# Asymptotic distribution of resonances in one dimension

Richard Froese[1]
Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada
V6T 1Y4

AMS Mathematics Subject Classification

primary: 47A10 47A40 81U05

secondary: 81U05

keywords: resonances, scattering poles

[1] Research partially supported by NSERC grant

*Abstract*

*We determine the leading asymptotics of the resonance counting function for a class of Schrödinger operators in one dimension whose potentials may have non-compact support.*

## Introduction

Although there has been much work in proving upper and lower bounds for the number of resonances (or scattering poles) in various situations (for a recent excellent review of this subject, see [Z1]), to date the only results giving asymptotics are Zworski's theorems for Schrödinger operators in one dimension [Z2] and for radial potentials [Z3]. In both these theorems the potentials are required to have compact support. The purpose of this note is to give a new (and perhaps simpler) proof of the one dimensional result, and to extend it to a class of non-compactly supported potentials.

Resonances are defined as poles in the meromorphic continuation of the resolvent $R(k) = (-D^2 + V - k^2)^{-1}$ from the upper half plane $\{\mathrm{Im}\, k > 0\}$ to some larger region, in our case the whole complex plane. Of course, the continuation will not exist as an operator in $L^2(\mathbb{R})$. Instead, the resolvent must be considered as a map between suitable spaces of distributions, or between exponentially weighted spaces. In this paper we will take the potential itself as a weight, and use the following definition.

*Definition:* A *resonance* is a pole in the meromorphic continuation of $V^{\frac{1}{2}} R(k)|V|^{\frac{1}{2}}$. Here $V^{\frac{1}{2}}$ denotes the function $\mathrm{sign}(V)|V|^{\frac{1}{2}}$.

Resonances can also be defined as poles in the continuation of the scattering operator. In physics, the main interest is in resonances close to the real axis. These show up as bumps in the scattering cross section, which can be measured in experiments. Resonances also show up in the analysis of the Laplace operator on non-compact asymptotically hyperbolic manifolds. Here they can play the rôle of discrete spectral data, for example in the Selberg trace formula [M]. In this situation it is important to know their asymptotic distribution.

For Schrödinger operators in dimensions greater than one, although sharp upper bounds on the number of resonances are known [Z4], the best lower bounds to date are results which assert the existence of infinitely many resonances. Such results are known in three dimensions [SaB-Z] and for restricted classes of potentials [C-P-S].

Our first theorem concerns compactly supported potentials. This theorem was first proven by Zworski [Z2].

**Theorem 1.1** *Let $V \in L^\infty$ be a potential with compact support. Then, apart from a set of density zero, all the resonances of $-D^2+V$ are contained arbitrarily small sectors about the real axis. Let $n_+(r)$ ($n_-(r)$) denote the number of resonances of modulus less than $r$ contained in some sector about the positive (negative) real axis. Then*

$$n_\pm(r) = \pi^{-1}\left(\sup_{x,y\in \mathrm{supp}(V)} |x-y|\right) r + o(r)$$

Zworki's proof of his theorem proceeds via Melin's scattering theory. He also gives an example (based on an example of Titchmarch [T]) of a potential with infinitely many resonances on the imaginary axis, which shows that the set of density zero in the theorem need not be finite.

When the potential does not have compact support, we need to make some assumptions about its decay, to ensure that $V^{\frac{1}{2}}R(k)|V|^{\frac{1}{2}}$ has a meromorphic continuation.

*Definition:* The function $V(x)$ is called *super exponentially decreasing* if for every $N \in \mathbb{R}$, there exist a constant $C_N$ such that

$$V(x) \leq C_N e^{-N|x|}. \tag{1.1}$$

We make the following conjecture about the distribution of resonances for super exponentially decreasing potentials.

**Conjecture 1.2** *Let $V$ be a super exponentially decreasing potential. Suppose that $\widehat{V}$ is of order $\rho$ and of completely regular growth (see the following section for definitions). Then the asymptotic distribution of resonances is identical to the asymptotic distribution of zeros for the function $\widehat{V}(2k)\widehat{V}(-2k)+1$. In particular, the number of resonances $n(r,\theta_1,\theta_2)$ in the sector $\{|k|e^{i\theta} : |k| < r, \theta \in (\theta_1,\theta_2)\}$ is given by*

$$n(r,\theta_1,\theta_2) = \frac{s(\theta_1,\theta_2)}{2\pi\rho} r^\rho + o(r^\rho).$$

*where $s(\theta_1,\theta_2)$ is defined in terms of the growth of $\widehat{V}$ (as is explained below).*

This conjectured distribution results if the full scattering matrix is replaced by the Born approximation when computing resonances. We are able to prove this conjecture for a class of potentials which includes Gaussians $V(x) = e^{-ax^2}$, and sums of Gaussians with potentials of compact support. (In this case $n(r) = (2a/\pi)r^2 + o(r^2)$, and the resonances are concentrated near the diagonal rays $\{k : |\operatorname{Re} k| = |\operatorname{Im} k|\}$in the lower half plane.) Of course, the conjecture also agrees with Theorem 1.1.

**Theorem 1.3** *If $V$ is a super exponentially decreasing potential satisfying Hypothesis 5.1, then Conjecture 1.2 holds.*

To prove the theorems in this section, we identify the resonances with the zeros of an analytic function (a determinant), and use standard theorems relating the growth of such a function to the distribution of its zeros. The results we need are summarized in the next section.

## Theorems on zeros of entire functions

We will use the following classical theorems on the relationship between the growth of entire functions and the distribution of zeros. References are [L] and [B].

In all the following definitions, $F(z)$ will denote an entire function of the complex variable $z$.

*Definition:* Let

$$M(r) = \sup_{|z|=r} |F(z)|.$$

Then $F$ is *of order* $\rho$ if

$$\limsup_{r\to\infty} \frac{\ln \ln M(r)}{\ln r} = \rho.$$

A function of order $\rho > 0$ is *of type* $\tau$ if

$$\limsup_{r\to\infty} \frac{\ln M(r)}{r^\rho} = \tau.$$

If $0 < \tau < \infty$ then $F$ is said to be *of normal type.* A function of order 1 and type $\tau < \infty$ is said to be *of exponential type.*

*Definition:* The *indicator function,* $h(\theta)$, of a function of order $\rho$ and normal type is defined by

$$h(\theta) = \limsup_{r\to\infty} \frac{\ln |F(re^{i\theta})|}{r^\rho}.$$

For functions of exponential type, it can be shown that $h(\theta)$ is the supporting function of a convex body. (This means that if $R$ is the ray making an angle $\theta$ with the real axis, and $L$ is the supporting line of the convex body perpendicular to $R$, then $h(\theta)$ is the distance from $L$ to the origin.) This convex body is called *indicator diagram* of $F$.

*Definition:* The function $F(z)$ is of *completely regular growth* in the angle $(\theta_1, \theta_2)$ if $\ln|F(re^{i\theta})|/r^\rho$ converges uniformly to $h(\theta)$ for $\theta \in (\theta_1, \theta_2)$ as $r$ tends to infinity along a set of density one. ([L] p. 139)

In this definition, the set of density one is the same for all angles. It's complement contains, for example, the radii of all the zeros of $F$ in the sector.

*Definition:* The angular counting function $n(R, \theta_1, \theta_2)$ is defined to be the number of zeros of $F$ in the sector $\{re^{i\theta} : r < R, \theta \in (\theta_1, \theta_2)\}$.

**Theorem 2.1** ([L] p. 152) *If a holomorphic function of order $\rho$ is of completely regular growth in $(\theta_1, \theta_2)$, then for all but countably many values of $\theta$ and $\vartheta$ in this interval (which must be points of discontinuity of $h'(\theta)$) the following limit exists*

$$\frac{1}{2\pi\rho} s(\theta, \vartheta) = \lim_{r\to\infty} \frac{n(r, \theta, \vartheta)}{r^\rho}.$$

*The function $s$ has the representation*

$$s(\theta, \vartheta) = h'(\theta) - h'(\vartheta) + \rho^2 \int_\theta^\vartheta h(\phi)d\phi.$$

**Theorem 2.2** ([L], p. 251) *If $F(z)$ is of exponential type and*

$$\int_{-\infty}^{\infty} \frac{\ln_+ |F(x)|}{1 + x^2} dx < \infty$$

*then $F$ is of completely regular growth, and the indicator diagram of $F$ is an interval on the imaginary axes. All zeros of $F$, except for a set of zero density, lie in arbitrarily small sectors about the real axis, and if $\Delta_\pm = \lim_{r\to\infty} n_\pm(r)/r$, $i = 1, 2$ denote the densities in either direction, then $\Delta_\pm = d/2\pi$, where $d$ is the length of the indicator diagram.*

Notice that the last sentence of this theorem is a consequence what precedes it, given Theorem 2.1.

We will need to use the fact that Theorem 2.2 also holds if $F(z)$ is defined and of exponential type only in a half plane ([L] p. 243).

**Growth of $D(k)$**

To begin, we use standard resolvent identities to identify resonances as the zeros of a determinant $D(k)$. Let $R_0(k) = (-D^2 - k^2)^{-1}$ denote the free resolvent. Beginning with the resolvent formula

$$R(k) = R_0(k) - R_0(k)VR(k)$$

and multiplying from the right and left with $V^{\frac{1}{2}}$ and $|V|^{\frac{1}{2}}$ respectively, we obtain

$$(1 + \mathbf{R}_V(k))V^{\frac{1}{2}}R(k)|V|^{\frac{1}{2}} = \mathbf{R}_V(k),$$

where

$$\mathbf{R}_V(k) = V^{\frac{1}{2}}R_0(k)|V|^{\frac{1}{2}}.$$

This can be viewed as an equation on $L^2(\mathrm{supp}(V))$. On this space, $\mathbf{R}_V(k)$ is analytic and invertible for all $k$, apart from a pole at $k = 0$. Therefore, resonances are exactly those values of $k$ for which $(1 + \mathbf{R}_V(k))$ is not invertible. Thus the resonances are the zeros of the determinant $D(k)$ defined by

$$D(k) = \det(1 + \mathbf{R}_V(k)).$$

To apply the results of the previous section, we must show that $D(k)$ is of completely regular growth, and determine its indicator function.

In the appendix we prove the following elementary lemma

**Lemma 3.1** *Suppose that $V$ is super exponentially decreasing. Then the operator $\mathbf{R}_V(k)$ is trace class for $\operatorname{Im} k > 0$, and has a trace class operator valued analytic extension to $\mathbb{C}$, apart from a single pole at $k = 0$. For $k \neq 0$ we have the representation*

$$\mathbf{R}_V(k) = \mathbf{R}_V(-k) + F_V(k), \tag{3.1}$$

*where $F_V(k)$ is the rank two operator*

$$F_V(k) = \frac{i}{2k}\left(V^{\frac{1}{2}}e^{-ikx} \otimes |V|^{\frac{1}{2}}e^{ikx} + V^{\frac{1}{2}}e^{ikx} \otimes |V|^{\frac{1}{2}}e^{-ikx}\right)$$

*The operator $\mathbf{R}_V(k)$ obeys the following estimates. For $\operatorname{Im} k > 0$*

$$\|\mathbf{R}_V(k)\|_1 \le C/\operatorname{Im} k. \tag{3.2}$$

*For* $\operatorname{Im} k = 0$

$$\|\mathbf{R}_V(k)\|_1 \leq C(1 + 1/|k|) \tag{3.3}$$

We are using the notation $\phi \otimes \psi$ to denote the operator with integral kernel $\phi(x)\psi(y)$ (not $\phi(x)\overline{\psi}(y)$ !).

It follows from this lemma that $D(k) = \det(1 + \mathbf{R}_V(k))$ is well defined and entire, except for a pole at $k = 0$. It also follows from (3.1) that

$$1 + \mathbf{R}_V(k) = \Big(1 + \mathbf{R}_V(-k)\Big)\Big(1 + \big(1 + \mathbf{R}_V(-k)\big)^{-1} F_V(k)\Big).$$

Taking determinants, this gives

$$D(k) = D(-k)E(-k)$$

where

$$E(-k) = \det\Big(1 + \big(1 + \mathbf{R}_V(-k)\big)^{-1} F_V(k)\Big) \tag{3.4}$$

(In fact, although we won't use this, $E(-k)$ is equal to the determinant of the scattering matrix, as we will see below. This explains the identification of resonances with "scattering poles.")

We now estimate the growth of $D(k)$ in the upper half plane, and along the real axis, using the bound ([S], p. 48)

$$|\det(1 + A) - \det(1 + B)| \leq \|A - B\|_1 e^{1+\|A\|_1+\|B\|_1} \tag{3.5}$$

**Lemma 3.2** *For $k$ tending to infinity along a ray in the upper half plane,*

$$\lim_{|k|\to\infty} D(k) = 1.$$

*For $k$ real, $D(k)$ is bounded for large $k$.*

*Proof:* We can apply (3.5) with $A = \mathbf{R}_V(k)$ and $B = 0$. This gives

$$|D(k) - 1| \leq \|\mathbf{R}_V(k)\|_1 e^{1+\|\mathbf{R}_V(k)\|_1}.$$

The lemma follows from (3.2) and (3.3). $\square$

We will now examine the growth of $D(k) = D(-k)E(-k)$ in the lower half plane. The first factor $D(-k)$ tends to one as $k \to \infty$ along any ray in the lower half plane, by the estimates above. Thus it suffices to study the growth of the second term, $E(-k)$. This function is the determinant of a rank two perturbation of the identity, i.e., the determinant of an operator of the form $1 + (i/2k)(\phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2)$, with

$$\begin{aligned} \phi_1 &= \big(1 + \mathbf{R}_V(-k)\big)^{-1} V^{\frac{1}{2}} e^{-ikx} \\ \phi_2 &= \big(1 + \mathbf{R}_V(-k)\big)^{-1} V^{\frac{1}{2}} e^{ikx} \\ \psi_1 &= |V|^{\frac{1}{2}} e^{ikx} \\ \psi_2 &= |V|^{\frac{1}{2}} e^{-ikx} \end{aligned} \tag{3.6}$$

Thus

$$E(-k) = \det\left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i}{2k} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right),$$

where

$$T_{ij}(-k) = \int \psi_i \phi_j dx.$$

It is not hard to see, using the resolvent formula, that

$$\big(1 + \mathbf{R}_V(-k)\big)^{-1} = 1 - V^{\frac{1}{2}} R(-k) |V|^{\frac{1}{2}}.$$

Thus we obtain

$$T_{11}(-k) = \langle |V|^{\frac{1}{2}}, e^{ikx} \Big(1 - V^{\frac{1}{2}} R(-k) |V|^{\frac{1}{2}}\Big) e^{-ikx} V^{\frac{1}{2}} \rangle,$$

and similar expressions for the remaining integrals. Thus, if we define

$$\begin{aligned} f_+(x,k) &= e^{ikx} R(-k) e^{-ikx} V \\ f_-(x,k) &= e^{-ikx} R(-k) e^{ikx} V, \end{aligned} \tag{3.7}$$

then

$$\begin{aligned} T_{11}(-k) &= \int V(x)\big(1 - f_+(x,k)\big) dx, \\ T_{12}(-k) &= \int e^{2ikx} V(x)\big(1 - f_-(x,k)\big) dx, \\ T_{21}(-k) &= \int e^{-2ikx} V(x)\big(1 - f_+(x,k)\big) dx, \\ T_{22}(-k) &= \int V(x)\big(1 - f_-(x,k)\big) dx. \end{aligned} \tag{3.8}$$

(These expressions show that $[T_{ij}]$ is the $T$ matrix, which implies that $E(-k)$ is the determinant of the scattering matrix, as claimed.)

**Lemma 3.3** *Let $f_\pm$ be defined by (3.7). Suppose $k$ lies in the lower half plane. If $V$ is $L^1$ then*

$$|f_\pm(x,k)| \le C/|k|. \tag{3.9}$$

*Proof:* We will give the proof for $f_-$. Expanding in a Neumann series, we obtain

$$f_-(x,k) = e^{-ikx}R(-k)e^{ikx}V = \sum_{n=0}^{\infty}(-1)^n B(VB)^n V$$

where $B = e^{-ikx}R_0(-k)e^{ikx}$. The operator $B$ has integral kernel

$$B(x,y) = -\frac{i}{2k}\begin{cases} e^{-2ik(x-y)} & \text{if} \quad x \ge y \\ 1 & \text{if} \quad x \le y \end{cases}$$

Since $|B(x,y)| < 1/(2|k|)$, we see that $\|BV\|_\infty \le \|V\|_1/(2|k|)$, and inductively that $\|B(VB)^nV\|_\infty \le \left(\|V\|_1/(2|k|)\right)^{n+1}$. Thus the Neumann series can be summed to give the result. □

**Corollary 3.4** *For $k$ in the lower half plane, both $T_{11}(k)$ and $T_{22}(k)$ equal $\widehat{V}(0)+O(1/|k|)$, and thus*

$$D(k) = 1 + \frac{1}{4k^2}T_{12}(-k)T_{21}(-k) + O(1/|k|) \tag{3.10}$$

Thus, to determine the growth of $D(k)$ in the lower half plane, it suffices to determine the growth of the product $T_{12}(-k)T_{21}(-k)$. These functions are perturbations of $\widehat{V}(2k)$ and $\widehat{V}(-2k)$. For compactly supported $V$, the growth of the Fourier transform is determined by the endpoints of the support of $V$. The same is true for $T_{12}(-k)$ and $T_{21}(-k)$, as we will see in the next section

## Compactly supported potentials

We will need the following variant of the Paley Wiener theorem.

**Lemma 4.1** *Suppose $V \in L^\infty$ has compact support contained in $[-1,1]$, but in no smaller interval. Suppose $f(x,k)$ is analytic for $k$ in the lower half plane, and satisfies (3.9). Then $\int e^{\pm ikx}V(1-f(x,k))dx$ has exponential type at least 1 for $k$ in the lower half plane.*

*Proof:* We give the proof for the case $\pm = +$. Suppose that $\int e^{ikx}V(1-f(x,k))dx$ has exponential type $1-\delta$ for some positive $\delta$ and for $k$ in the lower half plane. Let $\chi_\epsilon$ denote the characteristic function of $[1-\epsilon,1]$. Then, for $\epsilon < \delta$, the function $\phi_\epsilon =$

$\int e^{ikx} V\chi_\epsilon(1-f(x,k))dx$ has exponential type at most $1-\epsilon$, for if its type were greater, we could not obtain a function of type $1-\delta$ by adding back the $1-\chi_\epsilon$ term. By the Paley-Wiener theorem, the Fourier transform of $\phi_\epsilon$ vanishes for $x > 1-\epsilon$. Thus, for $x > 1-\epsilon$, we have

$$V\chi_\epsilon(x) = \frac{1}{2\pi}\int e^{-ikx}\int e^{iky} V\chi_\epsilon(x) f(y,k) dy dk.$$

Thus, by the Plancherel theorem and the Cauchy-Schwartz inequality

$$\begin{aligned}
\|V\chi_\epsilon\|_{L^2} &\leq \frac{1}{2\pi}\|\int e^{iky} V\chi_\epsilon(x) f(y,k) dy\|_{L^2(dk)} \\
&\leq \frac{1}{2\pi}\|\|V\chi_\epsilon\|_{L^2}\|\chi_\epsilon(x) f(y,k)\|_{L^2(dy)}\|_{L^2(dk)} \\
&= \frac{1}{2\pi}\|V\chi_\epsilon\|_{L^2}\|\chi_\epsilon(x) f(y,k)\|_{L^2(dy,dk)}.
\end{aligned}$$

Since the second term tends to zero for small $\epsilon$, we conclude that $\|V\chi_\epsilon\|_{L^2}$ must vanish for small $\epsilon$. This contradicts the assumption about the support of $V$. □

**Corollary 4.2** *Suppose $V \in L^\infty$ has compact support contained in $[-1,1]$, but in no smaller interval. Then the functions $T_{12}(-k)$ and $T_{21}(-k)$ are of completely regular growth, and have exponential type two for $k$ in the lower half plane.*

*Proof:* The exponential upper bound is easily obtained from (3.8) . It is also easy to see from this formula that $T_{12}(-k)$ and $T_{21}(-k)$ are bounded for $k$ real, and thus satisfy the hypotheses of Theorem 2.2 in the lower half plane. Thus we may conclude that they are of completely regular growth. Thus the lower bound on the type follows from Lemma 4.1.

□

*Proof of Theorem 1.1:* We assume (by scaling and translating if neccesary) that the support of $V$ is contained in $[-1,1]$ but in no smaller interval. The function $D(k)$ is a meromorphic function with a single pole at $k=0$. Since multiplying $D(k)$ by a polynomial doesn't change the growth properties defined in the appendix, nor the density of zeros, we may safely ignore this pole. We will show that $D(k)$ satisfies the hypotheses of Theorem 2.2 with indicator diagram $[-4i, 0]$. Then Theorem 1.1 follows from Theorem 2.2.

We first must show that $D(k)$ is of exponential type. In the upper half plane, $D(k)$ is bounded, and thus certainly of exponential type. In the lower half plane we can use the representation (3.10) for $E$ together with elementary bounds on (3.8) to conclude that in the lower half plane $D(k)$ is of exponential type at most four.

Next, we must show that the integral appearing in the hypothesis of Theorem 2.2 converges. This follows from the boundedness of $D(k)$ for real $k$.

Finally, we must show that the indicator diagram of $D(k)$ is $[-4i, 0]$. By Theorem 2.2 the indicator diagram is some interval on the imaginary axis. The fact that $D(k)$ tends to one along rays in the upper half plane implies that the upper endpoint of the diagram is zero. Thus, the indicator diagram must equal $[-i\tau, 0]$, where $\tau$ is the exponential type of $D(k)$. From the representation (3.10), we see that it suffices to show that the product $T_{12}(-k)T_{21}(-k)$ has exponential type four in the lower half plane. Since the type of a product of two functions of completely regular growth is the sum of the types of the factors, this follows from Corollary 4.2. □

#### Non compactly supported potentials

When $V$ does not have compact support, it is much more difficult to determine the growth of $T_{12}(-k)$ and $T_{21}(-k)$. Roughly speaking, Conjecture 1.2 follows if we assume that the growth of these integrals is the same as their formal leading terms given by the Born approximation, namely $\widehat{V}(2k)$ and $\widehat{V}(-2k)$, in the sectors where the Fourier transforms are large. In this case the indicator function for $D(k)$ would be given by

$$h(\theta) = \begin{cases} 0 & \text{for} \quad 0 \le \theta \le \pi \\ \max\{0, 2^\rho\big(h_{\widehat{V}}(2\theta) + h_{\widehat{V}}(-2\theta)\big)\} & \text{for} \quad -\pi \le \theta \le 0 \end{cases} \tag{5.1}$$

(in this formula $h_{\widehat{V}}$ is the indicator function for $\widehat{V}$), and the conjecture would follow from Theorem 2.1 with

$$s(\theta_1, \theta_2) = h'(\theta_1) - h'(\theta_2) + \rho^2 \int_{\theta_1}^{\theta_2} h(\phi) d\phi.$$

To explain our additional assumptions, consider again the case of compactly supported $V$. Here, we used the fact that $\widehat{V}(2k)$, $T_{12}(-k)$, $T_{21}(-k)$ and $D(k)$ all turn out to be functions of exponential type, and bounded on the real axis. Thus, by Theorem 2.2, their indicator diagrams are line segments on the imaginary axis, or equivalently, that the indicator functions have the form $\tau\cos(\theta \pm \pi/2)$. Thus the problem of determining the indicator function is reduced to finding a single number. This number is easily estimated from above, and is estimated from below by the Paley Wiener theorem.

In the non-compact case, we will consider potentials whose Fourier transforms are "small" (exponential type) in sectors about the positive and negative real semi-axes. We then show that $T_{12}(-k)$ and $T_{21}(-k)$ inherit this property. These sectors are chosen large enough to insure that in the complementary sectors about the imaginary semi-axes, the integrals have sinusoidal indicator functions. This means that once a again there is a single number to determine, and it suffices to make an estimate on the imaginary axis. To identify the rate of growth here in terms of $\widehat{V}$, we will assume in *(iii)* below that $\widehat{|V|}$ does not grow too much more quickly than $\widehat{V}$ on the imaginary axis.

**Hypothesis 5.1**

*(i)* *$\widehat{V}(k)$ has order $\rho > 1$ and is of completely regular growth.*

*(ii)* *Let $\alpha = \pi/\rho$ and let $\Gamma$ denote the sector $\{k : |\arg k| \leq (\pi - \alpha)/2\}$. Then there exists a positive number $b$ such that*

$$|\widehat{V}(k)| + |\widehat{V}'(k)| + |\widehat{V}''(k)| \leq e^{b|\operatorname{Im} k|}$$

*for $k \in \pm\Gamma$.*

*(iii)* *Let $C$ denote the constant in (3.9). There exists $\delta > 0$ such that for a set of real $\lambda$ of density one,*

$$\widehat{|V|}\,(2i\lambda) \leq \frac{1-\delta}{C}|\lambda||\widehat{V}(2i\lambda)|$$

Note that that *(iii)* is satisfied if $V = \pm|V|$, and can be thought of as a sort of positivity (or negativity) condition on $V$. We now restate Theorem 1.3 in greater detail.

**Theorem 1.3** *Suppose $V$ is a very rapidly decreasing potential satisfying Hypothesis 5.1. Then the indicator function for $D(k)$ is given by*

$$h_D(\theta) = \begin{cases} 2^\rho\big(h_{\widehat{V}}(\pi/2) + h_{\widehat{V}}(-\pi/2)\big)\cos(\rho(\theta + \pi/2)) & \text{for} \quad |\theta + \pi/2| < \alpha/2 \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

*In any sector about one of the two rays in the lower half plane $\{re^{i\theta} : \theta = (-\pi \pm \alpha)/2\}$, the number of resonances of modulus less than $r$ is given by*

$$n(r) = \frac{2^\rho\big(h_{\widehat{V}}(\pi/2) + h_{\widehat{V}}(-\pi/2)\big)}{2\pi} r^\rho + o(r^\rho).$$

*In any other sector the number of resonances of modulus less than $r$ is $o(r^\rho)$.*

To analyze $T_{12}(-k)$, we once again expand the resolvent occurring in the definition in a Neumann series. This yields

$$T_{21}(-k) = \widehat{V}(2k) + \sum_{n=1}^{\infty}(-1)^n\langle V, e^{-ikx}R_0(-k)(VR_0(-k))^{n-1}e^{-ikx}V\rangle$$

By the Plancherel theorem, the $n^{\text{th}}$ term in the sum above may be rewritten as

$$\begin{aligned}
&\langle V, e^{-ikx} R_0(-k)(VR_0(-k))^{n-1} e^{-ikx} V\rangle \\
&\qquad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \widehat{V}(k-p)\frac{1}{p^2-k^2}\widehat{V}(p-t_1)\frac{1}{t_1^2-k^2}\widehat{V}(t_1-t_2)\cdots \\
&\qquad\qquad\qquad \cdots\widehat{V}(t_n-q)\frac{1}{q^2-k^2}\widehat{V}(q+k)dpdt_1\dots dt_n dq \qquad (5.3)\\
&\qquad = \big(A_k^n\widehat{V}\big)(2k)
\end{aligned}$$

where the operator $A_k$ is defined by

$$\big(A_k f\big)(t) = \int_{-\infty}^{\infty}\widehat{V}(t-k-q)\frac{1}{q^2-k^2}f(q+k)dq. \qquad (5.4)$$

Thus

$$T_{21}(-k) = \widehat{V}(2k) + \sum_{n=1}^{\infty}(-1)^n\big(A_k^n\widehat{V}\big)(2k),$$

and similarly,

$$T_{12}(-k) = \widehat{V}(-2k) + \sum_{n=1}^{\infty}(-1)^n\big(A_{-k}^n\widehat{V}\big)(-2k).$$

To consider the growth of $T_{21}(-k)$ and $T_{12}(-k)$ in the lower half plane, in sectors with angle $(\pi-\alpha)/2$ below the real axis, we must consider $\big(A_k^n\widehat{V}\big)(2k)$, for $k$ both below and above the real axis.

**Lemma 5.2** *Fix $k\in\Gamma$, $k\notin\mathbb{R}$, with $|\operatorname{Im} k|$ sufficiently large (say $>1$). Let $\pm\Gamma_k$ be the smallest closed sectors containing the real axis and $k$, and symmetric about the real axis. For a function $f(z)$ analytic in $\Gamma_k$ define*

$$|||f||| = \sup_{t\in\Gamma_k} e^{-b|\operatorname{Im} k|}\Big(|f(t)|+|f'(t)|\Big).$$

*Then*

$$|||A_k f||| \le C\frac{\ln|\operatorname{Im} k|}{|\operatorname{Im} k|}|||f|||. \qquad (5.5)$$

*Proof:* Fix $t\in\Gamma_k$. We will shift the contour in the integral defining $A_k$, depending on the position of $t$.

To begin, we consider the case where $|t-2k|>1$ and $|t|>1$. We shift the contour to a contour $C$ consisting of three line segments: a horizontal line from infinity on the left to $-k$, a line from $-k$ to $t-k$ followed from a line from $t-k$ to infinity on the right. Actually, since the integrand has a singularity at $q=-k$, we take a sequence of contours,

$C_\epsilon$ approaching $C$ from below as illustrated in figure 1. Notice that the second singularity at $q = k$ is avoided, since $|t - 2k| > 1$. The contour $C$ is chosen to so that the contours where $\widehat{V}$ is evaluated, namely $C + k$ and $t - k - C$ are equal (as sets) and stay away from the sectors about the imaginary axes where $\widehat{V}$ is big. In our estimates we break up the integrals depending on the size of the denominator.

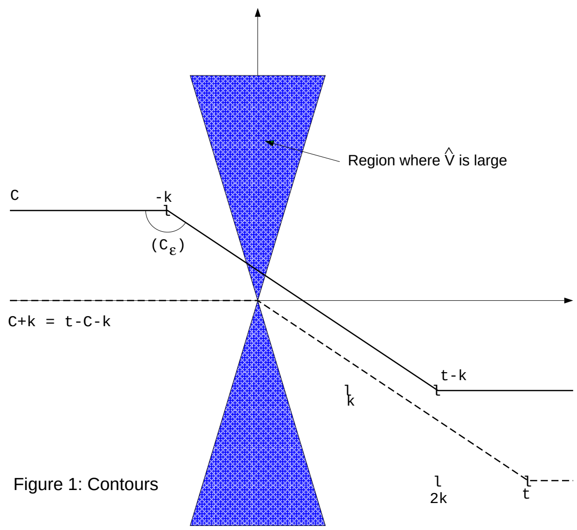


Figure 1: Contours

To estimate the triple norm we must consider $|\big(A_k f\big)(t)|$ and $|\big(A_k f\big)'(t)|$. We begin with $|\big(A_k f\big)(t)|$. Let $F(q) = (q-k)^{-1}\widehat{V}(t-k-q)f(q+k)$, let $\theta$ be the angle in the first bend of the contour $C$ at $-k$, and let $\hat{t} = t/|t|$. Then

$$
\begin{aligned}
\big(A_k f\big)(t) &= \lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{1}{q+k}F(q)dq \\
&= -i\theta F(-k) \\
&\quad + \int_{-\infty}^{-1}\frac{1}{x}F(-k+x)dx + \int_0^\infty \frac{1}{t+x}F(t-k+x)dx \\
&\quad + \int_1^{|t|}\frac{1}{x}F(-k+\hat{t}x)dx \\
&\quad + \lim_{\epsilon\to 0}\left(\int_{-1}^{-\epsilon}\frac{1}{x}F(-k+x)dx + \int_\epsilon^1 \frac{1}{x}F(-k+\hat{t}x)dx\right)
\end{aligned}
$$

We estimate each of these terms. We will use Hypothesis 5.1 (ii) and the assumption that $|||f|||$ is finite. To begin, we estimate

$$
|-i\theta F(-k)| = \frac{\theta}{|2k|}\widehat{V}(t)f(0) \le \frac{C}{|k|}e^{b|\operatorname{Im} k|}|||f|||.
$$

Next, we see that

$$\begin{aligned}\left|\int_{-\infty}^{-1}\frac{1}{x}F(-k+x)dx\right| &\le \int_{-\infty}^{-1}\frac{1}{|x(x-2k)|}\left|\widehat{V}(t-x)f(x)\right|dx\\ &\le e^{b|\operatorname{Im} k|}|||f|||\int_{-\infty}^{-1}\frac{1}{|x(x-2k)|}dx\\ &\le C\frac{\ln|\operatorname{Im} k|}{|\operatorname{Im} k|}e^{b|\operatorname{Im} k|}|||f|||\end{aligned}$$

The integrals $\int_0^\infty$ and $\int_1^{|t|}$ are handled similarly. Finally, to estimate the principal value integrals, we first subtract $F(-k)/x$ from each integrand and use the estimate

$$\begin{aligned}&\int_{-1}^{-\epsilon}\frac{1}{x}\big(F(-k+x)-F(-k)\big)dx\\ &\quad=\int_{-1}^{-\epsilon}\frac{1}{x}\int_0^x F'(-k+u)dudx\\ &\quad\le \sup_{-1\le u\le 0}|F'(-k+u)|\\ &\quad\le \sup_{-1\le u\le 0}\left(\left|\frac{\widehat{V}(t-u)f(u)}{(-2k+u)^2}\right|+\left|\frac{\widehat{V}'(t-u)f(u)}{-2k+u}\right|+\left|\frac{\widehat{V}(t-u)f'(u)}{-2k+u}\right|\right)\\ &\quad\le C\frac{1}{|\operatorname{Im} k|}e^{b|\operatorname{Im} k|}|||f|||\end{aligned}$$

and a similar one for $\int_\epsilon^1$. These estimates imply that

$$e^{-b|\operatorname{Im} k|}|\big(A_k f\big)(t)| \le C\frac{\ln|\operatorname{Im} 2k|}{|\operatorname{Im} 2k|}|||f|||$$

To get the same estimate for $|\big(A_k f\big)'(t)|$, note that in (5.4) we need only replace $\widehat{V}$ with $\widehat{V}'$. This completes the estimate in the case where $|t-2k|>1$ and $|t|>1$.

When $|t-2k|\le 1$ (in which case $|t|\ge|2k|-|t-2k|\ge 1$), we cannot avoid the second singularity, and therefore run the contour right through it, i.e., from $-\infty$ to $-k$ to $k$ to $\infty$. This produces a second residue term, namely $i\theta\widehat{V}(t-2k)f(2k)/(2k)$, which is readily estimated. The estimates will contain $e^{b|\operatorname{Im} k|}$ instead of $e^{b|\operatorname{Im} t-k|}$, but this doesn't matter since $\operatorname{Im} k$ and $\operatorname{Im} t-k$ differ by at most 1.

Finally, when $|t|<1$, we can use the contour $C=\mathbb{R}-k$, and miss both singularities.

□

**Corollary 5.3** *For $k$ in the lower half plane, $k\in\Gamma$ with $|\operatorname{Im} k|$ sufficiently large*

$$\left.\begin{array}{r}|T_{21}(-k)|\\ |T_{21}(-k)|\end{array}\right\}\le Ce^{2b|\operatorname{Im} k|}$$

*Proof:* Iterating (5.5) we find that

$$|||A_{\pm k}\widehat{V}||| \le \left(C\frac{\ln|\operatorname{Im} k|}{|\operatorname{Im} k|}\right)^n |||\widehat{V}|||.$$

Thus

$$|A_{\pm k}(\pm 2k)| \le \left(C\frac{\ln|\operatorname{Im} k|}{|\operatorname{Im} k|}\right)^n |||\widehat{V}|||e^{2b|\operatorname{Im} k|}$$

Inserting this into the Neumann expansion we have

$$|T_{21}(-k)| \le \sum_{n=0}^{\infty}\left|A_k^n\widehat{V}(2k)\right| \le |||\widehat{V}|||\sum_{n=0}^{\infty}\left(\frac{C\ln|\operatorname{Im} k|}{|\operatorname{Im} k|}\right)^n e^{2b|\operatorname{Im} t|},$$

and a similar bound for $|T_{12}(-k)|$. □

It remains to examine the growth of the $T_{21}(-k)$ and $T_{12}(-k)$ in the sector $\Gamma_-$ of width $\alpha$ about the negative imaginary axis. Let $F(k)$ denote one of these two functions. Then $F$ is of order $\rho$ in $\Gamma_-$ and satisfies the estimate $|F(k)| \le e^{2b|\operatorname{Im} k|}$ on the boundary of $\Gamma_-$. It follows that the function $G(k) = F(-i(ik)^{1/\rho})$ is analytic in the lower half plane, of exponential type, and satisfies the hypotheses of Theorem 2.2. Thus $G(k)$ is a function of completely regular growth with indicator function $a\cos(\theta+\pi/2)$ for $-\pi \le \theta \le 0$. This implies that $F(k)$ has completely regular growth in $\Gamma_-$, with indicator function $a\cos(\rho(\theta+\pi/2))$.

To determine the value of the constant $a$ when $F = T_{21}$, we begin with (3.8) and (3.9), to conclude

$$|\widehat{V}(2i\lambda)| - \frac{C}{|k|}\widehat{|V|}(2i\lambda) \le |T_{21}(-i\lambda)| \le |\widehat{V}(2i\lambda)| + \frac{C}{|k|}\widehat{|V|}(2i\lambda)$$

Using Hypothesis 5.1 (*iii*), this implies that for a set of $\lambda$ of density one,

$$\delta|\widehat{V}(2i\lambda)| \le |T_{21}(-i\lambda)| \le (1+\delta)|\widehat{V}(2i\lambda)|$$

This implies that $a$ must equal $2^\rho h_{\widehat{V}}(-\pi/2)$. Similarly, when $F = T_{12}$, we find that $a = 2^\rho h_{\widehat{V}}(\pi/2)$.

We will need the following proposition to obtain a lower bound.

**Proposition 5.4** *Suppose that $f(k)$ is holomorphic in a sector with angle $\pi/\sigma$ less than $\pi$ (i.e., $\sigma > 1$), and satisfies $f(k) < e^{b|k|}$ everywhere in the sector (including the boundary). Then for $\rho > \sigma$, there exists a set of density one, such that for $r$ in this set*

$$\lim_{r\to\infty}\frac{\ln|f(re^{i\theta})|}{r^\rho} = 0.$$

*Proof:* We may assume that the sector is given by $\{re^{i\theta} : |\theta| < \pi/(2\sigma)\}$. Consider $F(k) = f(k^{1/\sigma})$. Then $F$ is of exponential type in the right half plane. On the imaginary axis we have $\ln|F(re^{i\pi/2})|/(1+r^2) = O(r^{1/\sigma-2})$ which is integrable. Thus Theorem 2.2 applies and we conclude that there is a set $\Sigma$ of density one such that

$$\lim_{\substack{r\to\infty \\ r\in\Sigma}} \frac{\ln|F(re^{i\theta})|}{r} = \tau\cos(\theta)$$

Thus

$$\begin{aligned}\lim_{\substack{r\to\infty \\ r\in\Sigma^{1/\sigma}}} \frac{\ln|f(re^{i\theta})|}{r^\rho} &= \lim_{\substack{r^\sigma\to\infty \\ r^\sigma\in\Sigma}} \frac{\ln|F(r^\sigma e^{i\sigma\theta})|}{r^\sigma r^{(\rho-\sigma)}} \\ &= 0.\end{aligned}$$

The lemma now follows from the fact that $\Sigma^{1/\sigma}$ is a set of density one if $\Sigma$ is. □

*Proof of Theorem 1.3:* For rays in the upper half plane $D(k) \to 1$ by Lemma 3.2. Thus $D$ has completely regular growth and $h_D(\theta) = 0$ for $0 < \theta < \pi$. For $k$ in the lower half plane, we use (3.10) to establish the growth. If either $-(\pi-\alpha)/2 < \arg k < 0$ or $-\pi < \arg k < -\pi + (\pi-\alpha)/2$, the functions $T_{12}$ and $T_{21}$ are increasing at most exponentially, by Corollary 5.3. Thus $D(k)$ too is increasing at most exponentially in these sectors. By including parts of the upper half plane, we can obtain exponential upper bounds on $D(k)$ in sectors with any angle approaching $\pi$. Thus Proposition 5.4 applies, and we conclude that in these sectors, $D$ has completely regular growth and $h_D(\theta) = 0$.

It remains to consider $\theta$ in the sector of width $\alpha$ centred on the negative imaginary axis. In this sector, the product $T_{12}(-k)T_{21}(-k)$, being the product of two functions of completely regular growth, has indicator function $2^\rho\left(h_{\widehat{V}}(\pi/2) + h_{\widehat{V}}(-\pi/2)\right)$. From (3.10) we conclude that this is the indicator function of $D(k)$ too.

This establishes the formula (5.2) for $h_D(k)$. The rest of Theorem 1.3 now follows from Theorem 2.1. □

### Further remarks

The half line problem with a boundary condition at $x = 0$ can be analyzed in a similar way. For example, in the case of Dirichlet boundary conditions, $F_V$ is replaced by the rank one operator $-(2i/k)V^{\frac{1}{2}}\sin(kx) \otimes |V|^{\frac{1}{2}}\sin(kx)$ and resonances are given by the zeroes in the lower half plane of

$$E(k) = 1 - (2i/k)\langle V^{\frac{1}{2}}, \sin(kx)\big(1 - V^{\frac{1}{2}}R(-k)|V|^{\frac{1}{2}}\big)\sin(kx)|V|^{\frac{1}{2}}\rangle.$$

When $V$ is not very rapidly decreasing, the operator $\mathbf{R}_V(k)$ will not have a continuation to the lower half plane. However, the function $E(k)$ defined by (3.4) (or by the formula above in the half line case) may well be well defined. A similar situation occurs in higher dimensions. There $\mathbf{R}_V(k)$ is not trace class, so the determinant $D(k)$ does not exist. However, the determinant $E(k)$ still makes sense and can be used to count resonances.

### Appendix: Analysis of the weighted free resolvent

In this appendix we give proof of Lemma 3.1.

We begin with a standard Green function identity. Let $G_0(x, y, k)$ denote the integral kernel of the free resolvent $(-D^2 - k^2)^{-1}$ or Green function. Initially the Green function is defined for $\mathrm{Im}\, k > 0$. However the explicit representation

$$G_0(x, y, k) = \frac{i}{2k}e^{ik|x-y|}$$

shows that $G_0$ as an analytic continuation to $\mathbb{C}$, except for a pole at zero.

**Lemma 7.1** *Let $L$ be a positive number. Then for $-L < x, z < L$, and $k_1, k_2$ non-zero complex numbers,*

$$\begin{aligned} G_0(x,z,k_1) - G_0(x,z,k_2) = (k_1^2 - k_2^2)&\int_{-L}^{L} G_0(x,y,k_1)G_0(y,z,k_2)dy \\ &+\frac{i}{4}e^{i(k_1+k_2)L}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\big(e^{-ik_1x}e^{-ik_2z} + e^{ik_1x}e^{ik_2z}\big) \end{aligned} \tag{7.1}$$

*Remark:* If both $k_1$ and $k_2$ lie in the upper half plane, then the second term vanishes in the limit $L \to \infty$, and we recover the resolvent formula. On the other hand, if $k_1 = -k_2$ then the first term vanishes, and we obtain the functional equation for the Green function.

*Proof:* The first term on the right of (7.1) can be rewritten

$$\int_{-L}^{L} G(x, y, k_1)\big(D^2 + k_1^2 - D^2 - k_2^2\big)G(y, z, k_2)dy.$$

Integration by parts and the formula

$$(-D^2 - k^2)G(x, y, k) = \delta(x - y)$$

yield (7.1) □

We will begin by considering the operator $\mathbf{R}_{\chi_L}(k)$ where $\chi_L$ denotes the characteristic function of $[-L, L]$. When $L = 1$ we will use the special notation

$$\mathbf{R}(k) = \mathbf{R}_{\chi_1}(k).$$

**Proposition 7.2** *For any two non-zero complex numbers $k_1$ and $k_2$,*

$$\mathbf{R}(k_1) = \mathbf{R}(k_2) + (k_1^2 - k_2^2)\mathbf{R}(k_1)\mathbf{R}(k_2) + F(k_1, k_2) \tag{7.2}$$

*where $F(k_1, k_2)$ is the rank two operator*

$$F(k_1, k_2) = \frac{i}{4}e^{i(k_1+k_2)}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\left(\chi e^{-ik_1x} \otimes \chi e^{-ik_2x} + \chi e^{ik_1x} \otimes \chi e^{ik_2x}\right)$$

*Here $\chi$ denotes the characteristic function of $[-1, 1]$.*

*Proof:* This follows immediately from (7.1) □

**Proposition 7.3** *The operator $\mathbf{R}(k)$ is trace class for every $k \neq 0$. For $\operatorname{Im} k > 0$ we have*

$$\|\mathbf{R}(k)\|_1 \leq 1/\operatorname{Im} k \tag{7.3}$$

*For $\operatorname{Im} k = 0$ we have*

$$\|\mathbf{R}(k)\|_1 \leq C(1 + 1/|k|) \tag{7.4}$$

*Finally, for $\operatorname{Im} k < 0$ we have*

$$\|\mathbf{R}(k)\|_1 \leq Ce^{4|\operatorname{Im} k|}/\operatorname{Im} k. \tag{7.5}$$

*Proof:* For $\mathrm{Im}\, k > 0$, $\mathbf{R}(k)$ is the product of two Hilbert Schmidt operators, namely $\chi(p+k)^{-1}$ and $(p-k)^{-1}\chi$ (here $p = -iD$), and hence trace class. We have

$$\|\mathbf{R}(k)\|_1 \leq \|\chi(p+k)^{-1}\|_2 \|(p-k)^{-1}\chi\|_2 = 1/(\mathrm{Im}\, k),$$

which proves (7.3).

When $k$ is real, we first observe that $\mathbf{R}(k)$ is Hilbert Schmidt, with

$$\|\mathbf{R}(k)\|_2 \leq 1/|k|.$$

Thus, using (7.2) with $k_1 = k$ and $k_2 = i|k|$, we find that

$$\begin{aligned}\|\mathbf{R}(k)\|_1 &\leq \|\mathbf{R}(i|k|)\|_1 + (k^2 + |k|^2)\|\mathbf{R}(k)\|_2 \|\mathbf{R}(i|k|)\|_1 + \|F(k, i|k|)\|_1 \\ &\leq 1/|k| + (k^2 + |k|^2)/|k|^2 + C/|k|\end{aligned}$$

Here we used $\|\phi \otimes \psi\|_1 = \|\phi\|\|\psi\|$ to estimate $F(k, i|k|)$.

For $\mathrm{Im}\, k < 0$, the identity (7.2) with $k = k_1 = -k_2$ shows that $\mathbf{R}(k)$ is trace class plus rank two, hence trace class. The estimate (7.5) for $\mathrm{Im}\, k$ negative, follows from (7.2) with $k = k_1 = -k_2$ and the calculation

$$\|F(k, -k)\|_1 \leq \frac{C}{|\lambda|} e^{4|\lambda|}.$$

□

The operator $\mathbf{R}_{\chi_L}(k)$ is related to $\mathbf{R}(k) = \mathbf{R}_{\chi_1}(k)$ by scaling.

**Proposition 7.4** *Let $U_L$ denote the unitary dilation defined by*

$$(U_L \psi)(x) = L^{-\frac{1}{2}} \psi(x/L).$$

*Then*

$$\mathbf{R}_{\chi_L}(k) = L^2 U_L \mathbf{R}(Lk) U_L^{-1}.$$

We omit the simple proof.

**Corollary 7.5** *For negative $\lambda$*

$$\|\chi_L R_0(\mu + i\lambda)\chi_L\|_1 \leq C L e^{4L|\lambda|}/|\lambda|. \tag{7.6}$$

*Proof:* This follows from the previous proposition and (7.5). □

**Lemma 7.6** *If $V$ is very rapidly decreasing, then*

$$\lim_{L\to\infty} V^{\frac{1}{2}}\mathbf{R}_{\chi_L}(k)|V|^{\frac{1}{2}} = \mathbf{R}_V(k). \tag{7.7}$$

*Here the limit is taken in the trace norm.*

*Proof:* Fix $N > 0$. If $V$ is very rapidly decreasing, then so is $|V|^{\frac{1}{2}}$. Let $C_N$ be the constants given by this condition for $|V|^{\frac{1}{2}}$. We begin by defining a step function approximation to $C_N e^{-N|x|}$. Let $\beta_k = C_N e^{-kN}$ and define $\alpha_1 = \beta_1$, $\alpha_k = \beta_k - \beta_{k-1}$ for $k > 1$. Let

$$w(x) = \sum_{k=1}^{\infty} \alpha_k \chi_k(x),$$

where $\chi_k(x)$ is the characteristic function of $[-k, k]$. Note that if $|x| \in (j-1, j]$, then $\chi_k(x) = 0$ for $k < j$. Thus $w(x) = \sum_{k\geq j} \alpha_k = \beta_j$, and so

$$\begin{aligned}|V|^{\frac{1}{2}}(x)w^{-1}(x) &= |V|^{\frac{1}{2}}(x)/\beta_j \\ &\leq C_N e^{j|N|}/C_N e^{j|N|} \\ &= 1.\end{aligned}$$

This estimate implies that it suffices to prove (7.7) with $V^{\frac{1}{2}}$ and $|V|^{\frac{1}{2}}$ replaced with $w(x)$.

We now estimate

$$\begin{aligned}\|w\chi_L R_0 \chi_L w - wR_0 w\|_1 &= \|w\chi_L R_0\chi_L w - wR_0\chi_L w + wR_0\chi_L w - wR_0 w\|_1 \\ &\leq 2\|wR_0(1-\chi_L)w\|_1 \\ &\leq 2\|\sum_i \alpha_i\chi_i R_0 \sum_{j>L}\alpha_j\chi_j\|_1 \\ &\leq 2\sum_i\sum_{j>L}\alpha_i\alpha_j\|\chi_i R_0\chi_j\|_1 \\ &\leq 2\sum_{i<j}\sum_{j>L}\alpha_i\alpha_j\|\chi_j R_0\chi_j\|_1 + 2\sum_{i>j}\sum_{j>L}\alpha_i\alpha_j\|\chi_i R_0\chi_i\|_1 \\ &\leq C\sum_i\alpha_i\sum_{j>L} je^{-(N-4|\lambda|)j} + C\sum_i ie^{-(N-4|\lambda|)i}\sum_{j>L}\alpha_j\end{aligned}$$

Here $R_0$ denotes $R_0(\mu + i\lambda)$. We used the estimate (7.6). Since $\sum\alpha_i = \beta_1 < \infty$, the left side tends to zero for large $L$, provided $\lambda < N/4$. But $N$ can be chosen arbitrarily large, so the lemma follows. □

*Proof of Lemma 3.1:* That $\mathbf{R}_V(k)$ is trace class follows from Lemma 7.6. To establish (3.1) we begin with (7.2) with $k = k_1 = -k_2$, conjugate by $U_L$, and take $L$ to infinity.

The estimate (3.2) can be proven just like (7.3), by factoring the operator into two Hilbert Schmidt operators. To prove (3.3) we use the notation introduced in Lemma 7.6. Let $k \in \mathbb{R}$.

$$\begin{aligned}
\|\mathbf{R}_V(k)\|_1 &\leq \|wR_0(k)w\|_1 \\
&\leq \sum_{i,j} \alpha_i\alpha_j \|\chi_i R_0(k)\chi_j\|_1 \\
&\leq 2\sum_i \alpha_i \sum_j \alpha_j \|\chi_j R_0(k)\chi_j\|_1 \\
&\leq 2\beta_1 \sum_j \alpha_j j^2 \|\mathbf{R}(jk)\|_1 \\
&\leq C\sum_j \alpha_j j^2 (1 + 1/(jk)) \\
&\leq C(1 + 1/|k|)
\end{aligned}$$

Here we used (7.4). $\square$


## Acknowledgements

This paper began as a simple proof of Theorem 1.1 using the spectral representation for the self-adjoint operators $\mathbf{R}(i\lambda)$, $\lambda \in \mathbb{R}$. The author wishes to thank Maciej Zworski for suggesting that the proof might be extended to non-compact potentials. This suggestion lead to the present version. Thanks are also due to Plamen Stefanov for the reference [C-P-S].


## References


[B] Ralph Philip Boas, Jr., *Entire Functions,* Academic Press, 1954

[C-P-S] J. Cooper, G. Perla-Menzala and W. Strauss, *On the scattering frequencies of time-dependent potentials,* Math. Meth. in the Appl. Sci. **8** (1986) 576–584

[L] B. Ja. Levin, *Distribution of Zeros of Entire Functions,* American Mathematical Society Translations of Mathematical Monographs, Volume 5, 1964

[M] W. Müller, *Spectral geometry and scattering theory for certain complete surfaces of finite volume,* Invent. Math. **109** (1992), 265–305

[SaB-Z] Antônio Sà Barreto and Maciej Zworski, Existence of resonances in three dimensions, preprint

[S] Barry Simon, *Trace ideals and their applications,* London Mathematical Society Lecture Note Series 35, Cambridge University Press (1979)

[T] E. C. Titchmarsh, *The zeros of certain classes of integral functions*, Proc. London Math. Soc. **25** (1926), 283–302

[Z1] Maciej Zworski, *Counting scattering poles*, to appear in *Spectral and Scattering Theory,* M. Ikawa ed., Marcel Dekker

[Z2] Maciej Zworski, *Distribution of poles for scattering on the real line,* J. of Funct. Anal., **73** (2) , (1987), 277–296

[Z3] Maciej Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials,* J. of Funct. Anal., **82** (2) , (1989), 370–403

[Z4] Maciej Zworski, *Sharp polynomial bounds on the number of scattering poles,* Duke Math. Jour, **59** (2) , (1989), 311–323