\phi_1(s)} \sin\theta_1(s), \cos\theta_1(s))$$
$$\alpha_2(s) = (e^{i\phi_2(s)} \sin\theta_2(s), \cos\theta_2(s))$$

where θ_j and ϕ_j are continuous functions of s for $j = 1, 2$. Let us take γ_j to be the parametrization of $\mathfrak{h}^{-1}(p_j)$ as in Proposition 4.3 and define for $j = 1, 2$, $H_j : [0, 2\pi] \times [0, 1] \to \mathbb{S}^3$ by

$$H_j(t_j, s) = (\sin(\theta_j(s)/2)e^{i(t_j - \phi_j(s)/2)}, \cos(\theta_j(s)/2)e^{i(t_j + \phi_j(s)/2)}).$$

Then H_j is a continuous function of t_j and s and

$$H_1(t_1, 0) = (0, e^{it_1}), \quad H_1(t_1, 1) = \gamma_1(t_1)$$
$$H_2(t_2, 0) = (e^{it_2}, 0), \quad H_1(t_2, 1) = \gamma_2(t_2)$$

Let F denote the steriographic projection $F : \mathbb{S}^3 \backslash \{(0, i)\} \to \mathbb{R}^3$. Now define for each $s \in [0, 1]$, $\psi_s : \mathbb{T} \to \mathbb{S}^2 \subset \mathbb{R}^3$ by,

$$\psi_s(t_1, t_2) = \frac{F \circ H_1(t_1, s) - F \circ H_2(t_2, s)}{|F \circ H_1(t_1, s) - F \circ H_2(t_2, s)|}$$

Then ψ_s is a one-parameter family of maps such that ψ_0 is Gauss map for the pair $(\mathfrak{h}^{-1}(0, 1), \mathfrak{h}^{-1}(0, -1))$ and ψ_1 is the Gauss map for the pair $(\mathfrak{h}^{-1}(p_1), \mathfrak{h}^{-1}(p_2))$. As ψ_s gives a homotopy between ψ_0 and ψ_1, we have $\deg(\psi_0) = \deg(\psi_1)$. Hence $Lk(\mathfrak{h}^{-1}(0, 1), \mathfrak{h}^{-1}(0, -1)) = Lk(\mathfrak{h}^{-1}(p_1), \mathfrak{h}^{-1}(p_2))$. ■

Now let us calculate the linking number of the Hopf fiber. By the previous Proposition we can find it by calculating $Lk(\mathfrak{h}^{-1}(0, -1), \mathfrak{h}^{-1}(0, 1))$. Let β_1 and β_2 be the stereographic projections of the parametrizations of $\mathfrak{h}^{-1}(0, -1)$ and $\mathfrak{h}^{-1}(0, 1)$ respectively. Then β_1 and β_2 are given by

$$\beta_1(t_1) = (\cos t_1, \sin t_1, 0)$$
$$\beta_2(t_2) = \left(0, 0, \frac{\cos t_2}{1 - \sin t_2}\right)$$

The normal vector $n(t_1, t_2)$ in the Gauss map ψ is given by

$$\begin{aligned} n(t_1, t_2) &= \frac{\beta_1(t_1) - \beta_2(t_2)}{|\beta_1(t_1) - \beta_2(t_2)|} \\ &= \frac{\big(\cos t_1, \sin t_1, -\frac{\cos t_2}{1-\sin t_2}\big)}{c} \end{aligned} \tag{4.8}$$

where $c = \big|\big(\cos t_1, \sin t_1, \frac{\cos t_2}{1-\sin t_2}\big)\big| = \frac{\sqrt{2}}{\sqrt{1-\sin t_2}}$. Hence

$$n(t_1, t_2) = \Big(\frac{(\sqrt{1-\sin t_2})\cos t_1}{\sqrt{2}}, \frac{(\sqrt{1-\sin t_2})\sin t_1}{\sqrt{2}}, -\frac{(\sqrt{1-\sin t_2})\cos t_2}{\sqrt{2}(1-\sin t_2)}\Big).$$

According to Eq. 4.5,

$$\deg(\psi) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} sgn\Big(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\Big)\Big|\frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2}\Big| dt_1 dt_2.$$

After some tedious but straightforward calculations we find that

$$\frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2} = \Big(\frac{\cos t_1(1-\sin t_2)}{4}, \frac{\sin t_1(1-\sin t_2)}{4}, -\frac{\cos t_2}{4}\Big)$$

and comparing it with Eq. 4.8, we see that $\frac{\partial n}{\partial t_1} \times \frac{\partial n}{\partial t_2}$ is an outward normal to the surface $\psi(\mathbb{T})$. Hence $sgn\big(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\big) = 1$ and

$$\begin{aligned} \deg(\psi) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \Big|\Big(\frac{\cos t_1(1-\sin t_2)}{4}, \frac{\sin t_1(1-\sin t_2)}{4}, -\frac{\cos t_2}{4}\Big)\Big| dt_1 dt_2 \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \sqrt{\frac{1}{16}(\cos^2 t_1(1-\sin t_2)^2 + \sin^2 t_1(1-\sin t_2)^2 + \cos^2 t_2)} dt_1 dt_2 \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{1}{4}\sqrt{(1-\sin t_2)^2 + \cos^2 t_2} dt_1 dt_2 \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1}{4}\sqrt{2 - 2\sin t_2} dt_2 \\ &= 1. \end{aligned}$$

Hence $Lk(\mathfrak{h}^{-1}(0,-1), \mathfrak{h}^{-1}(0,1)) = 1$.

Here we have done our calculation with the yellow circle and the blue straight line in Figure 4.4. There is a slight computational advantage if we take these two specific points. The advantage in this case is that $sgn\big(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\big)$ is constant 1 or -1 on the torus depending on orientation of $\psi(\mathbb{T})$. So, we can integrate on the

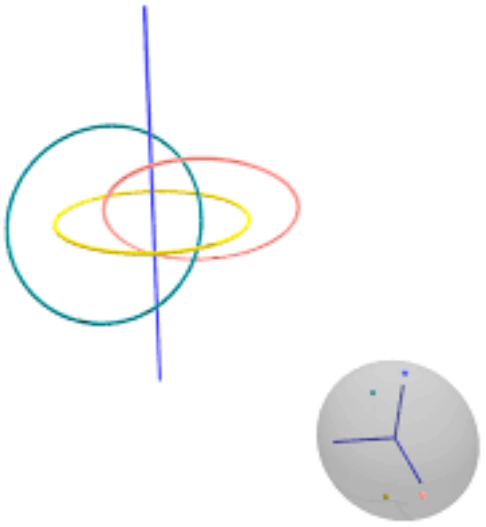


Figure 4.4: Some Hopf Fibers
Source: A visualization of the Hopf fibration [17]

whole torus without any worries. But if we had chosen any other pair of points, we would not have $sgn\big(n, \frac{\partial n}{\partial t_1}, \frac{\partial n}{\partial t_2}\big)$ is constant on the torus. Then we would have to identify the regions where the sign is positive and where the sign is negative and integrate on those regions separately to calculate the oriented area.
The linking number for the Hopf fibers is 1 and it is same for any two fibers. The linking number thus gives us an invariant for the Hopf map. It actually gives us an invariant for any map $f : \mathbb{S}^3 \to \mathbb{S}^2$. To see a formulation of this concept, one can look at [22]. This notion of Hopf invariant is same as our notion that we defined in Chapter-2. We will not go into the details of the equivalence of the two notions. One thing that we would like to mention is to generate maps such that the linking number of its fibers is k, we only need to pre-compose Hopf map with a degree k map from $\mathbb{S}^3$ to $\mathbb{S}^3$. Also, if we have a map f of degree k from $\mathbb{S}^2$ to $\mathbb{S}^2$, then the fibers of the map $f \circ \mathfrak{h}$ have linking number k^2. These facts are immediate from Proposition 2.5 and the equivalence of the two notions of Hopf invariant.
The Hopf fibration gave us a way of viewing $\mathbb{S}^3$ in terms of disjoint circles- one for each point of $\mathbb{S}^2$. Using disjoint circles and one straight line, can you fill up $\mathbb{R}^3$ in such a way that each pair of circles is linked and the line passes through the interior of each circle? our discussion says that the answer to this question is the Hopf fibration and it is the only answer (up to a homotopy)!!!
We can perform a similar study on the other Hopf bundle $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$. For

this we need to extend the notion of linking number to higher dimensions. For sub-manifolds A^k and B^l of dimension k and l of $\mathbb{R}^{k+l+1}$, we can define a map $\psi : A^k \times B^l \to \mathbb{S}^{k+l}$ by $\psi(a,b) = (a-b)/|a-b|$. This is analogous to our Gauss map. We define the linking number $Lk(A^k, B^l) = \deg(\psi)$. Using this description, one can find the linking number for two fibers of the Hopf bundle $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$.

Appendix A

CW Complexes

To start with, we first define what is known as a CW complex or cell complex. Most of the content in this Appendix is taken from [2].

Definition A.1. (CW complex)

1. Start with a discrete set X^0, whose points are regarded as 0-cells.
2. Inductively, form the n-skeleton X^n form the $(n-1)$-skeleton X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : \mathbb{S}^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha \mathbb{D}^n_\alpha$ under the identification $x \sim \varphi_\alpha(x)$ for $x \in \partial\mathbb{D}^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ dor some $n < \infty$, or continue indefinitely, setting $X = \cup_n X^n$. In the later case X is given the weak topology: A set $A \subseteq X$ is open if and only if $A \cap X^n$ is open in X^n for all n.

Example A.1. *Let us see couple of examples of CW complex of our interest.*

1. ***CW Complex Structure of*** $\mathbb{S}^n$*: The n-sphere $\mathbb{S}^n$ has the structure of a cell complex with a 0-cell e^0 and an n-cell e^n. The attaching map $\varphi : \mathbb{S}^{n-1} \to e^0$ is the constant map. This is basically the quotient space $\mathbb{D}^n/\partial\mathbb{D}^n$. As, $\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$, we conclude that the CW topology of $\mathbb{S}^n$ is same as the usual euclidean topology of $\mathbb{S}^n$.*

2. ***CW Complex Structure of*** $\mathbb{CP}^n$*: The complex projective space* $\mathbb{CP}^n$ *is defined as the quotient space of* $\mathbb{C}^{n+1}\backslash\{0\}$ *under the equivalence relation* $z \sim \lambda z$ *for* $\lambda \in \mathbb{C}\backslash\{0\}$*. Equivalently,* $\mathbb{CP}^n$ *is the quotient of the unit sphere* $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ *under the equivalence relation* $z \sim \lambda z$ *for* $|\lambda| = 1$*. There is another way of obtaining* $\mathbb{CP}^n$ *as a quotient space of the disc* $\mathbb{D}^{2n}$ *under the identification* $z \sim \lambda z$ *for* $z \in \partial\mathbb{D}^{2n}$ *in the following way. The points in* $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ *with last coordinate real and non-negative are precisely the points of the form* $(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{C}$ *with* $|w| \leqslant 1$*. These points form the graph of the function* $w \mapsto \sqrt{1-|w|^2}$*. This is a disc* $\mathbb{D}^{2n}_+$ *bounded by* $\mathbb{S}^{2n-1} \subset \mathbb{S}^{2n+1}$ *consisting of points* $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ *with* $|w| = 1$*. Each vector in* $\mathbb{S}^{2n+1}$ *is equivalent under the identification* $z \sim \lambda z$ *to a point in* $\mathbb{D}^{2n}_+$*, and the later point is unique if its last coordinate is non-zero. If the last coordinate is zero, we have the identification* $z \sim \lambda z$ *for* $z \in \mathbb{S}^{2n-1}$*.*
From this description of $\mathbb{CP}^n$*, it follows tha* $\mathbb{CP}^n$ *is obtained from* $\mathbb{CP}^{n-1}$ *by attaching a* $2n$*-cell* e^{2n} *via the quotient map* $\mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$*. So, by induction on* n *we obtain a cell complex structure* $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ *with cells in even dimensions only.*

Proposition A.1. *The space* $\mathbb{CP}^1$ *and* $\mathbb{S}^2$ *are homeomorphic.*

Proof. From Example A.1, the cell complex structure of $\mathbb{S}^2$ is $e^0 \cup e^{2n}$ with attaching map $\varphi_1 : \mathbb{S}^1 \to e^0$ is the constant map. Also the cell complex structure of $\mathbb{CP}^1$ is also $e^0 \cup e^{2n}$ with attaching map $\varphi_2 : \mathbb{S}^1 \to \mathbb{CP}^0 = e^0$ is the constant map. Hence we see that $\mathbb{S}^2$ and $\mathbb{CP}^1$ are homeomorphic as CW complex. As the usual topology and the CW topology coincide, we have $\mathbb{CP}^1 \cong \mathbb{S}^2$. ■

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