AN INVERSE SATAKE ISOMORPHISM IN CHARACTERISTIC p

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Abstract. Let $\mathfrak{F}$ be a local field with finite residue field of characteristic p and k an algebraic closure of the residue field. Let G be the group of $\mathfrak{F}$-points of a $\mathfrak{F}$-split connected reductive group. In the apartment corresponding to a maximal split torus of T, we choose a hyperspecial vertex and denote by K the corresponding maximal compact subgroup of G.

Given an irreducible smooth k-representation ρ of K, we construct an isomorphism from the affine semigroup k-algebra $k[\mathrm{X}_*^+(\mathrm{T})]$ of the dominant cocharacters of T onto the spherical k-algebra $\mathcal{H}(\mathrm{G},\rho)$. In the case when the derived subgroup of G is simply connected, we prove furthermore that our isomorphism is the inverse to the mod p Satake isomorphism constructed by Herzig in [19].

Contents

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1. Introduction

The smooth mod p representation theory of a split p-adic reductive group G and its number theoretic interpretation *via* an expected mod p version of the Langlands program is currently only understood in the case of $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_p)$ (see [4] for an overview). In the general case, the irreducible representations of G have been classified up to the supersingular representations, about which very little is known ([20], generalized by [1]). The main tool used in this classification is the mod p Satake isomorphism established in [19] and comparison theorems between compact and parabolic inductions for mod p representations.

In the classical setting, *i.e* for complex representations of G, the Satake transform provides a description of the convolution algebra $\mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}]$: it is an isomorphism

$$S : \mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}] \xrightarrow{\simeq} (\mathbb{C}[\mathrm{X}_*(\mathrm{T})])^{\mathfrak{W}}$$

where $\mathfrak{W}$ denotes the finite Weyl group corresponding to the split torus T ([31], see also [14]) and $\mathrm{X}_*(\mathrm{T})$ the group of cocharacters of T. This isomorphism says that the characters of $\mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}]$ parametrize the isomorphism classes of a family of smooth irreducible representations of G called unramified. Generically, an unramified representation is isomorphic to an irreducible principal series representation with an Iwahori-fixed vector (in general it is the unique unramified subquotient of such a principal series representation [9, 4.4]).

On the other hand, all unramified representations lie in the Bernstein block of the category of smooth representations of G called *the Iwahori block*. By Bernstein, Borel [2] and Matsumoto [25], this block is equivalent to the category of modules over the Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}}(\mathrm{G},\mathrm{I})$ *via* the functor $\mathrm{V} \mapsto \mathrm{V}^{\mathrm{I}}$. By this equivalence, the isomorphism classes of unramified representations are parametrized by the ($\mathfrak{W}$-orbits of) characters of the commutative Bernstein subalgebra of $\mathcal{H}_{\mathbb{C}}(\mathrm{G},\mathrm{I})$ (as defined in [24, 3.5]), or equivalently, by the characters of the center $\mathcal{Z}(\mathcal{H}_{\mathbb{C}}(\mathrm{G},\mathrm{I}))$ of $\mathcal{H}_{\mathbb{C}}(\mathrm{G},\mathrm{I})$.

This discussion reveals a correspondence between characters of $\mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}]$ and characters of $\mathcal{Z}(\mathcal{H}_{\mathbb{C}}(\mathrm{G},\mathrm{I}))$, which is realized by the following statement (proved and discussed in [15, Proposition 10.1] and [13, Corollary 3.1]): the *Bernstein isomophism* (see [23], [24])

$$B : (\mathbb{C}[\mathrm{X}_*(\mathrm{T})])^{\mathfrak{W}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{H}_{\mathbb{C}}(\mathrm{G},\mathrm{I}))$$

is compatible with S in the sense that the composition $(e_{\mathrm{K}}\star .)B$ is an inverse for S, where $(e_{\mathrm{K}}\star .)$ is the convolution by the characteristic function of K.

The current article provides a link between the Iwahori-Hecke algebra and the spherical Hecke algebra with coefficients in a field k of characteristic p. In fact, more accurately, the pro-p Iwahori-Hecke k-algebra $\tilde{\mathrm{H}}$ comes naturally into play in the mod p setting. It has been studied by [36]. On the other hand, instead of focusing on the trivial representation of K, we will consider the more general Hecke algebra $\mathcal{H}(\mathrm{G}, \rho)$ of an irreducible k-representation ρ of K.

We construct in $\tilde{\mathrm{H}}$ a commutative subalgebra $\mathscr{A}_\rho$ which is isomorphic to $k[\mathrm{X}_*^+(\mathrm{T})]$, where $\mathrm{X}_*^+(\mathrm{T})$ denotes the monoid of the dominant cocharacters. The notation $\mathscr{A}_\rho$ does not appear in the body of the article: $\mathscr{A}_\rho$ is the image of $k[\mathrm{X}_*^+(\mathrm{T})]$ by the modified Bernstein map $\mathcal{B}^+_{F_\chi}$ described below in 1.2.1. Using a theorem by Cabanes that relates categories of k-representations of parahoric subgroups of G and Hecke modules ([8]), we then define a natural isomorphism from $\mathscr{A}_\rho$ onto $\mathcal{H}(\mathrm{G}, \rho)$ and therefore obtain an isomorphism $k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\sim} \mathcal{H}(\mathrm{G}, \rho)$. The fact that these two algebras are isomorphic was shown by Herzig in [19]. We do not rely on that result. Instead, we prove that, if the derived subgroup of G is simply connected, our isomorphism is an inverse to the Satake transform of [19].

We give below, after some notations, a more detailed description of our methods and results.

1.1. **Framework and notations.** Let $\mathfrak{F}$ be a nonarchimedean locally compact field with ring of integers $\mathfrak{O}$, maximal ideal $\mathfrak{P}$ and residue field $\mathbb{F}_q$ where q is a power of a prime number p. We fix a uniformizer ϖ of $\mathfrak{O}$ and choose the valuation $val_{\mathfrak{F}}$ on $\mathfrak{F}$ normalized by $val_{\mathfrak{F}}(\varpi) = 1$. Let $\mathrm{G} := \mathbf{G}(\mathfrak{F})$ be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$ which we assume to be $\mathfrak{F}$-split. We fix an algebraic closure k of $\mathbb{F}_q$: it is the field of coefficients of (most of) the representations we consider. All representations of G and its subgroups are smooth.

Let $\mathscr{X}$ (resp. $\mathscr{X}^{ext}$) be the semisimple (resp. extended) building of G and $\mathrm{pr} : \mathscr{X}^{\mathrm{ext}} \to \mathscr{X}$ the canonical projection map. We fix a maximal $\mathfrak{F}$-split torus T in G which is equivalent to choosing an apartment $\mathscr{A}$ in $\mathscr{X}$ (see 2.2.1). We fix a chamber C in $\mathscr{A}$ as well as a hyperspecial vertex x_0 of C. The stabilizer of x_0 in G contains a good maximal compact subgroup K of G which in turns contains an Iwahori subgroup I that fixes C pointwise. Let $\mathbf{G}_{x_0}$ and $\mathbf{G}_C$ denote the Bruhat-Tits group schemes over $\mathfrak{O}$ whose $\mathfrak{O}$-valued points are K and I respectively. Their reductions over the residue field $\mathbb{F}_q$ are denoted by $\overline{\mathbf{G}}_{x_0}$ and $\overline{\mathbf{G}}_C$. By [35, 3.4.2, 3.7 and 3.8], $\overline{\mathbf{G}}_{x_0}$ is connected reductive and $\mathbb{F}_q$-split. Therefore we have $\mathbf{G}_C^\circ(\mathfrak{O}) = \mathbf{G}_C(\mathfrak{O}) = \mathrm{I}$ and $\mathbf{G}_{x_0}^\circ(\mathfrak{O}) = \mathbf{G}_{x_0}(\mathfrak{O}) = \mathrm{K}$.

Denote by $\overline{\mathbf{B}}$ the Borel subgroup of $\overline{\mathbf{G}}_{x_0}$ image of the natural morphism $\overline{\mathbf{G}}_C \longrightarrow \overline{\mathbf{G}}_{x_0}$ and by $\overline{\mathbf{N}}$ the unipotent radical of $\overline{\mathbf{B}}$ and $\overline{\mathbf{T}}$ its Levi subgroup. Set

$$\mathrm{K}_1 := \mathrm{Ker}\big(\mathbf{G}_{x_0}(\mathfrak{O}) \xrightarrow{proj} \overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)\big) \quad \text{and} \quad \tilde{\mathrm{I}} := \{g \in \mathrm{K} : proj(g) \in \overline{\mathbf{N}}(\mathbb{F}_q)\}.$$

Then we have a chain $\mathrm{K}_1 \subseteq \tilde{\mathrm{I}} \subseteq \mathrm{I} \subseteq \mathrm{K}$ of compact open subgroups in G such that

$$\mathrm{K}/\mathrm{K}_1 = \overline{\mathbf{G}}_{x_0}(\mathbb{F}_q) \supseteq \mathrm{I}/\mathrm{K}_1 = \overline{\mathbf{B}}(\mathbb{F}_q) \supseteq \tilde{\mathrm{I}}/\mathrm{K}_1 = \overline{\mathbf{N}}(\mathbb{F}_q)\ .$$

The subgroup $\tilde{\mathrm{I}}$ is pro-p and is called the pro-p Iwahori subgroup. It is a maximal pro-p subgroup in K. The quotient $\mathrm{I}/\tilde{\mathrm{I}}$ identifies with $\overline{\mathbf{T}}(\mathbb{F}_q)$.

Let $\tilde{\mathbf{X}} := \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}}(1)$ denote the compact induction of the trivial character of $\tilde{\mathrm{I}}$ (with values in k). We see it as the space of k-valued functions with compact support in $\tilde{\mathrm{I}}\backslash\mathrm{G}$, endowed with the action of G by right translation. The Hecke k-algebra of the G-equivariant endormorphisms of $\tilde{\mathbf{X}}$ will be denoted by $\tilde{\mathrm{H}}$.

Remark 1.1. Throughout the article, we will use accented letters such as $\tilde{\mathbf{X}}$, $\tilde{\mathrm{H}}$, $\tilde{\mathfrak{H}}$, $\tilde{\mathrm{W}}$, $\tilde{\mathrm{X}}_*(\mathrm{T})$ even when their non accented versions do not necessarily come into play: in doing so, we want to emphasize the fact that we work with the pro-p Iwahori subgroup $\tilde{\mathrm{I}}$ and the attached objects. The non accented letters are kept for the classical root data, universal representations, affine Hecke algebra *etc.* attached to the chosen Iwahori subgroup I.

The algebra $\tilde{\mathrm{H}}$ is relatively well understood: an integral Bernstein basis has been described by Vignéras ([36]) who underlines the existence of a commutative subalgebra denoted by $\mathcal{A}^{+,(1)}$ in $\tilde{\mathrm{H}}$ that contains the center of $\tilde{\mathrm{H}}$ and such that $\tilde{\mathrm{H}}$ is finitely generated over $\mathcal{A}^{+,(1)}$.

Let ρ be an irreducible k-representation of K. Such an object is called a *weight.* It descends to an irreducible representation of $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$ because K_1 is a pro-p group. Its compact induction to G is denoted by $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$. The k-algebra of the G-endomorphisms of the latter is denoted by $\mathcal{H}(\mathrm{G},\rho)$ and will be called the spherical Hecke algebra attached to ρ. It is described by Herzig in [19] (remark that the results of [19] are equally valid when $\mathfrak{F}$ has characteristic p). In particular, $\mathcal{H}(\mathrm{G},\rho)$ is a commutative noetherian algebra. For example, if $\mathbf{G} = \mathrm{GL}_n$ (for $n \geq 1$) then $\mathcal{H}(\mathrm{G},\rho)$ is an algebra of polynomials in n variables localized at one of them (Example 1.6, *loc. cit.*). More precisely, for general $\mathbf{G}$, let $\mathrm{X}_*(\mathrm{T})$ denote the set of cocharacters of the split torus T and $\mathrm{X}_*^+(\mathrm{T})$ the monoid of the dominant ones, then there is an isomorphism

$$\mathcal{S} : \mathcal{H}(\mathrm{G},\rho) \xrightarrow{\simeq} k[\mathrm{X}_*^+(\mathrm{T})]$$

given by [19, Thm 1.2] (see our remark 2.5 for our choice of the "dominant" normalization).

1.2. **Results.**

1.2.1. Let ρ be a weight. We prove independently from [19] that there is an isomorphism between $k[\mathrm{X}_*^+(\mathrm{T})]$ and $\mathcal{H}(\mathrm{G},\rho)$ (depending on the choice of a uniformizer ϖ and of a set of positive roots) by constructing a map in the opposite direction

$$\mathcal{T} : k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\simeq} \mathcal{H}(\mathrm{G},\rho) \tag{1.1}$$

and proving that it is an isomorphism (Theorem 4.11).

Under the hypothesis that the derived subgroup of $\mathbf{G}$ is simply connected, we give in 5 an explicit description of $\mathcal{T}$ and prove that it is an inverse for $\mathcal{S}$ which, under the same hypothesis, is explicitly computed in [20].

Our method to construct $\mathcal{T}$ is based on the following result: it is well known that there is a one-to-one correspondence between the weights and the characters of the (finite dimensional) pro-p Iwahori-Hecke algebra $\tilde{\mathfrak{H}}$ of the maximal compact K ([11]). In fact, we have more than this: by a theorem of Cabanes ([8], recalled in 3.2), there is an equivalence of categories between $\tilde{\mathfrak{H}}$-modules and a certain category (denoted here by $\mathscr{B}(x_0)$) of representations of K. Using this theorem, we prove (Corollary 3.14) that passing to $\tilde{\mathrm{I}}$-invariant vectors gives natural isomorphisms of k-algebras

$$\mathcal{H}(\mathrm{G},\rho) \cong \mathrm{Hom}_{\tilde{\mathrm{H}}}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}) \cong \mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}) \tag{1.2}$$

where χ is the character of $\tilde{\mathfrak{H}}$ corresponding to ρ. Therefore, it remains to describe the k-algebra $\mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}})$. The necessary tools are introduced in Section 4.

In Section 4, we consider F a standard facet, that is to say a facet of the standard chamber C containing x_0 in its closure. We attach to F a Weyl chamber $\mathscr{C}^+(F)$ (for example, if $F = C$, it is simply the set of dominant cocharacters) as well as a k-linear "Bernstein-type" map

$$\mathcal{B}_F^+ : k[\tilde{\mathrm{X}}_*(\mathrm{T})] \to \tilde{\mathrm{H}}$$

defined on the group algebra of the extended cocharacters $\tilde{\mathrm{X}}_*(\mathrm{T})$ (2.2.2 and Remark 1.1). Restricted to the dominant monoid $k[\tilde{\mathrm{X}}_*^+(\mathrm{T})]$, the map $\mathcal{B}_F^+$ respects the product and its image is a commutative subalgebra of $\tilde{\mathrm{H}}$. For example, if $F = C$, then $\mathcal{B}_C^+$ coincides on $k[\tilde{\mathrm{X}}_*^+(\mathrm{T})]$ with the map sending a dominant cocharacter onto the characteristic function of the corresponding double coset modulo $\tilde{\mathrm{I}}$, and the image of $\mathcal{B}_C^+$ coincides with the subalgebra $\mathcal{A}^{+,(1)}$ of [36].

To the character χ of $\tilde{\mathfrak{H}}$ we attach a standard facet F_χ as well as its restriction $\bar{\chi}$ to the finite torus $\overline{\mathbf{T}}(\mathbb{F}_q)$ (3.4). (For example, if ρ is the Steinberg representation, then $F_\chi = C$ and $\bar{\chi}$ is the trivial character.) We prove in 4.2 that the map $\mathcal{B}_{F_\chi}^+$ induces an isomorphism of k-algebras

$$k[\mathrm{X}_*^+(\mathrm{T})] \;\cong\; \bar{\chi} \otimes_{k[\mathrm{T}^0/\mathrm{T}^1]} k[\tilde{\mathrm{X}}_*^+(\mathrm{T})] \;\xrightarrow{\simeq}\; \mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}) \tag{1.3}$$

which, combined with (1.2), yields the isomorphism $\mathcal{T}$.

1.3. **Perspectives.** We started this introduction by recalling some facts about complex representations of G and the attached Hecke algebras. It suggested that constructing an inverse to the mod p Satake transform by means of commutative Bernstein-type subalgebras of $\tilde{\mathrm{H}}$ is motivated by understanding families of mod p representations of G. As opposed to the complex case however, functors of the form $\mathrm{V} \mapsto \mathrm{V}^U$ where U is an open compact subgroup of G (for example I or $\tilde{\mathrm{I}}$) do not in general provide equivalences between the expected categories of representations of G and of Hecke modules in characteristic p ([26], [30]). It is one of the obstacles to the classification of the irreducible supercuspidal representations of G.

The notion of supersingularity for characters of $\mathcal{H}(\mathrm{G},\rho)$ has been defined in [20], and related to the supercuspidality for representations of G. The notion of supersingularity for $\tilde{\mathrm{H}}$-modules has been defined in [36]. The current article provides a way to unify these two notions: consequences of our theorem and applications to the study of blocks of $\tilde{\mathrm{H}}$-modules in relation with mod p representations of G are analyzed in a separate article ([28]). We point out in particular the importance of the Bernstein-type maps $\mathcal{B}_F^\sigma$ defined in the current paper for the classification of the simple supersingular $\tilde{\mathrm{H}}$-modules in [28].

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2. Root data and associated affine Hecke rings

We first give notations and basic results about "abstract" reduced root data. In 2.2 (respectively 2.3), we will describe some aspects of the construction of the reduced root data of G (respectively, of a semi-standard Levi subgroup of G) associated to the choice of the torus T. In both the settings of 2.2 and of 2.3, the results of 2.1 apply.

2.1. **Root datum.** We refer to [24, §1]. We consider an affine root datum $(\Phi, \mathrm{X}^*, \check\Phi, \mathrm{X}_*)$ where Φ is the set of roots and $\check\Phi$ the set of coroots. We suppose that it is reduced. An element of the free abelian group X_* is called a coweight. We denote by $\langle\, .\,,\, .\rangle$ the perfect pairing on $\mathrm{X}_*\times\mathrm{X}^*$ and by $\alpha\leftrightarrow\check\alpha$ the correspondence between roots and coroots satisfying $\langle\alpha,\check\alpha\rangle = 2$. We choose a basis Π for Φ and denote by Φ^+ (resp. Φ^-) the set of roots which are positive (resp. negative) with respect to Π. There is a partial order on Φ given by $\alpha\preceq\beta$ if and only if $\beta-\alpha$ is a linear combination with (integral) nonnegative cœfficients of elements in Π.

To the root $\check\alpha$ corresponds the reflection $s_\alpha : \lambda\mapsto\lambda - \langle\lambda,\alpha\rangle\,\check\alpha$ defined on X_*. It leaves $\check\Phi$ stable. The finite Weyl group $\mathfrak{W}$ is the subgroup of $\mathrm{GL}(\mathrm{X}_*)$ generated by the simple reflections s_α for $\alpha\in\Pi$. It is a Coxeter system with generating set $S=\{s_\alpha,\ \alpha\in\Pi\}$. We will denote by $(w_0,\lambda)\mapsto {}^{w_0}\lambda$ the natural action of $\mathfrak{W}$ on the set of coweights. It induces a natural action of $\mathfrak{W}$ on the weights which stabilizes the set of roots. The set X_* acts on itself by translations: for any coweight $\lambda\in\mathrm{X}_*$, we denote by e^λ the associated translation. The (extended) Weyl group W is the semi-direct product $\mathfrak{W}\ltimes\mathrm{X}_*$.

2.1.1. Define the set of affine roots by $\Phi_{aff}=\Phi\times\mathbb{Z}=\Phi^+_{aff}\coprod\Phi^-_{aff}$ where

$$\Phi^+_{aff} := \{(\alpha, r),\ \alpha\in\Phi,\ r>0\}\cup\{(\alpha,0),\ \alpha\in\Phi^+\}.$$

The Weyl group W acts on Φ_{aff} by $we^\lambda : (\alpha, r)\mapsto (w\alpha,\ r-\langle\lambda,\alpha\rangle)$ where we denote by $(w,\alpha)\mapsto w\alpha$ the natural action of $\mathfrak{W}$ on the roots. Denote by Π_m the set of roots that are minimal elements for $\preceq$. Define the set of simple affine roots by $\Pi_{aff} := \{(\alpha,0),\alpha\in\Pi\}\cup\{(\alpha,1),\ \alpha\in\Pi_m\}$. Identifying α with $(\alpha,0)$, we consider Π a subset of Π_{aff}. For $A\in\Pi_{aff}$, denote by s_A the following associated reflection: $s_A=s_\alpha$ if $A=(\alpha,0)$ and $s_A=s_\alpha e^{\check\alpha}$ if $A=(\alpha,1)$. The length on the Coxeter system $\mathfrak{W}$ extends to W in such a way that, the length of $w\in\mathrm{W}$ is the number of affine roots $A \in \Phi_{aff}^+$ such that $w(A) \in \Phi_{aff}^-$. It satisfies the following formula, for any $A \in \Pi_{aff}$ and $w \in \mathrm{W}$:

$$\ell(ws_A) = \begin{cases} \ell(w)+1 & \text{if } w(A) \in \Phi_{aff}^+, \\ \ell(w)-1 & \text{if } w(A) \in \Phi_{aff}^-. \end{cases} \tag{2.1}$$

The affine Weyl group is defined as the subgroup $\mathrm{W}_{aff} := < s_A,\ A \in \Phi_{aff} >$ of W. Let $S_{aff} := \{s_A : A \in \Pi_{aff}\}$. The pair $(\mathrm{W}_{aff}, S_{aff})$ is a Coxeter system ([3, V.3.2 Thm. 1(i)]), and the length function ℓ restricted to W_{aff} coincides with the length function of this Coxeter system. Recall ([24, 1.5]) that W_{aff} is a normal subgroup of W: the set Ω of elements with length zero is an abelian subgroup of W and W is the semi-direct product $\mathrm{W} = \Omega \ltimes \mathrm{W}_{aff}$. The length ℓ is constant on the double cosets $\Omega w \Omega$ for $w \in \mathrm{W}$. In particular Ω normalizes S_{aff}.

We extend the Bruhat order $\leq$ on the Coxeter system $(\mathrm{W}_{aff}, S_{aff})$ to W by defining

$$\omega_1 w_1 \leq \omega_2 w_2 \text{ if } \omega_1 = \omega_2 \text{ and } w_1 \leq w_2$$

for $w_1, w_2 \in \mathrm{W}_{aff}$ and $\omega_1, \omega_2 \in \Omega$ (see [15, §2.1]). We write $w < w'$ if $w \leq w'$ and $w \neq w'$ for $w, w' \in \mathrm{W}$. Note that $w \leq w'$ and $\ell(w) = \ell(w')$ implies $w = w'$.

2.1.2. Let X_*^+ denote the set of dominant coweights that is to say the subset of all $\lambda \in \mathrm{X}_*$ such that

$$\langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+.$$

The set of antidominant coweights is $\mathrm{X}_*^- := -\mathrm{X}_*^+$. The extended Weyl group W is the disjoint union of all $\mathfrak{W} e^\lambda \mathfrak{W}$ where λ ranges over X_*^+ (resp. X_*^-) (see for example [22, 2.2]).

Remark 2.1. We have $\ell(we^\lambda) = \ell(w) + \ell(e^\lambda)$ for all $w \in \mathfrak{W}$ and $\lambda \in \mathrm{X}_*^+$.

There is a partial order on X_*^+ given by $\lambda \preceq \mu$ if and only if $\lambda - \mu$ is a non-negative integral linear combination of the simple coroots.

2.1.3. *Distinguished coset representatives.* The following statement is [29, Proposition 4.6] (see [27, Lemma 2.6] for ii).

Proposition 2.2. *Let $\mathcal{D}$ be the subset of the elements d in* W *such that*

$$d^{-1}\Phi^+ \subset \Phi_{aff}^+. \tag{2.2}$$

i. *It is a system of representatives of the right cosets $\mathfrak{W}\backslash\mathrm{W}$. Any $d \in \mathcal{D}$ is the unique element with minimal length in $\mathfrak{W}d$ and for any $w \in \mathfrak{W}$, we have*

$$\ell(wd) = \ell(w) + \ell(d). \tag{2.3}$$

ii. *An element $d \in \mathcal{D}$ can be written uniquely $d = e^\lambda w$ with $\lambda \in \mathrm{X}_*^+$ and $w \in \mathfrak{W}$. We then have $\ell(e^\lambda) = \ell(d) + \ell(w^{-1})$.*

iii. *Let $s \in S$ and $d \in \mathcal{D}$. If $\ell(ds) = \ell(d) - 1$ then $ds \in \mathcal{D}$. If $\ell(ds) = \ell(d) + 1$ then either $ds \in \mathcal{D}$, or $ds \in \mathfrak{W}d$.*

Remark 2.3. Let $\lambda \in \mathrm{X}_*^+$.
- Then $e^\lambda \in \mathcal{D}$ and $\mathcal{D} \cap \mathfrak{W} e^\lambda \mathfrak{W} = \mathcal{D} \cap e^\lambda \mathfrak{W}$.
- There is a unique element with maximal length in $\mathfrak{W} e^\lambda \mathfrak{W}$: it is $w_\lambda := w_0 e^\lambda$ where w_0 is the unique element with maximal length in $\mathfrak{W}$.

Lemma 2.4. *Let $\lambda, \mu \in \mathrm{X}_*^+$ and $d \in \mathcal{D} \cap e^\lambda \mathfrak{W}$.*

i. *$d \leq e^\lambda$ and in particular $\ell(d) < \ell(e^\lambda)$ if $d \neq e^\lambda$.*
ii. *$d \leq e^\mu$ is equivalent to $e^\lambda \leq e^\mu$.*
iii. *Let $w \in \mathfrak{W} e^\lambda \mathfrak{W}$. If $w \leq w_\mu$ then $e^\lambda \leq e^\mu$. In particular, $w_\lambda \leq w_\mu$ is equivalent to $e^\lambda \leq e^\mu$.*

Proof. The first assertion comes from ii. of Proposition 2.2. To prove the second assertion, write $d = e^\lambda w$ with $w \in \mathfrak{W}$ and suppose that $d \leq e^\mu$. If $w \neq 1$, then ${}^{w^{-1}}\lambda$ is not a dominant coweight otherwise by Remark 2.1 we would have $\ell(d) > \ell(e^\lambda)$. Therefore, there is $\beta \in \Pi$ such that $\langle {}^{w^{-1}}\lambda, \beta \rangle < 0$, that is to say $d(\beta, 0) = (w\beta, -\langle {}^{w^{-1}}\lambda, \beta\rangle) \in \Phi_{aff}^+ - \Phi^+$. This implies that $\ell(ds_\beta) = \ell(d) + 1$ by (2.1) and that $ds_\beta d^{-1} \notin \mathfrak{W}$ so that $ds_\beta \in \mathcal{D}$ after Proposition 2.2 iii. Note that applying Proposition 2.2 ii. to d and ds_β shows that $\ell(ws_\beta) = \ell(w) - 1$. By Lemma [15, 4.3] (repeatedly) we get from $d \leq e^\mu$ that $ds_\beta \leq e^\mu$ (we have either $ds_\beta \leq e^\mu$ or $ds_\beta \leq e^\mu s_\beta$. In the latter case, $ds_\beta \leq e^\mu s_\beta \leq e^\mu$ if $\langle \mu, \beta \rangle > 0$; otherwise $\langle \mu, \beta \rangle = 0$ and e^μ and s commute: we have $ds_\beta \leq s_\beta e^\mu$ which implies that either $ds_\beta \leq e^\mu$ or $s_\beta ds_\beta \leq e^\mu$, but $ds_\beta \leq s_\beta d s_\beta$ because $ds_\beta \in \mathcal{D}$, so in any case $ds_\beta \leq e^\mu$). We then complete the proof of the second assertion by induction on $\ell(w)$.
To prove the last assertion, it is enough to consider the case $w = d \in \mathcal{D}$. We prove by induction on $\ell(u)$ for $u \in \mathfrak{W}$ that $d \leq ue^\mu$ implies $d \leq e^\mu$: let $s \in S$ such that $\ell(su) = \ell(u) - 1$; by Lemma [15, 4.3] we have $d \leq sd \leq sue^\mu$ or $d \leq sue^\mu$; conclude. Therefore, $d \leq w_\mu$ implies $d \leq e^\mu$ and by ii., $e^\lambda \leq e^\mu$. □

One easily deduces from the previous Lemma (see also [23, §1] for the compatibility between the partial orderings $\preceq$ and $\leq$ on the dominant coweights) the following well known result ([17, 7.8], [22, (4.6)]). Let $\lambda \in \mathrm{X}_*^+$. We have

$$\{w \in \mathrm{W},\ w \leq w_\lambda\} = \coprod_\mu \mathfrak{W} e^\mu \mathfrak{W} \tag{2.4}$$

where $\mu \in \mathrm{X}_*^+$ ranges over the dominant coweights such that $e^\mu \leq e^\lambda$ or equivalently $\mu \preceq \lambda$.

2.2. **Root datum attached to $\mathbf{G}(\mathfrak{F})$.** We refer for example to [34, I.1] and [35] for the description of the root datum $(\Phi, \mathrm{X}^*(\mathrm{T}), \check{\Phi}, \mathrm{X}_*(\mathrm{T}))$ associated to the choice (§1.1) of a maximal $\mathfrak{F}$-split torus T in G (or rather, T is the group of $\mathfrak{F}$-points of a maximal torus in $\mathbf{G}$). This root datum is reduced because the group $\mathbf{G}$ is $\mathfrak{F}$-split.

2.2.1. *Apartment attached to a maximal split torus.* The set $\mathrm{X}^*(\mathrm{T})$ (resp. $\mathrm{X}_*(\mathrm{T})$) is the set of algebraic characters (resp. cocharacters) of T. The cocharacters will also be called the coweights. Let $\mathrm{X}^*(\mathrm{Z})$ and $\mathrm{X}_*(\mathrm{Z})$ denote respectively the set of algebraic characters and cocharacters of the connected center Z of G.

As before, we denote by $\langle\,.\,,.\,\rangle : \mathrm{X}_*(\mathrm{T}) \times \mathrm{X}^*(\mathrm{T}) \to \mathbb{Z}$ the natural perfect pairing. The vector space

$$\mathbb{R} \otimes_{\mathbb{Z}} (\mathrm{X}_*(\mathrm{T})/\mathrm{X}_*(\mathrm{Z}))$$

considered as an affine space on itself identifies with an apartment $\mathscr{A}$ of the building $\mathscr{X}$ that we will call standard. We choose the hyperspecial vertex x_0 as an origin of $\mathscr{A}$. Note that the corresponding apartment in the extended building $\mathscr{X}^{ext}$ as described in [35, 4.2.16] is the affine space $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathrm{T})$. Let $\alpha \in \Phi$. Since $\langle\,.\,,\alpha\rangle$ has value zero on $\mathrm{X}_*(\mathrm{Z})$, it extends to a function $\alpha(\,.\,)$ on $\mathscr{A}$ which we will sometimes still denote by $\langle\,.\,,\alpha\rangle$. The reflection s_α associated to a root $\alpha \in \Phi$ can be seen as a reflection on the affine space $\mathscr{A}$ given by $s_\alpha : x \mapsto x - \alpha(x)\check{\alpha}$. The natural action on $\mathscr{A}$ of the normalizer $N_{\mathrm{G}}(\mathrm{T})$ of T in G yields an isomorphism between $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$ and the subgroup $\mathfrak{W}$ of the transformations of $\mathscr{A}$ generated by these reflections.

Together with the choice of the vertex x_0, the choice of the chamber C (§1.1) of the standard apartment implies in particular the choice of the subset Φ^+ of the positive roots, that is to say the set of all $\alpha \in \Phi$ that take non negative values on C. Set $\Phi^- := -\Phi$. We fix Π a basis for Φ^+. We denote by Φ_{aff} (resp. Φ^+_{aff}, resp. Φ^-_{aff}) the set of affine (resp. positive affine, resp. negative affine) roots, and by Π_{aff} the corresponding basis for Φ_{aff} as in 2.1. Denote by $\mathrm{X}_*^+(\mathrm{T})$ (resp. $\mathrm{X}_*^-(\mathrm{T})$) the set of dominant (resp. antidominant) coweights. The partial ordering on $\mathrm{X}_*^+(\mathrm{T})$ associated to Π is denoted by $\preceq$.

The extended Weyl group W is the semi-direct product of $\mathfrak{W} \ltimes \mathrm{X}_*(\mathrm{T})$. It contains the affine Weyl group W_{aff}. We denote by ℓ the length function and by $\leq$ the Bruhat ordering on W. They extend the ones on the Coxeter system $(\mathrm{W}_{aff}, \mathrm{S}_{aff})$.

To an element $g \in \mathrm{T}$ corresponds a vector $\nu(g) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathrm{T})$ defined by

$$\langle \nu(g),\, \chi\rangle = -\, val_{\mathfrak{F}}(\chi(g)) \qquad \text{for any } \chi \in \mathrm{X}^*(\mathrm{T}). \tag{2.5}$$

The kernel of ν is the maximal compact subgroup T^0 of T. The quotient of T by T^0 is a free abelian group with rank equal to dim(T), and ν induces an isomorphism $\mathrm{T}/\mathrm{T}^0 \cong \mathrm{X}_*(\mathrm{T})$. The group T/T^0 acts by translation on $\mathscr{A}$ via ν. The actions of $\mathfrak{W}$ and T/T^0 combine into an action of the quotient of $N_{\mathrm{G}}(\mathrm{T})$ by T^0 on $\mathscr{A}$ as recalled in [34, page 102]. Since x_0 is a special vertex of the building, this quotient identifies with W ([35, 1.9]) and from now on we identify W with $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}^0$. In particular, a simple reflection $s_A \in \mathrm{S}_{aff}$ corresponding to the affine root $A = (\alpha, r)$ can be seen as the reflection at the hyperplane with equation $\langle\,.\,,\alpha\rangle + r = 0$ in the affine space $\mathscr{A}$.

We denote by $\mathcal{D}$ the distinguished set of representatives of the cosets $\mathfrak{W}\backslash\mathrm{W}$ as defined in 2.1.3.

Remark 2.5. In [19] the chosen isomorphism between T/T^0 and $\mathrm{X}_*(\mathrm{T})$ is not the same as (2.5). Here we chose to follow [35, 1.1] and [34, I.1]. The consequence is that the image in T/T^0 of the submonoid $\mathrm{T}^- := \{t \in \mathrm{T},\, val_{\mathfrak{F}}(\alpha(t)) \leq 0 \text{ for all } \alpha \in \Phi^+\}$ (cf [19, Definition 1.1]) corresponds, in our normalization, to the submonoid $\mathrm{X}_*^+(\mathrm{T})$ of $\mathrm{X}_*(\mathrm{T})$. It explains why the dominant coweights appear naturally in our setting.

2.2.2. *Tame extended Weyl group.* Let T^1 be the pro-p Sylow subgroup of T^0. We denote by $\tilde{\mathrm{W}}$ the quotient of $N_{\mathrm{G}}(\mathrm{T})$ by T^1 and obtain the exact sequence

$$0 \to \mathrm{T}^0/\mathrm{T}^1 \to \tilde{\mathrm{W}} \to \mathrm{W} \to 0.$$

We fix a lift $\tilde{w} \in \tilde{\mathrm{W}}$ of any $w \in \mathrm{W}$.

The length function ℓ on W pulls back to a length function ℓ on $\tilde{\mathrm{W}}$ ([36, Prop. 1]). For $u, v \in \tilde{\mathrm{W}}$ we write $u \leq v$ if their projections $\bar{u}$ and $\bar{v}$ in W satisfy $\bar{u} \leq \bar{v}$.

For any $A \subseteq \mathrm{W}$ we denote by $\tilde{A}$ its preimage in $\tilde{\mathrm{W}}$. In particular, we have the set $\tilde{\mathrm{X}}_*(\mathrm{T})$: as well as those of $\mathrm{X}_*(\mathrm{T})$, its elements will be denoted by λ or e^{λ}. For $\alpha \in \Phi$, we inflate the function $\alpha(\,.\,)$ defined on $\mathrm{X}_*(\mathrm{T})$ to $\tilde{\mathrm{X}}_*(\mathrm{T})$. We will write $\langle x, \alpha\rangle := \alpha(x)$ for $x \in \tilde{\mathrm{X}}_*(\mathrm{T})$. We still call *dominant coweights* the elements in the preimage $\tilde{\mathrm{X}}_*^+(\mathrm{T})$ of $\mathrm{X}_*^+(\mathrm{T})$.

2.2.3. *Bruhat decomposition.* We have the decomposition $\mathrm{G} = \mathrm{I}N_{\mathrm{G}}(\mathrm{T})\mathrm{I}$ and two cosets $\mathrm{I}n_1\mathrm{I}$ and $\mathrm{I}n_2\mathrm{I}$ are equal if and only if n_1 and n_2 have the same projection in W. In other words, a system of representatives in $N_{\mathrm{G}}(\mathrm{T})$ of the elements in W provides a system of representatives of the double cosets of G modulo I. This follows from [35, 3.3.1]. We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \mathrm{W}$ (resp. $w \in \tilde{\mathrm{W}}$). In 2.2.5 we will introduce specifically chosen lifts for the elements $\tilde{s}$, where $s \in \mathrm{S}_{aff}$. By [36, Theorem 1] the group G is the disjoint union of the double cosets $\tilde{\mathrm{I}}\hat{w}\tilde{\mathrm{I}}$ for all $w \in \tilde{\mathrm{W}}$.

Remark 2.6. For $w \in \tilde{\mathrm{W}}$, we will sometimes write $w\tilde{\mathrm{I}}w^{-1}$ instead of $\hat{w}\tilde{\mathrm{I}}\hat{w}^{-1}$ since it does not depend on the chosen lift.

2.2.4. *Cartan decomposition.* The double cosets of G modulo K are indexed by the coweights in a chosen Weyl chamber: for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, the element $\lambda(\varpi)$ is a lift for $e^{-\lambda} \in \mathrm{W}$ (see Remark 2.5) and G is the disjoint union of the double cosets $\mathrm{K}\lambda(\varpi)\mathrm{K}$ for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$.

2.2.5. *Root subgroups and Chevalley basis.* For $\alpha \in \Phi$, we consider the attached unipotent subgroup $\mathcal{U}_\alpha$ of G as in ([5, 6.1]). To an affine root $(\alpha, r) \in \Phi_{aff}$ corresponds a subgroup $\mathcal{U}_{(\alpha,r)}$ of $\mathcal{U}_\alpha$ ([35, 1.4]) the following properties of which we are going to use ([34, p. 103]):
- For $r \leq r'$ we have $\mathcal{U}_{(\alpha,r')} \subseteq \mathcal{U}_{(\alpha,r)}$.
- For $w \in \mathrm{W}$, the group $\hat{w}\mathcal{U}_{(\alpha,r)}\hat{w}^{-1}$ does not depend on the lift $\hat{w} \in \mathrm{G}$ and is equal to $\mathcal{U}_{w(\alpha,r)}$.

We fix an épinglage for G as in SGA3 Exp. XXIII, 1.1 (see [12]). In particular, to $\alpha \in \Phi$ is attached a central isogeny $\phi_\alpha : \mathrm{SL}_2(\mathfrak{F}) \to \mathrm{G}_\alpha$ where G_α is the subgroup of G generated by $\mathcal{U}_\alpha$ and $\mathcal{U}_{-\alpha}$ ([12, Thm 1.2.5]).

We set $n_{s_\alpha} := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and, for $u \in \mathfrak{F}^*$, $h_\alpha(u) := \phi_\alpha \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. Then T contains $h_\alpha(\mathfrak{F}^*)$ for all $\alpha \in \Phi$. After embedding $\mathbb{F}_q^*$ into $\mathfrak{F}^*$ by Teichmüller lifting, we consider the subgroup T_α of T^0 equal to the image of $\mathbb{F}_q^*$ by h_α. It identifies with a subgroup of $\mathrm{T}^0/\mathrm{T}^1$.

For $\alpha \in \Pi_m$, set $h_{(\alpha,1)} := h_\alpha$, $\mathrm{T}_{(\alpha,1)} := \mathrm{T}_\alpha$ and $n_{s_{(\alpha,1)}} := \phi_\alpha \begin{pmatrix} 0 & \varpi \\ -\varpi^{-1} & 0 \end{pmatrix}$.

For $A \in \Pi_{aff}$, the element $n_{s_A} \in N_G(\mathrm{T})$ is a lift for $s_A \in \mathrm{S}_{aff}$ ([12, Proof of Proposition 1.3.2]). The normalizer $N_{\mathrm{G}}(\mathrm{T})$ of T is generated by T and all n_{s_α} for $\alpha \in \Phi$. For all $w \in \tilde{\mathrm{W}}$ with length ℓ, there is $\omega \in \tilde{\mathrm{W}}$ with length zero and $s_1, ..., s_\ell \in \mathrm{S}_{aff}$ such that the product $n_{s_1}...n_{s_\ell} \in N_{\mathrm{G}}(\mathrm{T})$ is a lift for $\omega w \in \tilde{\mathrm{W}}$.

2.3. **Root datum attached to a standard facet.**

2.3.1. Let $F \subseteq \overline{C}$ be a facet containing x_0 in its closure. Such a facet will be called *standard*. Attached to it is the subset Π_F of the roots in Π taking value zero on F, or equivalently the subset S_F of the reflections in S fixing F pointwise.

Remark 2.7. The closure $\overline{F}$ of a facet F consists exactly of the points of $\overline{C}$ that are fixed by the reflections in S_F ([3, V.3.3 Proposition 1]).

We let Φ_F denote the set of roots in Φ taking value zero on F and set $\Phi_F^+ := \Phi_F \cap \Phi^+$, $\Phi_F^- := \Phi_F \cap \Phi^-$. We consider the root datum $(\Phi_F, \mathrm{X}^*(\mathrm{T}), \check{\Phi}_F, \mathrm{X}_*(\mathrm{T}))$. The corresponding finite Weyl group $\mathfrak{W}_F$ is the subgroup of $\mathfrak{W}$ generated by all s_α for $\alpha \in \Phi_F$. The pair $(\mathfrak{W}_F, \mathrm{S}_F)$ is a Coxeter system. The restriction $\ell|\mathfrak{W}_F$ coincides with its length function ([3, IV.1.8 Cor. 4]). The extended Weyl group is $\mathrm{W}_F = \mathfrak{W}_F \ltimes \mathrm{X}_*(\mathrm{T})$. Its action on the affine roots $\Phi_{F,aff} := \Phi_F \times \mathbb{Z}$ coincides with the restriction of the action of W. Denote by $\underset{F}{\preceq}$ the partial order on $\mathrm{X}_*(\mathrm{T})$ with respect to Π_F, by $\mathrm{W}_{F,aff}$ the affine Weyl group with generating set $\mathrm{S}_{F,aff}$ defined as in 2.1. It comes with a length function denoted by ℓ_F and a Bruhat order denoted by $\underset{F}{\leq}$, which can both be extended to W_F.

2.3.2. The restriction $\ell|\mathrm{W}_F$ does not coincide with ℓ_F in general, and likewise the restriction to W_F of the Bruhat order on W does not coincide with $\underset{F}{\leq}$. We call F-positive the elements w in W_F satisfying

$$w^{-1}(\Phi^+ - \Phi_F^+) \subset \Phi_{aff}^+.$$

For $\lambda \in \mathrm{X}_*(\mathrm{T})$, the element e^λ is F-positive if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+ - \Phi_F^+$. In this case, we will say that the coweight λ itself is F-positive. We observe that if μ and $\nu \in \mathrm{X}_*(\mathrm{T})$ are F-positive coweights such that $\mu - \nu$ is also F-positive, then we have the equality.

$$\ell(e^{\mu-\nu}) + \ell(e^\nu) - \ell(e^\mu) = \ell_F(e^{\mu-\nu}) + \ell_F(e^\nu) - \ell_F(e^\mu) \tag{2.6}$$

Its left hand side is indeed by definition $\sum_{\alpha\in\Phi^+} |\langle \mu - \nu, \alpha \rangle| + |\langle \nu, \alpha \rangle| - |\langle \mu, \alpha \rangle|$ but the contribution to this sum of the roots in $\Phi^+ - \Phi_F^+$ is zero since $\mu - \nu$, μ and ν are F-positive.

Since the elements in $\mathfrak{W}_F$ stabilize the set $\Phi^+ - \Phi_F^+$, an element in W_F is F-positive if and only if it belongs to $\mathfrak{W}_F e^\lambda \mathfrak{W}_F$ for some F-positive coweight λ. The F-positive elements in W_F form a semi-group. A coweight λ is said strongly F-positive if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Phi^+ - \Phi_F^+$ and $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Phi_F^+$. By [7, Lemma 6.14], strongly F-positive elements do exist.

Remark 2.8. If $F = x_0$, then $\mathrm{W}_{x_0} = \mathrm{W}$. If $F = C$ then $\mathrm{W}_C = \mathrm{X}_*(\mathrm{T})$ and the C-positive elements are the dominant coweights. A strongly C-positive element will be called strongly dominant.

Lemma 2.9. i. *Let $\mu \in \mathrm{X}_*(\mathrm{T})$ and $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ such that $\mu \preceq_F \lambda$. Suppose that for all $\alpha \in \Phi_F^+$ we have $\langle \mu, \alpha \rangle \geq 0$, then $\mu \in \mathrm{X}_*^+(\mathrm{T})$.*
ii. *Let $v \in \mathrm{W}_F$ such that $v \underset{F}{\leq} e^\lambda$ for some $\lambda \in \mathrm{X}_*^+(\mathrm{T})$. Then v is F-positive and there is $\mu \in \mathrm{X}_*^+(\mathrm{T})$ with $\mu \preceq_F \lambda$ such that $\mathfrak{W}_F v \mathfrak{W}_F = \mathfrak{W}_F e^\mu \mathfrak{W}_F$.*

Proof. i. Let $\alpha \in \Pi \backslash \Pi_F$. For all $\beta \in \Pi_F$, we have $\langle \check{\beta}, \alpha \rangle \leq 0$ [3, Thm1 Ch VI, n° 1.3] so $\langle \lambda - \mu, \alpha \rangle \leq 0$ and $\langle \mu, \alpha \rangle \geq 0$. For ii., note that in particular, $v \underset{F}{\leq} w_{F,\lambda}$ where $w_{F,\lambda}$ denotes the element with maximal length in $\mathfrak{W}_F e^\lambda \mathfrak{W}_F$ (see Remark 2.3). By (2.4) applied to the root system associated to F, there is a unique $\mu \in \mathrm{X}_*(\mathrm{T})$ with $\langle \mu, \alpha \rangle \geq 0$ for all $\alpha \in \Phi_F^+$ and $\mu \underset{F}{\preceq} \lambda$ such that $v \in \mathfrak{W}_F e^\mu \mathfrak{W}_F$, and part i of this lemma implies that $\mu \in \mathrm{X}_*^+(\mathrm{T})$. In particular it is F-positive and v is also F-positive.

2.3.3. The root datum $(\Phi_F, \mathrm{X}^*(\mathrm{T}), \check{\Phi}_F, \mathrm{X}_*(\mathrm{T}))$ is in fact the one attached to the semi-standard Levi subgroup M_F of G corresponding to the facet F described below.

Consider the subtorus T_F of T with dimension $dim(\mathrm{T}) - |\Pi_F|$ equal to the connected component of $\bigcap_{\alpha \in \Pi_F} \ker \alpha \subseteq \mathrm{T}$ and the Levi subgroup M_F of G defined to be the centralizer of T_F. It is the group of $\mathfrak{F}$-points of a reductive connected algebraic group $\mathbf{M}_F$ which is $\mathfrak{F}$-split. The group M_F is generated by T and the root subgroups $\mathcal{U}_\alpha$ for $\alpha \in \Phi_F$.

The subgroup $(N_{\mathrm{G}}(\mathrm{T}) \cap \mathrm{M}_F)/\mathrm{T}^0$ of $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}^0$ identifies with W_F in the isomorphism $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}^0 \simeq \mathrm{W}$. It is generated by T and all n_α for $\alpha \in \Phi_F$. Denote by $\mathscr{X}_F^{ext}$ the extended building for M_F. It shares with $\mathscr{X}^{ext}$ the apartment corresponding to T but, in this apartment, the set of affine hyperplanes coming from the root system attached to M_F is a subset of those coming from the root system attached to G. Every facet in $\mathscr{X}^{ext}$ is contained in a unique facet of $\mathscr{X}_F^{ext}$ [16, §2.9]. Denote by $\mathbf{c}_F$ the unique facet in $\mathscr{X}_F^{ext}$ containing $\mathrm{pr}^{-1}(C)$. By [16, Lemma 2.9.1], the intersection $\mathrm{I} \cap \mathrm{M}_F$ is an Iwahori subgroup for M_F: it is the pointwise fixator in M_F of $\mathbf{c}_F$. Its pro-p Sylow subgroup is $\tilde{\mathrm{I}} \cap \mathrm{M}_F$. We have a Bruhat decomposition for M_F: it is the disjoint union of the double cosets $(\mathrm{I} \cap \mathrm{M}_F)\hat{w}(\mathrm{I} \cap \mathrm{M}_F)$ where $\hat{w}$ denotes the chosen lift for $w \in \mathrm{W}_F$ in G (2.2.3) which in fact belongs to M_F.

Denote by $\tilde{\mathrm{W}}_F$ the quotient $(N_{\mathrm{G}}(\mathrm{T}) \cap \mathrm{M}_F)/\mathrm{T}^1$. It is generated by $\mathrm{T}^0/\mathrm{T}^1$ and all $\tilde{w}$ for $w \in \mathrm{W}_F$. We have an exact sequence

$$0 \to \mathrm{T}^0/\mathrm{T}^1 \to \tilde{\mathrm{W}}_F \to \mathrm{W}_F \to 0.$$

The Levi subgroup M_F is the disjoint union of the double cosets $(\tilde{\mathrm{I}} \cap \mathrm{M}_F)\hat{w}(\tilde{\mathrm{I}} \cap \mathrm{M}_F)$ for all $w \in \tilde{\mathrm{W}}_F$. We denote by $\tilde{\mathfrak{W}}_F$ the preimage of $\mathfrak{W}_F$ in $\tilde{\mathrm{W}}_F$.

□

2.4. Generic Hecke rings.

2.4.1. For $g \in \mathrm{G}$ we denote by $\boldsymbol{\tau}_g$ the characteristic function of $\tilde{\mathrm{I}} g \tilde{\mathrm{I}}$. Since it only depends on the element $w \in \tilde{\mathrm{W}}$ such that $g \in \tilde{\mathrm{I}}\hat{w}\tilde{\mathrm{I}}$, we will also denote it by $\boldsymbol{\tau}_w$. We consider the convolution ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ of the functions with finite support in $\tilde{\mathrm{I}}\backslash \mathrm{G}/\tilde{\mathrm{I}}$ and values in $\mathbb{Z}$ with product defined by

$$f \star f' : \mathrm{G} \to \mathbb{Z}, g \mapsto \sum_{u \in \tilde{\mathrm{I}}\backslash \mathrm{G}} f(gu^{-1}) f'(u)$$

for $f, f' \in \tilde{\mathrm{H}}_{\mathbb{Z}}$. It is a free $\mathbb{Z}$-module with basis the set of all $\{\boldsymbol{\tau}_w\}_{w \in \tilde{\mathrm{W}}}$ satisfying the following braid and, respectively, quadratic relations ([36, Theorem 1]):

$$\boldsymbol{\tau}_{ww'} = \boldsymbol{\tau}_w \boldsymbol{\tau}_{w'} \text{ for } w, w' \in \tilde{\mathrm{W}} \text{ satisfying } \ell(ww') = \ell(w) + \ell(w'). \tag{2.7}$$

$$\boldsymbol{\tau}_{n_A}^2 = \nu_A \boldsymbol{\tau}_{n_A} + q \boldsymbol{\tau}_{h_A(-1)} \text{ for } A \in \Pi_{aff}, \text{ where } \nu_A := \sum_{t \in \mathrm{T}_A} \boldsymbol{\tau}_t. \tag{2.8}$$

The braid relations imply that $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is generated by all $\boldsymbol{\tau}_{n_A}$ for $A \in \Pi_{aff}$ and $\boldsymbol{\tau}_\omega$ for $\omega \in \tilde{\mathrm{W}}$ with length zero.

2.4.2. For any $w \in \mathrm{W}$, define $\boldsymbol{\tau}_w^*$ to be the element in $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ equal to $q^{\ell(w)} \boldsymbol{\tau}_w^{-1}$. It actually lies in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ and the ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is endowed with an involutive automorphism defined by ([36, Corollary 2])

$$\iota : \boldsymbol{\tau}_w \mapsto (-1)^{\ell(w)} \boldsymbol{\tau}_{w^{-1}}^*. \tag{2.9}$$

Remark 2.10. We have $\iota(\boldsymbol{\tau}_{n_A}) = -\boldsymbol{\tau}_{n_A} + \nu_A$.

The following fundamental Lemma is proved in [36, Lemma 13] which is a adaptation to the pro-p Hecke ring of the analogous results of [15, §5] established for the Iwahori-Hecke ring.

Lemma 2.11. *For $v, w \in \tilde{\mathrm{W}}$ we have in $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$*

$$q^{\frac{\ell(vw) + \ell(w) - \ell(v)}{2}} \boldsymbol{\tau}_v \boldsymbol{\tau}_{w^{-1}}^{-1} = \boldsymbol{\tau}_{vw} + \sum_x a_x \boldsymbol{\tau}_x$$

where $a_x \in \mathbb{Z}$ and x ranges over a finite set of elements in $\tilde{\mathrm{W}}$ with length $< \ell(vw)$. More precisely, these elements satisfy $x < vw$ (see 2.2.2).

2.4.3. Following [36, §1.3, page 9], we suppose in this paragraph that R is a ring containing an inverse for $(q.1_{\mathrm{R}} - 1)$ and a primitive $(q-1)^{\text{th}}$ root of 1. We denote by $\mathrm{R}^\times$ the group of invertible elements of R. Recall that $\overline{\mathbf{T}}(\mathbb{F}_q)$ identifies with $\mathrm{T}^0/\mathrm{T}^1$ and can therefore be seen as a subgroup of $\tilde{\mathrm{W}}$. The finite Weyl group $\mathfrak{W}$ identifies with the Weyl group of $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$ ([35, 3.5.1]): it acts on $\overline{\mathbf{T}}(\mathbb{F}_q)$ and its R-character. Inflate this action to an action of the extended Weyl group W. Let $\xi : \overline{\mathbf{T}}(\mathbb{F}_q) \to \mathrm{R}^\times$ be a R-character of $\overline{\mathbf{T}}(\mathbb{F}_q)$. We attach to it the following idempotent element

$$\epsilon_\xi := \frac{1}{|\overline{\mathbf{T}}(\mathbb{F}_q)|} \sum_{t \in \overline{\mathbf{T}}(\mathbb{F}_q)} \xi^{-1}(t) \boldsymbol{\tau}_t \ \in \ \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}.$$

Note that for $t \in \overline{\mathbf{T}}(\mathbb{F}_q)$, we have $\epsilon_\xi \boldsymbol{\tau}_t = \boldsymbol{\tau}_t \epsilon_\xi = \xi(t)\epsilon_\xi$. It implies that the quadratic relations in $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}$ have the (simpler) form: let $A \in \Pi_{aff}$

$$\begin{array}{l} \text{- if } \xi \text{ is trivial on } \mathrm{T}_A \text{ then } \xi(h_A(-1)) = 1 \text{ and } \epsilon_\xi \boldsymbol{\tau}_{n_A}^2 = \epsilon_\xi((q-1)\boldsymbol{\tau}_{n_A} + q). \\ \text{- otherwise } \epsilon_\xi \boldsymbol{\tau}_{n_A}^2 = q\epsilon_\xi \xi(h_A(-1)). \end{array} \tag{2.10}$$

2.4.4. Let F be a standard facet. The definitions of the previous paragraphs apply to the Levi subgroup M_F and its root system (2.3). In particular, for $w \in \tilde{\mathrm{W}}_F$, denote by $\boldsymbol{\tau}_w^F$ the characteristic function of $(\tilde{\mathrm{I}} \cap \mathrm{M}_F)\hat{w}(\tilde{\mathrm{I}} \cap \mathrm{M}_F)$ and by $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ the Hecke ring as defined in 2.4.1. It has $\mathbb{Z}$-basis the set of all $\boldsymbol{\tau}_w^F$ for $w \in \tilde{\mathrm{W}}_F$ and the braid relations are controlled by the length function ℓ_F on $\tilde{\mathrm{W}}_F$. The $\mathbb{Z}$-linear involution of $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ as defined in 2.4.2 is denoted by ι^F. Note that when $F = x_0$ then $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ is in fact $\tilde{\mathrm{H}}_{\mathbb{Z}}$ and we do not write the F exponents.

The algebra $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ does not inject in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ in general. However, there is a *positive* subring $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+$ of $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ with $\mathbb{Z}$-basis the set of all $\boldsymbol{\tau}_w^F$ for $w \in \tilde{\mathrm{W}}_F$ that are F-positive, and an injection

$$\begin{array}{rccc} j_F^+ : & \tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+ & \longrightarrow & \tilde{\mathrm{H}}_{\mathbb{Z}} \\ & \boldsymbol{\tau}_w^F & \longmapsto & \boldsymbol{\tau}_w \end{array}$$

which, if R is a ring containing an inverse for $q.1_R$, extends to a R-linear injection $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathrm{R} \to \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}$ denoted by j_F. The proof in the case of complex Hecke algebras can be found in [7, (6.12)]; it goes through for pro-p Iwahori-Hecke rings over $\mathbb{Z}$. We point out that what we call the positive subring of $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ is called negative in [7].

3. Representations of spherical and pro-p Hecke algebras

3.1. **Hecke algebras attached to parahoric subgroups of $\mathbf{G}(\mathfrak{F})$.**

3.1.1. *Parahoric subgroups.* Associated to each facet F of the (semi-simple) building is, in a G-equivariant way, a smooth affine $\mathfrak{O}$-group scheme $\mathbf{G}_F$ whose general fiber is $\mathbf{G}$ and such that $\mathbf{G}_F(\mathfrak{O})$ is the pointwise stabilizer in G of the preimage $\mathrm{pr}^{-1}(F)$ of F in the extended building. Its neutral component is denoted by $\mathbf{G}_F^\circ$ so that the reduction $\overline{\mathbf{G}}_F^\circ$ over $\mathbb{F}_q$ is a connected smooth algebraic group. The subgroup $\mathbf{G}_F^\circ(\mathfrak{O})$ of G is a parahoric subgroup. We consider

$$\tilde{\mathrm{I}}_F := \{g \in \mathbf{G}_F^\circ(\mathfrak{O}) : (g \mod \varpi) \in \text{ unipotent radical of } \overline{\mathbf{G}}_F^\circ\}.$$

The groups $\tilde{\mathrm{I}}_F$ are compact open pro-p subgroups in G such that $\tilde{\mathrm{I}}_C = \tilde{\mathrm{I}}$, $\tilde{\mathrm{I}}_{x_0} = \mathrm{K}_1$ and

$$g\tilde{\mathrm{I}}_F g^{-1} = \tilde{\mathrm{I}}_{gF} \qquad \text{for any } g \in \mathrm{G}, \text{ and } \tilde{\mathrm{I}}_{F'} \subseteq \tilde{\mathrm{I}}_F \qquad \text{whenever } F' \subseteq \overline{F}. \tag{3.1}$$

Let F be a standard facet. Then $\mathbf{G}_F^\circ(\mathfrak{O})$ is the distinct union of the double cosets $\tilde{\mathrm{I}}\hat{w}\tilde{\mathrm{I}}$ for all w in $\mathfrak{W}_F$ [29, Lemma 4.9 and §4.7].

Remark 3.1. Denote by $\mathfrak{U}_F$ the subgroup of K generated by all $\mathfrak{U}_{(\alpha, -\inf_F(\alpha))}$ for all $\alpha \in \Phi$. Then $\mathbf{G}_F^\circ(\mathfrak{O}) = \mathfrak{U}_F \mathrm{T}^0$ ([6, 5.2.1, 5.2.4]).

Since F is standard, the product map

$$\prod_{\alpha\in\Phi^-}\mathcal{U}_{(\alpha,1)}\times \mathrm{T}^1\times\prod_{\alpha\in\Phi_F^+}\mathcal{U}_{(\alpha,1)}\times\prod_{\alpha\in\Phi^+-\Phi_F^+}\mathcal{U}_{(\alpha,0)}\xrightarrow{\sim}\tilde{\mathrm{I}}_F \tag{3.2}$$

induces a bijection, where the products on the left hand side are ordered in some arbitrary chosen way ([34, Proposition I.2.2]). Denote by $\mathcal{U}_F^+$ the subgroup of $\tilde{\mathrm{I}}_F$ generated by all $\mathcal{U}_{(\alpha,0)}$ for α in $\Phi^+-\Phi_F^+$. Then $\tilde{\mathrm{I}}_F$ is generated by K_1 and $\mathcal{U}_F^+$.

Let $\mathcal{D}_F$ denote the set of elements in W such that $d^{-1}\Phi_F^+\subseteq\Phi_{aff}^+$. In particular, $\mathcal{D}_{x_0}$ coincides with $\mathcal{D}$ (defined in 2.2.1) and is contained in $\mathcal{D}_F$ for any standard facet F.

Lemma 3.2. i. *The set of all $\hat{\tilde{d}}$ for $d\in\mathcal{D}_F$ is a system of representatives of the double cosets $\mathbf{G}_F^\circ(\mathfrak{O})\backslash\mathrm{G}/\tilde{\mathrm{I}}$.*

ii. *For $d\in\mathcal{D}_F$, we have $\tilde{\mathrm{I}}_F(d\tilde{\mathrm{I}}d^{-1}\cap\mathbf{G}_F^\circ(\mathfrak{O}))=\tilde{\mathrm{I}}$.*

iii. *If $d\in\mathcal{D}_F\cap\mathrm{W}_F$ is F-positive, then $d\in\mathcal{D}$.*

Proof. Point i is [29, Remark 4.17]. Point ii is given by [29, Proposition 4.13] and its proof. iii. Let $d\in\mathcal{D}_F\cap\mathrm{W}_F$. Then $d^{-1}\Phi_F^+\subseteq\Phi_{aff}^+$. Suppose furthermore that d is F-positive, then there is an F-positive $\mu\in\mathrm{X}_*(\mathrm{T})$ such that $d\in e^\mu\mathfrak{W}_F$ and we deduce that $d^{-1}(\Phi^+-\Phi_F^+)\subseteq\Phi_{aff}^+$ because $\mathfrak{W}_F$ stabilizes $\Phi^+-\Phi_F^+$.

□

Remark 3.3. - The intersection of $\mathcal{D}_F$ with W_F is the distinguished set of representatives of $\mathfrak{W}_F\backslash\mathrm{W}_F$ (see 2.1.3).

- The set of all $\hat{\tilde{d}}$ for $d\in\mathfrak{W}\cap\mathcal{D}_F$ is a system of representatives of the double cosets $\mathbf{G}_F^\circ(\mathfrak{O})\backslash\mathrm{K}/\tilde{\mathrm{I}}$.

3.1.2. *Hecke algebras.* The universal representation $\tilde{\mathbf{X}}$ for G was defined in 1.1. Recall that it is a left module for the pro-p Iwahori-Hecke k-algebra $\tilde{\mathrm{H}}$ which is isomorphic to

$$\tilde{\mathrm{H}}_{\mathbb{Z}}\otimes_{\mathbb{Z}}k$$

where $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is the Hecke ring described in 2.4. Remark that the results of 2.4.3 apply.

For $w\in\tilde{\mathrm{W}}$ (resp. $g\in\mathrm{G}$) we still denote by $\boldsymbol{\tau}_w$ (resp. $\boldsymbol{\tau}_g$) its natural image in $\tilde{\mathrm{H}}$. Let F be a standard facet. Extending functions on $\mathbf{G}_F^\circ(\mathfrak{O})$ by zero to G induces a $\mathbf{G}_F^\circ(\mathfrak{O})$-equivariant embedding

$$\mathbf{X}_F:=\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathbf{G}_F^\circ(\mathfrak{O})}(1)\hookrightarrow\mathbf{X}\ .$$

The k-algebra

$$\tilde{\mathfrak{H}}_F:=\mathrm{End}_{k[\mathbf{G}_F^\circ(\mathfrak{O})]}(\mathbf{X}_F)\cong[\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathbf{G}_F^\circ(\mathfrak{O})}(1)]^{\tilde{\mathrm{I}}}\ .$$

is naturally a subalgebra of $\tilde{\mathrm{H}}$ via the extension by zero embedding $[\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathbf{G}_F^\circ(\mathfrak{O})}(1)]^{\tilde{\mathrm{I}}}\hookrightarrow\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}}(1)$.

Proposition 3.4. i. *The finite Hecke algebra $\tilde{\mathfrak{H}}_F$ has basis the set of all $\boldsymbol{\tau}_w$ for $w\in\tilde{\mathfrak{W}}_F$.*

ii. *It is a Frobenius algebra over k. In particular, for any (left or right) $\tilde{\mathfrak{H}}_F$-module $\mathfrak{m}$, we have an isomorphism of vector spaces $\mathrm{Hom}_{\tilde{\mathfrak{H}}_F}(\mathfrak{m},\tilde{\mathfrak{H}}_F)=\mathrm{Hom}_k(\mathfrak{m},k)$.*

iii. *The Hecke algebra $\tilde{\mathrm{H}}$ is a free left $\tilde{\mathfrak{H}}_F$-module with basis the set of all $\boldsymbol{\tau}_{\tilde{d}}$, $d\in\mathcal{D}_F$.*

Proof. The first point is clear. For iii, see [29, Proposition 4.21]. Note that both i and iii are valid for the generic Hecke algebras defined over $\mathbb{Z}$. For ii. see [33, Thm 2.4] (or [10, Proposition 6.11]). Recall that $\tilde{\mathfrak{H}}_F$ being Frobenius means that it is finite dimensional over k and that it is endowed with a k-linear form δ such that the bilinear form $(a,b)\longmapsto\delta(ab)$ is nondegenerate. In particular, there is a unique map $\iota:\tilde{\mathfrak{H}}_F\longrightarrow\tilde{\mathfrak{H}}_F$ satisfying $\delta(\iota(a)b)=\delta(ba)$ for any $a,b\in\tilde{\mathfrak{H}}_F$ and one can check that ι is an automorphism of the k-algebra $\tilde{\mathfrak{H}}_F$. For any left or right $\tilde{\mathfrak{H}}_F$-module $\mathfrak{m}$ we let $\iota^*\mathfrak{m}$, resp. $\iota_*\mathfrak{m}$, denote $\mathfrak{m}$ with the new $\tilde{\mathfrak{H}}_F$-action through the automorphism ι, resp. ι^{-1}. Then for any left, resp. right, $\tilde{\mathfrak{H}}_F$-module $\mathfrak{m}$, we see that the map $f\mapsto\delta\circ f$ is an isomorphism of right, resp. left, $\tilde{\mathfrak{H}}_F$-modules between $\mathrm{Hom}_{\tilde{\mathfrak{H}}_F}(\mathfrak{m},\tilde{\mathfrak{H}}_F)$ and $\mathrm{Hom}_k(\iota^*\mathfrak{m},k)$ (resp. $\mathrm{Hom}_k(\iota_*\mathfrak{m},k)$). □

Remark 3.5. The previous definitions and results are valid when replacing G by a semi-standard Levi subgroup. We will denote by $\tilde{\mathrm{H}}(\mathrm{M}_F)$ the pro-p Iwahori-Hecke algebra of M_F with coefficients in k. It is isomorphic to $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)\otimes_{\mathbb{Z}}k$.

As for the finite dimensional Hecke algebras associated to parahoric subgroups of M_F, we will only consider the following situation. Let F be a standard facet and M_F the associated Levi-subgroup. By [16, Lemma 2.9.1], $\mathrm{M}_F\cap\mathrm{K}$ is the parahoric subgroup of M_F corresponding to an hyperspecial point x_F of the building of M_F. The corresponding finite Hecke algebra $\tilde{\mathfrak{H}}_{x_F}(\mathrm{M}_F)$ has basis the set of all $\boldsymbol{\tau}^F_w$ for $w\in\tilde{\mathfrak{W}}_F$.

3.1.3. When $F=x_0$, we write $\tilde{\mathfrak{H}}$ instead of $\tilde{\mathfrak{H}}_{x_0}$. Consider a simple $\tilde{\mathfrak{H}}$-module. By [32, (2.11)] it is one dimensional and we denote by $\chi:\tilde{\mathfrak{H}}\to k$ the corresponding character. Let $\bar\chi$ be the character of $\mathrm{T}^0/\mathrm{T}^1\simeq\overline{\mathbf{T}}(\mathbb{F}_q)$ given by

$$\bar\chi(t):=\chi(\boldsymbol{\tau}_t)$$

and $\epsilon_{\bar\chi}$ the corresponding idempotent (§2.4.3). We have $\chi(\epsilon_{\bar\chi})=1$. Let $\Pi_{\bar\chi}$ denote the set of simple roots $\alpha\in\Pi$ such that $\bar\chi$ is trivial on T_α. For $\alpha\in\Pi$, we have (by the quadratic relations (2.10)): $\chi(\boldsymbol{\tau}_{n_\alpha})=0$ if $\alpha\in\Pi-\Pi_{\bar\chi}$ and $\chi(\boldsymbol{\tau}_{n_\alpha})\in\{0,-1\}$ otherwise. Define Π_χ to be the set of all $\alpha\in\Pi_{\bar\chi}$ such that $\chi(\boldsymbol{\tau}_{n_\alpha})=0$.

A k-character χ of $\tilde{\mathfrak{H}}$ is parameterized by the following data:
- a k-character $\bar\chi$ of $\overline{\mathbf{T}}(\mathbb{F}_q)$ and the attached $\Pi_{\bar\chi}$ as above.
- a subset Π_χ of $\Pi_{\bar\chi}$ such that for all $\alpha\in\Pi$, we have $\chi(\boldsymbol{\tau}_{n_\alpha})=-1$ if and only if $\alpha\in\Pi_{\bar\chi}-\Pi_\chi$.

3.1.4. Let (ρ,V) be a weight. The compact induction $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ is the space of compactly supported functions $f:G\to\mathrm{V}$ such that $f(kg)=\rho(k)f(g)$ for all $k\in\mathrm{K}$, $g\in\mathrm{G}$ and with action of G given by $(g,f)\mapsto f(\,.\,g)$. Let $\mathcal{H}(\mathrm{G},\rho)$ denote the corresponding spherical Hecke algebra that is to say the k-algebra of the G-endomorphisms of $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$. Choose and fix v a basis for $\rho^{\tilde{\mathrm{I}}}$ (it is known that this space is one dimensional [11, Corollary 6.5], see also Theorem 3.10 below which is drawn from [8]). Denote by $\mathbf{1}_{\mathrm{K},v}$ the function of $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ with support K and value v at 1. It is $\tilde{\mathrm{I}}$-invariant. Since (ρ,V) is irreducible, an element T of $\mathcal{H}(\mathrm{G},\rho)$ is determined by the image

$T(\mathbf{1}_{\mathrm{K},v})$ of $\mathbf{1}_{\mathrm{K},v}$. The restriction to $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}$ therefore yields an injective morphism of k-algebras

$$\mathcal{H}(\mathrm{G},\rho) \longrightarrow \mathrm{Hom}_{\tilde{\mathrm{H}}}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}) \tag{3.3}$$

In 3.3, we will prove that this is an isomorphism. We first identify the structure of the $\tilde{\mathrm{H}}$-module $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}$.

Lemma 3.6. *We have a $\tilde{\mathrm{H}}$-equivariant isomorphism given by*

$$\begin{array}{ccc} \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}} & \cong & (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}} \\ 1\otimes 1 & \mapsto & \mathbf{1}_{\mathrm{K},v} \end{array} \tag{3.4}$$

Proof. Recall that for $g \in \mathrm{G}$, the right action of τ_g on an $\tilde{\mathrm{I}}$-invariant function $f \in (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}$ is given by

$$f.\boldsymbol{\tau}_g = \sum_{x\in \tilde{\mathrm{I}}\cap g^{-1}\tilde{\mathrm{I}}g\backslash\tilde{\mathrm{I}}} (gx)^{-1}f.$$

In particular, when $g\in\mathrm{K}$, we may consider $\boldsymbol{\tau}_g$ as an element in $\tilde{\mathfrak{H}}$ and we have $\mathbf{1}_{\mathrm{K},v}.\boldsymbol{\tau}_g = \mathbf{1}_{\mathrm{K},v.\boldsymbol{\tau}_g}$. Therefore, the morphism of $\tilde{\mathrm{H}}$-modules of the lemma is well-defined since $\mathbf{1}_{\mathrm{K},v}$ is an eigenvector for the action of $\tilde{\mathfrak{H}}$ and the character χ.

Fact 1. *For $d\in\mathcal{D}$, the action of $\boldsymbol{\tau}_{\tilde{d}}$ on the right on $\mathbf{1}_{\mathrm{K},v}$ gives the unique $\tilde{\mathrm{I}}$-invariant element of* $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ *with support in* $\mathrm{K}\hat{\tilde{d}}\,\tilde{\mathrm{I}}$ *and value v at $\hat{\tilde{d}}$; the set of all such elements when d ranges over $\mathcal{D}$ is a basis for* $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}$.

By Proposition 3.4, a basis for $\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}$ is given by all $1\otimes\boldsymbol{\tau}_{\tilde{d}}$ for $d\in\mathcal{D}$. Therefore the fact ensures that the morphism of the lemma is bijective.

Now we prove the fact. The first point follows easily from the identity $\mathrm{K}_1(\tilde{d}\tilde{\mathrm{I}}\tilde{d}^{-1}\cap\mathrm{K}) = \tilde{\mathrm{I}}$ in Lemma 3.2ii. Furthermore, by Lemma 3.2i, an $\tilde{\mathrm{I}}$-invariant function $f\in(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}$ is determined by its values at all $\hat{\tilde{d}}$'s for $d\in\mathcal{D}$ which (using the above identity again) are $\tilde{d}\tilde{\mathrm{I}}\tilde{d}^{-1}\cap\mathrm{K}$-invariant vectors of V: these vectors are $\tilde{\mathrm{I}}$-invariants and therefore proportional to v. It proves the second statement of the fact.

□

Remark 3.7. Recall that an element of $\mathcal{H}(\mathrm{G},\rho)$ can be seen as a function with compact support $f:\mathrm{G}\to\mathrm{End}_k(\mathrm{V})$ such that $f(kgk') = \rho(\kappa)f(g)\rho(\kappa')$ for any $g\in\mathrm{G}$, $\kappa,\kappa'\in\mathrm{K}$. To such a function f corresponds the Hecke operator $T_f\in\mathcal{H}(\mathrm{G},\rho)$ that sends $\mathbf{1}_{\mathrm{K},v}$ on the element of $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ defined by $g\mapsto f(g)\,.\,v$. Reciprocally, to an element $T\in\mathcal{H}(\mathrm{G},\rho)$ is associated the function $f_T:\mathrm{G}\to\mathrm{End}_k(\mathrm{V})$ defined by $f_T(g): w\mapsto T(\mathbf{1}_{\mathrm{K},w})[g]$ for any $g\in\mathrm{G}$, where $\mathbf{1}_{\mathrm{K},w}\in\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ is the unique function with support K and value $w\in\mathrm{V}$ at 1. For $\lambda\in\mathrm{X}_*(\mathrm{T})$, the function f_T has support in $\mathrm{K}\lambda(\varpi)\mathrm{K}$ if and only if $T(\mathbf{1}_{\mathrm{K},v})\in\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ has support in $\mathrm{K}\lambda(\varpi)\mathrm{K}$.

3.2. **Categories of Hecke modules and of representations of parahoric subgroups.** Let F be a standard facet. We consider the abelian category of (smooth) representations of $\mathbf{G}_F^\circ(\mathfrak{O})$. Define the functor $\dagger$ that associates to a smooth representation V of $\mathbf{G}_F^\circ(\mathfrak{O})$ the subrepresentation $\mathrm{V}^\dagger$ generated by $\mathrm{V}^{\tilde{\mathrm{I}}}$. Consider the following categories of representations:

a) $\mathscr{R}(F)$ is the category of finite dimensional representations of $\mathbf{G}_F^\circ(\mathfrak{O})$ with trivial action of the normal subgroup $\tilde{\mathrm{I}}_F$. It is equivalent to the (abelian) category of finite dimensional representations of the finite reductive group $\overline{\mathbf{G}}_F^{red}(\mathbb{F}_q) = \mathbf{G}_F^\circ(\mathfrak{O})/\tilde{\mathrm{I}}_F$ (see [29, Proof of Lemma 4.9]). The irreducible representations of $\mathbf{G}_F^\circ(\mathfrak{O})$ are the simple objects in $\mathscr{R}(F)$.

   Note that $\dagger$ induces a functor $\dagger : \mathscr{R}(F) \to \mathscr{R}(F)$. The category $\mathscr{R}(F)$ is also equipped with the endofunctor $\vee : \ \mathrm{V} \mapsto \mathrm{V}^\vee$ associating to V the contragredient representation $\mathrm{V}^\vee = \mathrm{Hom}_k(\mathrm{V}, k)$. Since $\vee$ is anti-involutive, $\mathrm{V}^\vee$ is irreducible if and only if V is irreducible.

b) $\mathscr{R}^\dagger(F)$ is the full subcategory of $\mathscr{R}(F)$ image of the functor $\dagger$. Any irreducible representation of $\mathbf{G}_F^\circ(\mathfrak{O})$ is an object in $\mathscr{R}^\dagger(F)$. By adjunction, a representation $\mathrm{V} \in \mathscr{R}(F)$ is an object of $\mathscr{R}^\dagger(F)$ if and only if V sits in an exact sequence in $\mathscr{R}(F)$ of the form

$$\tilde{\mathbf{X}}_F^\ell \to \mathrm{V} \to 0 \tag{3.5}$$

   for some $\ell \in \mathbb{N}$, $\ell \geq 1$.

c) $\mathscr{B}(F)$ is the full (additive) subcategory of $\mathscr{R}^\dagger(F)$ whose objects are the $\mathrm{V} \in \mathscr{R}^\dagger(F)$ such that $\mathrm{V}^\vee$ is also an object in $\mathscr{R}^\dagger(F)$. Any irreducible representation of $\mathbf{G}_F^\circ(\mathfrak{O})$ is an object in $\mathscr{B}(F)$. By the following proposition, this definition coincides with [8, Definition 1].

Proposition 3.8. i. *In $\mathscr{R}(F)$, we have $\tilde{\mathbf{X}}_F \cong \tilde{\mathbf{X}}_F^\vee$.*

ii. *A representation $\mathrm{V} \in \mathscr{R}(F)$ is an object of $\mathscr{B}(F)$ if and only if there are $\ell, m \geq 1$ and $f \in \mathrm{Hom}_{\mathscr{R}(F)}(\tilde{\mathbf{X}}_F^m, \tilde{\mathbf{X}}_F^\ell)$ such that $\mathrm{V} = \mathrm{Im}(f)$.*

Proof. i. Let $\phi : \tilde{\mathbf{X}}_F \to \tilde{\mathbf{X}}_F^\vee$ be the unique $\mathbf{G}_F^\circ(\mathfrak{O})$-equivariant map sending the characteristic function of $\tilde{\mathrm{I}}$ onto $\tilde{\mathbf{X}}_F \to k, f \mapsto f(1)$. One easily checks that it is well-defined, injective, and therefore surjective. ii. Let $\mathrm{V} \in \mathscr{R}(F)$. We deduce the claim from by i. by observing that, $\mathrm{V}^\vee \in \mathscr{R}^\dagger(F)$ if and only if V sits in an exact sequence in $\mathscr{R}(F)$ of the form

$$0 \to \mathrm{V} \to \tilde{\mathbf{X}}_F^\ell \tag{3.6}$$

for some $\ell \in \mathbb{N}$, $\ell \geq 1$.

∎

Remark 3.9. An irreducible representation V of $\mathbf{G}_F^\circ(\mathfrak{O})$ is an object in $\mathscr{B}(F)$. The work of Carter and Lusztig [11] describes V explicitly as the image of a $\mathbf{G}_F^\circ(\mathfrak{O})$-equivariant morphism $\tilde{\mathbf{X}}_F \to \tilde{\mathbf{X}}_F$.

Consider the category $\mathrm{Mod}(\tilde{\mathfrak{H}}_F)$ of finite dimensional modules over $\tilde{\mathfrak{H}}_F$. The functor

$$\mathscr{R}^\dagger(F) \to \mathrm{Mod}(\tilde{\mathfrak{H}}_F),\ \mathrm{V} \mapsto \mathrm{V}^{\tilde{\mathrm{I}}} \tag{3.7}$$

is faithful. The following theorem is [8, Theorem 2] the proof of which relies on the fact that $\tilde{\mathfrak{H}}_F$ is self-injective (see Proposition 3.4).

Theorem 3.10. *The functor* (3.7) *induces an equivalence between* $\mathscr{B}(F)$ *and* $\mathrm{Mod}(\tilde{\mathfrak{H}}_F)$*.*

Remark 3.11. In particular, (3.7) is faithful and essentially surjective. It is not full in general (see [30]).

For V in $\mathscr{R}(F)$ we consider the (compactly) induced representation $\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V})$.

Lemma 3.12.
i. $\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V})$ *is a representation in* $\mathscr{R}(x_0)$*.*
ii. *We have* $(\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V}))^\dagger = \mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V}^\dagger)$ *in* $\mathscr{R}(x_0)$*.*

Proof. It is clear that both $\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V})$ and $\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V}^\dagger)$ are in $\mathscr{R}(x_0)$ because K_1 is normal in K and contained in $\tilde{\mathrm{I}}_F$. Furthermore $\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V}^\dagger)$ is an object in $\mathscr{R}^\dagger(x_0)$: it is generated as a representation of K by the functions with support in $\mathbf{G}_F^\circ(\mathfrak{O})$ taking value in $\mathrm{V}^{\tilde{\mathrm{I}}}$ at 1. It remains to show that the natural injective morphism of representations of K

$$\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V}^\dagger) \to (\mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V}))^\dagger \tag{3.8}$$

is surjective: by Mackey decomposition, an $\tilde{\mathrm{I}}$-invariant function $f \in \mathrm{ind}_{\mathbf{G}_F^\circ(\mathfrak{O})}^{\mathrm{K}}(\mathrm{V})$ is completely determined by its values at all κ in a system of representatives of the double cosets $\mathbf{G}_F^\circ(\mathfrak{O})\backslash\mathrm{K}/\tilde{\mathrm{I}}$ and the value of f at κ can be any element in $\mathrm{V}^{\mathbf{G}_F^\circ(\mathfrak{O})\cap\kappa\tilde{\mathrm{I}}\kappa^{-1}} = \mathrm{V}^{\langle\tilde{\mathrm{I}}_F, \mathbf{G}_F^\circ(\mathfrak{O})\cap\kappa\tilde{\mathrm{I}}\kappa^{-1}\rangle}$. Choose the system of representatives given by Remark 3.3ii. Then by Lemma 3.2ii, the value of f at κ can be any value in $\mathrm{V}^{\tilde{\mathrm{I}}}$ and f lies in the image of (3.8).

∎

3.3. **Spherical Hecke algebra attached to a weight.** Let (ρ, V) be a weight and $\chi : \tilde{\mathfrak{H}} \to k$ the corresponding character. By Cartan decomposition (2.2.4), the compact induction $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ decomposes as a $k[[\mathrm{K}]]$-module into the direct sum

$$\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho = \bigoplus_{\lambda\in \mathrm{X}_*^+(\mathrm{T})} \mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho$$

of the spaces of functions with support in $\mathrm{K}\lambda(\varpi)\mathrm{K}$. The following proposition is proved after the subsequent corollary which is the main result of this section: it allows us to replace the study of the spherical algebra $\mathcal{H}(\mathrm{G}, \rho)$ by the one of $\mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}})$ which is achieved in Section 4 (see Proposition 4.9).

Proposition 3.13. *Let* $\lambda \in \mathrm{X}_*(\mathrm{T})$*.*
i. *The representation* $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho)^\dagger$ *of* K *lies in* $\mathscr{B}(x_0)$*.*
ii. *The space* $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho)^\dagger)$ *is at most one dimensional.*

Corollary 3.14. i. *The map* (3.3) *induces an isomorphism of k-algebras*

$$\mathcal{H}(\mathrm{G},\rho) \cong \mathrm{Hom}_{\tilde{\mathrm{H}}}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}) \cong \mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}). \tag{3.9}$$

ii. *For $\lambda \in \mathrm{X}_*(\mathrm{T})$, the subspace of $\mathcal{H}(\mathrm{G},\rho)$ of the functions with support in $\mathrm{K}\lambda(\varpi)\mathrm{K}$ is at most one dimensional.*

Remark 3.15. It will be a corollary of the proof of Proposition 4.9 that the subspace of $\mathcal{H}(\mathrm{G},\rho)$ of the functions with support in $\mathrm{K}\lambda(\varpi)\mathrm{K}$ is in fact one dimensional. This fact is proved and used in [19] (Step 1 of proof of Theorem 1.2) but our method is independent.

Proof of the Corollary. By adjunction, we have

$$\mathcal{H}(\mathrm{G},\rho) \cong \mathrm{Hom}_{\mathrm{K}}(\rho, \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho) \cong \oplus_{\lambda\in \mathrm{X}_*^+(\mathrm{T})}\mathrm{Hom}_{\mathrm{K}}(\rho, \mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho).$$

Recall that fixing an element $v \in \rho^{\tilde{\mathrm{I}}}$ yields an isomorphism of $\tilde{\mathfrak{H}}$-modules $\chi \simeq \rho^{\tilde{\mathrm{I}}}, 1 \mapsto 1_{\mathrm{K},v}$. Then, by Proposition 3.13i and Theorem 3.10 we have an isomorphism of vector spaces

$$\begin{aligned}\mathcal{H}(\mathrm{G},\rho) &\cong \oplus_{\lambda\in \mathrm{X}_*^+(\mathrm{T})}\,\mathrm{Hom}_{\mathrm{K}}(\rho, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}})^{\dagger})\\ &\cong \oplus_{\lambda\in \mathrm{X}_*^+(\mathrm{T})}\,\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}})^{\tilde{\mathrm{I}}}) \cong \mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}).\end{aligned}$$

sending $f \in \mathcal{H}(\mathrm{G},\rho)$ onto the element of $\mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}})$ defined by $1 \mapsto f(1_{\mathrm{K},v})$. Composing with the isomorphism of $\tilde{\mathrm{H}}$-modules $\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}} \simeq (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, 1\otimes h \mapsto 1_{\mathrm{K},v}h$ from Lemma 3.6, we get an isomorphism

$$\mathcal{H}(\mathrm{G},\rho) \cong \mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}) \cong \mathrm{Hom}_{\tilde{\mathrm{H}}}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}})$$

which is precisely the morphism of k-algebras (3.3). The second statement of the corollary then comes from the second statement of the proposition using Remark 3.7.

□

Proof of Proposition 3.13. It suffices to show the proposition for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$. We first describe the K_1-invariant subspace of $\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho$ because it contains $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho)^{\dagger}$. Set

$$\mathrm{K}_\lambda := \mathrm{K} \cap \lambda(\varpi)^{-1}\mathrm{K}\lambda(\varpi).$$

As a $k[[\mathrm{K}]]$-module, $\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho$ is isomorphic to the compact induction $\mathrm{ind}_{\mathrm{K}_\lambda}^{\mathrm{K}}\lambda_*(\rho)$ where $\lambda_*(\rho)$ denotes the space V with the group K_λ acting through the homomorphism $\mathrm{K}_\lambda \xrightarrow{\lambda(\varpi)\,.\,\lambda(\varpi)^{-1}} \mathrm{K}$.

Since K_1 is normal in K, we have the representation $(\lambda_*(\rho))^{\mathrm{K}_\lambda\cap \mathrm{K}_1}$ of K_λ on the following subspace of V:

$$\mathrm{V}_\lambda := \mathrm{V}^{\mathrm{K}\cap\lambda(\varpi)\mathrm{K}_1\lambda(\varpi)^{-1}} = \mathrm{V}^{\langle \mathrm{K}\cap\lambda(\varpi)\mathrm{K}_1\lambda(\varpi)^{-1}, \mathrm{K}_1\rangle}.$$

It can be extended to a representation $(\pi_\lambda, \mathrm{V}_\lambda)$ of $\mathcal{P}_\lambda := \mathrm{K}_\lambda\mathrm{K}_1$ that factors through $\mathcal{P}_\lambda/\mathrm{K}_1 \simeq \mathrm{K}_\lambda/\mathrm{K}_\lambda\cap\mathrm{K}_1$.

Denote by $\mathfrak{W}_\lambda$ the stabilizer of λ in $\mathfrak{W}$. Since $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, it is generated by the simple reflections s_α for all $\alpha \in \Phi$ such that $\langle\lambda,\alpha\rangle = 0$. Denote by F_λ the associated standard facet.

The attached subset Φ_{F_λ} of Φ consists in all the roots α such that $\langle\lambda,\alpha\rangle = 0$. The closure of F_λ is the set of points in $x \in \overline{C}$ such that $\alpha(x) = 0$ for all $\alpha \in \Phi_{F_\lambda}$.

Fact 2. *The subgroup $\mathcal{P}_\lambda$ of K is the parahoric subgroup associated to F_λ.*

Fact 3. *As $k[[\mathrm{K}]]$-modules, we have $(\mathrm{ind}_{\mathrm{K}_\lambda}^{\mathrm{K}}\lambda_*(\rho))^{\mathrm{K}_1} = \mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda$.*

Fact 4. *We have a) $\pi_\lambda \in \mathscr{R}(F_\lambda)$, b) $\pi_\lambda^\dagger$ is irreducible, c) $\mathrm{Hom}_{\tilde{\mathfrak{H}}_{F_\lambda}}(\chi, \pi_\lambda^{\tilde{\mathrm{I}}})$ is at most one dimensional.*

We deduce from Fact 3 that $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho)^\dagger = (\mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda)^\dagger$ so that, to prove the proposition, it remains to check that $(\mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda)^\dagger$ is a representation in $\mathscr{B}(x_0)$. By Fact 4 b) and Remark 3.9, there is an injective $\mathcal{P}_\lambda$-equivariant map $\pi_\lambda^\dagger \to \tilde{\mathbf{X}}_{F_\lambda}$ which, by exactness of compact induction, gives an injective K-equivariant map

$$\mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda^\dagger \to \mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\tilde{\mathbf{X}}_{F_\lambda} = \tilde{\mathbf{X}}_{x_0}. \tag{3.10}$$

Since furthermore, by Fact 4 a) and Lemma 3.12, we have $(\mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda)^\dagger = \mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}(\pi_\lambda^\dagger)$, we just proved that the K-representation $(\mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda)^\dagger$ injects in $\tilde{\mathbf{X}}_{x_0}$. By Proposition 3.8ii, the representation $(\mathrm{ind}_{\mathcal{P}_\lambda}^{\mathrm{K}}\pi_\lambda)^\dagger$ is therefore an object in $\mathscr{B}(x_0)$. It is the first statement of the proposition.

In passing, we deduce from (3.10) that there is a right $\tilde{\mathfrak{H}}$-equivariant injection

$$(\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho)^{\tilde{\mathrm{I}}} \longrightarrow \tilde{\mathfrak{H}}$$

so that $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{K}\lambda(\varpi)\mathrm{K}}\rho)^{\tilde{\mathrm{I}}})$ injects in $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi, \tilde{\mathfrak{H}})$ which is one dimensional by Proposition 3.4ii. It gives the second statement of the proposition.

We now prove the Facts. Recall that $\lambda(\varpi) \in \mathrm{T}$ is a lift for $e^{-\lambda} \in \mathrm{W}$ (Remark 2.5) and that for all $\alpha \in \Phi$, we have $\lambda(\varpi)\mathcal{U}_{(\alpha,0)}\lambda(\varpi)^{-1} = \mathcal{U}_{e^{-\lambda}(\alpha,0)} = \mathcal{U}_{(\alpha,\langle\lambda,\alpha\rangle)}$.

Proof of Fact 2: From (3.2) we deduce that the subgroup $\mathcal{U}_C^+$ of $\tilde{\mathrm{I}}$ generated by all the root subgroups $\mathcal{U}_{(\alpha,0)}$ for $\alpha \in \Phi^+$ is contained in K_λ: indeed, let $\alpha \in \Phi^+$, we have $\langle\lambda,\alpha\rangle \geq 0$ and $\lambda(\varpi)\mathcal{U}_{(\alpha,0)}\lambda(\varpi)^{-1} = \mathcal{U}_{(\alpha,\langle\lambda,\alpha\rangle)} \subset \tilde{\mathrm{I}}$, therefore $\mathcal{U}_{(\alpha,0)} \subseteq \mathrm{K} \cap \lambda(\varpi)^{-1}\mathrm{K}\lambda(\varpi)$.
Now recall that $\mathcal{P}_\lambda$ is the subgroup of K generated by K_λ and K_1. The pro-p Iwahori subgroup $\tilde{\mathrm{I}}$ which is generated by K_1 and $\mathcal{U}_C^+$ is contained in $\mathcal{P}_\lambda$, and so is I since $\mathrm{T}^0 \subseteq \mathcal{P}_\lambda$ We have proved that $\mathcal{P}_\lambda$ is the parahoric subgroup corresponding to a standard facet (this statement is in fact enough for the proof of the proposition). It remains to check that it is equal to $\mathbf{G}_{F_\lambda}^\circ(\mathfrak{O})$ which, by Remark 3.1 is the subgroup of K generated by T^0, all $\mathcal{U}_{(\alpha,0)}$ for $\alpha \in \Phi$ such that $\langle\lambda,\alpha\rangle \geq 0$ and all $\mathcal{U}_{(\alpha,1)}$ for $\alpha \in \Phi$ such that $\langle\lambda,\alpha\rangle < 0$. But $\lambda(\varpi)\mathcal{U}_{(\alpha,0)}\lambda(\varpi)^{-1} \subset \mathrm{K}$ if and only if $\langle\lambda,\alpha\rangle \geq 0$. It proves the required equality.

Proof of Fact 3: Since K_1 is normal in K, a K_1-invariant function f in $\mathrm{ind}_{\mathrm{K}_\lambda}^{\mathrm{K}}\lambda_*(\rho)$ is entirely determined by its values at the points of a system of representatives of the cosets $\mathrm{K}_\lambda\backslash\mathrm{K}/\mathrm{K}_1 = \mathcal{P}_\lambda\backslash\mathrm{K}$ and these values can be any elements in V_λ. Therefore, the $\mathcal{P}_\lambda$-equivariant map

$$\pi_\lambda \to (\mathrm{ind}_{\mathrm{K}_\lambda}^{\mathrm{K}}\lambda_*(\rho))^{\mathrm{K}_1}$$

carrying an element $v \in \mathrm{V}_\lambda$ to the unique K_1-invariant funtion $f \in \mathrm{ind}_{\mathrm{K}_\lambda}^{\mathrm{K}}\lambda_*(\rho)$ with value v at 1 induces the expected isomorphism of K-representations.

Proof of Fact 4: a) We want to prove that the pro-unipotent radical $\tilde{\mathrm{I}}_{F_\lambda}$ of $\mathcal{P}_\lambda$ acts trivially on π_λ. By (3.2), it is generated by K_1 and the root subgroups $\mathcal{U}_{(\alpha,0)}$ for $\alpha \in \Phi^+ - \Phi^+_{F_\lambda}$ that is to say for $\alpha \in \Phi^+$ satisfying $\langle \lambda, \alpha\rangle > 0$. Since K_1 acts trivially on π_λ, we only need to check that for $\alpha \in \Phi^+$ with $\langle \lambda, \alpha\rangle > 0$, the action of $\lambda(\varpi)\mathcal{U}_{(\alpha,0)}\lambda(\varpi)^{-1}$ on V_λ via ρ is trivial, but it is clear because this group is contained in K_1.
b) Since any nonzero k-representation of $\mathcal{P}_\lambda$ has a nonzero $\tilde{\mathrm{I}}$-fixed vector, proving that $\pi_\lambda^{\tilde{\mathrm{I}}}$ has dimension 1 is enough to prove that $\pi_\lambda^{\dagger}$ is an irreducible representation of $\mathcal{P}_\lambda$. We have $\{0\} \neq \pi_\lambda^{\tilde{\mathrm{I}}} = \pi_\lambda^{\mathcal{U}_C^+}$ and $\mathcal{U}_C^+ \subset \mathrm{K}_\lambda$ (see the definition of $\mathcal{U}_C^+$ in 3.1.1) so the space of $\pi_\lambda^{\tilde{\mathrm{I}}}$ is $\mathrm{V}^{\langle \mathrm{K}\cap\lambda(\varpi)\mathrm{K}_1\lambda(\varpi)^{-1}, \lambda(\varpi)\,\mathcal{U}_C^+\lambda(\varpi)^{-1}\rangle}$. Let w be the element with maximal length in $\mathfrak{W}_\lambda$. Denote by $\mathcal{U}_C^-$ the subgroup of K generated by all the root subgroups $\mathcal{U}_{(\alpha,0)}$ for $\alpha \in \Phi^-$. We claim that

$$\langle \mathrm{K}\cap\lambda(\varpi)\mathrm{K}_1\lambda(\varpi)^{-1}, \lambda(\varpi)\mathcal{U}_C^+\lambda(\varpi)^{-1}\rangle \supseteq \hat{w}\,\mathcal{U}_C^-\hat{w}^{-1}. \tag{3.11}$$

Indeed, let $\alpha \in \Phi^-$ and recall that $\langle \lambda, \alpha\rangle \leq 0$:
- if $\langle \lambda, \alpha\rangle = 0$ then $\alpha \in \Phi^-_{F_\lambda}$ and $w\alpha \in \Phi^+_{F_\lambda}$ so $\lambda(\varpi)^{-1}\hat{w}\mathcal{U}_{(\alpha,0)}\hat{w}\lambda(\varpi) = \mathcal{U}_{(w\alpha,-\langle\lambda,w\alpha\rangle)} = \mathcal{U}_{(w\alpha,0)}$ is contained in $\mathcal{U}_C^+$;
- if $\langle \lambda, \alpha\rangle < 0$ then $\lambda(\varpi)^{-1}\hat{w}\mathcal{U}_{(\alpha,0)}\hat{w}\lambda(\varpi) = \mathcal{U}_{(w\alpha,-\langle\lambda,w\alpha\rangle)} = \mathcal{U}_{(w\alpha,-\langle\lambda,\alpha\rangle)}$ is contained in K_1.

We deduce from (3.11) that

$$\mathrm{V}^{\langle \mathrm{K}\cap\lambda(\varpi)\mathrm{K}_1\lambda(\varpi)^{-1}, \lambda(\varpi)\,\mathcal{U}_C^+\lambda(\varpi)^{-1}\rangle} \subseteq \mathrm{V}^{\hat{w}\,\mathcal{U}_C^-\hat{w}^{-1}} = \mathrm{V}^{\langle \hat{w}\,\mathcal{U}_C^-\hat{w}^{-1},\,\mathrm{K}_1\rangle}$$

and the last space has dimension 1 because $\langle \hat{w}\,\mathcal{U}_C^-\hat{w}^{-1},\,\mathrm{K}_1\rangle$ is a K-conjugate of $\tilde{\mathrm{I}}$ (it is the pro-unipotent radical of the parahoric subgroup attached to the facet $\hat{w}\hat{w}_0C$ where w_0 denotes the longest element in $\mathfrak{W}$).

□

3.4. **Parameterization of the weights.** Recall that a weight is an irreducible representation of K that is to say a simple object in $\mathscr{R}(x_0)$. By [11, Corollary 7.5] (and also Theorem 3.10), the weights are in one-to-one correspondence with the characters of $\tilde{\mathfrak{H}}$. In 3.1.3, we recalled that a character $\chi : \tilde{\mathfrak{H}} \to k$ is parameterized by the data of a morphism $\bar{\chi} : \mathrm{T}^0/\mathrm{T}^1 \to k^\times$ such that $\bar{\chi}(t) = \chi(\boldsymbol{\tau}_t)$ for all $t \in \mathrm{T}^0/\mathrm{T}^1$, and of the subset Π_χ of $\Pi_{\bar{\chi}}$ such that $\chi(\boldsymbol{\tau}_{n_\alpha}) = -1$ if and only if $\alpha \in \Pi_{\bar{\chi}} - \Pi_\chi$.

To the subset Π_χ is attached a standard facet F_χ (Remark 2.7).

Remark 3.16. By [11, Theorem 7.1], the stabilizer of $\rho^{\tilde{\mathrm{I}}}$ in K is equal to the parahoric subgroup $\mathbf{G}^\circ_{F_\chi}(\mathfrak{O})$ with associated finite Weyl group generated by all s_α, $\alpha \in \Pi_\chi$. We will denote the latter by $\mathfrak{W}_\chi$.

4. Bernstein-type map attached to a weight and Satake isomorphism

4.1. **Commutative subrings attached to a standard facet.** We fix for a standard facet F.

4.1.1. Consider the subset of all $\lambda \in \mathrm{X}_*(\mathrm{T})$ such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in (\Phi^+ - \Phi_F^+) \cup \Phi_F^-$. If w_F denotes the element with maximal length in $\mathfrak{W}_F$, then this set is the w_F-conjugate of $\mathrm{X}_*^+(\mathrm{T})$. Bearing in mind the conventions introduced in 2.2.2, we introduce

$$\mathscr{C}^+(F) := \{\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T}) \text{ such that } \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in (\Phi^+ - \Phi_F^+) \cup \Phi_F^-\}$$
$$\mathscr{C}^-(F) := \{\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T}) \text{ such that } \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in (\Phi^- - \Phi_F^-) \cup \Phi_F^+\}.$$

Remark 4.1. For all λ, $\lambda' \in \mathscr{C}^+(F)$ (resp. $\mathscr{C}^-(F)$) we have $\ell(e^\lambda) + \ell(e^{\lambda'}) = \ell(e^{\lambda+\lambda'})$.

4.1.2. *Bernstein-type maps attached to a standard facet.*

Proposition 4.2. i. *There is a unique morphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras*

$$\Theta_F^+ : \mathbb{Z}[q^{\pm 1/2}][\tilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \tag{4.1}$$

such that $\Theta_F^+(\lambda) = q^{-\ell(e^\lambda)/2} \boldsymbol{\tau}_{e^\lambda}$ if $\lambda \in \mathscr{C}^+(F)$.

ii. *There is a unique morphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras*

$$\Theta_F^- : \mathbb{Z}[q^{\pm 1/2}][\tilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \tag{4.2}$$

such that $\Theta_F^-(\lambda) = q^{-\ell(e^\lambda)/2} \boldsymbol{\tau}_{e^\lambda}$ if $\lambda \in \mathscr{C}^-(F)$.

iii. *Both Θ_F^+ and Θ_F^- are injective.*

Proof of the proposition. It is the same proof as in the classical case for Iwahori-Hecke algebras and the dominant Weyl chamber. Let σ be a sign. By Remark 4.1, the formula $\Theta_F^\sigma(\lambda) = q^{-\ell(e^\lambda)/2} \boldsymbol{\tau}_{e^\lambda}$ for $\lambda \in \mathscr{C}^\sigma(F)$ defines a multiplicative map $\mathbb{Z}[q^{\pm 1/2}][\mathscr{C}^\sigma(F)] \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$. Let $\nu \in \mathscr{C}^\sigma(F)$ such that $\lambda + \nu \in \mathscr{C}^\sigma(F)$(if $\sigma = +$, choose for example ν to be the w_F-conjugate of a suitable strongly dominant coweight). We set $\Theta_F^\sigma(\lambda) := q^{(-\ell(e^{\lambda+\nu}) + \ell(e^\nu))/2} \tau_{\lambda+\nu} \tau_\nu^{-1}$. This formula does not depend on the choice on ν such that $\lambda + \nu \in \mathscr{C}^\sigma(F)$ as can be seen by applying again Remark 4.1.

To check the injectivity, use for example Lemma 2.11 that states that for all $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$, the element $\Theta_F^\sigma(\lambda)$ is equal to the sum $q^{-\ell(\lambda)/2} \boldsymbol{\tau}_{e^\lambda}$ and of a $\mathbb{Z}[q^{\pm 1/2}]$-linear combination of elements $\boldsymbol{\tau}_v \in \tilde{\mathrm{H}}_{\mathbb{Z}}$ such that $v \in \tilde{\mathrm{W}}$ and $\ell(v) < \ell(e^\lambda)$.

□

There is, more generally, for any $w \in \mathfrak{W}$, a unique morphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras

$$\Theta_w : \mathbb{Z}[q^{\pm 1/2}][\tilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \tag{4.3}$$

such that $\Theta_w(\lambda) = q^{-\ell(e^\lambda)/2} \boldsymbol{\tau}_{e^\lambda}$ if $\lambda \in w(\tilde{\mathrm{X}}_*^+(\mathrm{T}))$. Then for any standard facet F, our maps Θ_F^+ and Θ_F^- coincide respectively with Θ_{w_F} and $\Theta_{w_F w_0}$ where we recall that w_0 denotes the longest element in $\mathfrak{W}$. This remark was pointed out to me by the referee.

4.1.3. *Commutative subalgebras of* $\tilde{\mathrm{H}}_{\mathbb{Z}}$. For all $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$, we set

$$\mathcal{B}_F^+(\lambda) = q^{\ell(e^\lambda)/2}\Theta_F^+(\lambda) \text{ and } \mathcal{B}_F^-(\lambda) = q^{\ell(e^\lambda)/2}\Theta_F^-(\lambda) \tag{4.4}$$

and we remark in particular that by Lemma 2.11, all these elements lie in $\tilde{\mathrm{H}}_{\mathbb{Z}}$.

Remark 4.3. i. The maps $\mathcal{B}_F^+$ and $\mathcal{B}_F^-$ do not respect the product in general, but they are mulitplicative within Weyl chambers (see Remark 4.1).
ii. Consider the case $F = C$ or $F = x_0$. Then $\mathcal{B}_C^+ = \mathcal{B}_{x_0}^-$ (resp. $\mathcal{B}_C^- = \mathcal{B}_{x_0}^+$) coincides with the integral Bernstein map E^+ (resp. E) introduced in [36].

Lemma 4.4. *For* $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$ *we have* $\iota(\mathcal{B}_F^+(\lambda)) = (-1)^{\ell(e^\lambda)}\mathcal{B}_F^-(\lambda)$ *where* ι *is the involution defined by* (2.9).

Proof. Let $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$ and $\mu, \nu \in \mathscr{C}^+(F)$ such that $\lambda = \mu - \nu$. Then $\mathcal{B}_F^+(\lambda) = q^{(\ell(e^\lambda)+\ell(e^\nu)-\ell(e^\mu))/2}\tau_{e^\mu}\tau_{e^\nu}^{-1}$ and $\mathcal{B}_F^-(\lambda) = q^{(\ell(e^\lambda)+\ell(e^\mu)-\ell(e^\nu))/2}\tau_{e^{-\nu}}\tau_{e^{-\mu}}^{-1}$. Furthermore, $\ell(e^\mu) - \ell(e^\nu)$ and $\ell(e^\lambda)$ have the same parity and $\boldsymbol{\tau}_{e^{-\nu}}$ and $\boldsymbol{\tau}_{e^{-\mu}}^{-1}$ commute by Remark 4.1. $\square$

Using Lemma 2.11 we get the following:

Proposition 4.5. *Let* F *be a standard facet. The commutative ring*

$$\mathscr{A}_F^+ := \tilde{\mathrm{H}}_{\mathbb{Z}} \cap \mathrm{Im}(\Theta_F^+), \text{ and respectively } \mathscr{A}_F^- := \tilde{\mathrm{H}}_{\mathbb{Z}} \cap \mathrm{Im}(\Theta_F^-),$$

has $\mathbb{Z}$*-basis the set of all* $\mathcal{B}_F^+(\lambda)$*, respectively* $\mathcal{B}_F^-(\lambda)$*, for* $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$.

Proposition 4.6. *Let* $\lambda \in \tilde{\mathrm{X}}_*^+(\mathrm{T})$.

- *For any* $t \in \mathrm{T}^0/\mathrm{T}^1$*, the basis element* $\boldsymbol{\tau}_t \in \tilde{\mathrm{H}}_{\mathbb{Z}}$ *and* $\mathcal{B}_F^+(\lambda)$ *commute, as well as* $\boldsymbol{\tau}_t$ *and* $\mathcal{B}_F^-(\lambda)$.
- *Let* $\alpha \in \Pi$. *If* $\alpha \in \Pi_F$*, then*

$$a)\begin{cases} \mathcal{B}_F^-(\lambda)\boldsymbol{\tau}_{n_\alpha}^* \in q\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle > 0 \\ \mathcal{B}_F^-(\lambda)\boldsymbol{\tau}_{n_\alpha}^* \in \boldsymbol{\tau}_{n_\alpha}^*\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle = 0 \end{cases} \text{ and } a')\begin{cases} \mathcal{B}_F^+(\lambda)\boldsymbol{\tau}_{n_\alpha} \in q\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle > 0 \\ \mathcal{B}_F^+(\lambda)\boldsymbol{\tau}_{n_\alpha} \in \boldsymbol{\tau}_{n_\alpha}\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle = 0 \end{cases}$$

- *If* $\alpha \in \Pi - \Pi_F$*, then*

$$b)\begin{cases} \mathcal{B}_F^-(\lambda)\boldsymbol{\tau}_{n_\alpha} \in q\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle > 0 \\ \mathcal{B}_F^-(\lambda)\boldsymbol{\tau}_{n_\alpha} \in \boldsymbol{\tau}_{n_\alpha}\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle = 0 \end{cases} \text{ and } b')\begin{cases} \mathcal{B}_F^+(\lambda)\boldsymbol{\tau}_{n_\alpha}^* \in q\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle > 0 \\ \mathcal{B}_F^+(\lambda)\boldsymbol{\tau}_{n_\alpha}^* \in \boldsymbol{\tau}_{n_\alpha}^*\tilde{\mathrm{H}}_{\mathbb{Z}} & \text{if } \langle\lambda,\alpha\rangle = 0 \end{cases}$$

Proof. Let $\nu \in \tilde{\mathrm{X}}(\mathrm{T})$ be an element whose image in $\mathrm{X}(\mathrm{T})$ is the opposite of a strongly F-positive element (2.3.2) and such that $\lambda + \nu \in \mathscr{C}^-(F)$. Remark that $\nu \in \mathscr{C}^-(F)$ and e^ν is an element in $\tilde{\mathrm{W}}$ that commutes with all n_α, $\alpha \in \Pi_F$. We have

$$\mathcal{B}_F^-(\lambda) = q^{\frac{\ell(e^\lambda)}{2}}\Theta_F^-(\lambda+\nu)\Theta_F^-(-\nu) = q^{\frac{\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}}\boldsymbol{\tau}_{e^\nu}^{-1}.$$

a) Let $\alpha \in \Pi_F$. Recall that $\boldsymbol{\tau}^*_{n_\alpha} = q\boldsymbol{\tau}^{-1}_{n_\alpha}$. Since s_α and e^ν commute in W, we have $\ell(s_\alpha e^\nu) = \ell(e^\nu) + 1$ and $\boldsymbol{\tau}_{n_\alpha}$ and $\boldsymbol{\tau}_{e^\nu}$ commute:

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}^*_{n_\alpha} = q^{\frac{2+\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}}\boldsymbol{\tau}^{-1}_{n_\alpha}\boldsymbol{\tau}^{-1}_{e^\nu}$$

– First suppose that $\langle \lambda, \alpha\rangle > 0$. Then $\ell(e^\lambda s_\alpha) = \ell(e^\lambda) - 1$ and $\ell(e^{\lambda+\nu} s_\alpha) = \ell(e^{\lambda+\nu}) - 1$. Therefore, $\boldsymbol{\tau}_{e^{\lambda+\nu}} = \boldsymbol{\tau}_{e^{\lambda+\nu}n_\alpha^{-1}}\boldsymbol{\tau}_{n_\alpha}$ and

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}^*_{n_\alpha} = q^{\frac{2+\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}n_\alpha^{-1}}\boldsymbol{\tau}^{-1}_{e^\nu} = q^{\frac{2+\ell(e^\lambda n_\alpha^{-1})+\ell(e^\nu)-\ell(e^{\lambda+\nu}n_\alpha^{-1})}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}n_\alpha^{-1}}\boldsymbol{\tau}^{-1}_{e^\nu}$$

which is an element of $q\tilde{\mathrm{H}}_{\mathbb{Z}}$ by Lemma 2.11.

– Now suppose that $\langle \lambda, \alpha\rangle = 0$ so that e^λ, e^ν and $\boldsymbol{\tau}_{n_\alpha}$ commute. We have $\ell(s_\alpha e^{\lambda+\nu}) = \ell(e^{\lambda+\nu}) + 1$ so $\boldsymbol{\tau}_{e^{\lambda+\nu}}\boldsymbol{\tau}^{-1}_{n_\alpha} = \boldsymbol{\tau}^{-1}_{n_\alpha}\tau_{e^{\lambda+\nu}}$

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}^*_{n_\alpha} = q^{\frac{2+\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}^{-1}_{n_\alpha}\boldsymbol{\tau}_{e^{\lambda+\nu}}\boldsymbol{\tau}^{-1}_{e^\nu} = \boldsymbol{\tau}^*_{n_\alpha}\mathcal{B}^-_F(\lambda).$$

b) Let $\alpha \in \Pi - \Pi_F$. We have $\langle \nu, \alpha\rangle < 0$ so that $\ell(e^\nu s_\alpha) = \ell(e^\nu) + 1$ and $\boldsymbol{\tau}^{-1}_{e^\nu}\boldsymbol{\tau}_{n_\alpha} = \boldsymbol{\tau}_{n_\alpha}\boldsymbol{\tau}^{-1}_{e^{s_\alpha\nu}}$:

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}_{n_\alpha} = q^{\frac{\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}}\boldsymbol{\tau}_{n_\alpha}\boldsymbol{\tau}^{-1}_{e^{s_\alpha\nu}}.$$

Since $\langle \nu+\lambda, \alpha\rangle \leq 0$ we have $\ell(e^{\nu+\lambda}s_\alpha) = \ell(e^{\nu+\lambda}) + 1$ and

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}_{n_\alpha} = q^{\frac{\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}n_\alpha}\boldsymbol{\tau}^{-1}_{e^{s_\alpha\nu}}.$$

– First suppose that $\langle \lambda, \alpha\rangle > 0$. Then $\ell(e^\lambda s_\alpha) = \ell(e^\lambda) - 1$

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}_{n_\alpha} = q^{\frac{2+\ell(e^\lambda n_\alpha)+\ell(e^{s_\alpha\nu})-\ell(e^{\lambda+\nu}n_\alpha)}{2}}\boldsymbol{\tau}_{e^{\lambda+\nu}n_\alpha}\boldsymbol{\tau}^{-1}_{e^{s_\alpha\nu}}$$

which is an element of $q\tilde{\mathrm{H}}_{\mathbb{Z}}$ by Lemma 2.11.

– Now suppose that $\langle \lambda, \alpha\rangle = 0$ that it to say that e^λ and s_α commute. We have $\boldsymbol{\tau}_{e^{\lambda+\nu}}\boldsymbol{\tau}_{n_\alpha} = \boldsymbol{\tau}_{n_\alpha}\boldsymbol{\tau}_{e^{\lambda+s_\alpha\nu}}$ and

$$\mathcal{B}^-_F(\lambda)\boldsymbol{\tau}_{n_\alpha} = \boldsymbol{\tau}_{n_\alpha}q^{\frac{\ell(e^\lambda)+\ell(e^\nu)-\ell(e^{\lambda+\nu})}{2}}\boldsymbol{\tau}_{e^{\lambda+s_\alpha\nu}}\boldsymbol{\tau}^{-1}_{e^{s_\alpha\nu}}.$$

which is an element of $\boldsymbol{\tau}_{n_\alpha}\tilde{\mathrm{H}}_{\mathbb{Z}}$ by Lemma 2.11.

Statements a') and b') follow applying Lemma 4.4 since $\iota(\boldsymbol{\tau}_{n_\alpha}) = -\boldsymbol{\tau}^*_{n_\alpha}\boldsymbol{\tau}_{h_\alpha(-1)}$ and $\boldsymbol{\tau}_{h_\alpha(-1)}$ is invertible in $\tilde{\mathrm{H}}_{\mathbb{Z}}$. $\square$

4.1.4. Let M_F be the Levi subgroup of G corresponding to the facet F as in 2.3.3. We also refer to the notations introduced in 2.4.4.

Lemma 4.7. *For $\lambda \in \tilde{\mathrm{X}}_*^+(\mathrm{T})$, the element $(-1)^{\ell_F(e^\lambda)}\iota^F(\boldsymbol{\tau}^F_{e^\lambda}) \in \tilde{\mathrm{H}}_F(\mathrm{M}_F)$ is equal to the sum of $\boldsymbol{\tau}^F_{e^\lambda}$ and a linear combination with coefficients in $\mathbb{Z}$ of $\boldsymbol{\tau}^F_{\tilde{v}}$ for F-positive elements $v \in \mathrm{W}_F$ such that $v \underset{F}{<} e^\lambda$. Furthermore, we have*

$$j_F^+((-1)^{\ell_F(e^\lambda)}\iota^F(\boldsymbol{\tau}^F_{e^\lambda})) = \mathcal{B}^+_F(\lambda). \tag{4.5}$$

In particular for $F = x_0$,

$$(-1)^{\ell(e^\lambda)}\iota(\boldsymbol{\tau}_{e^\lambda}) = \mathcal{B}^+_{x_0}(\lambda). \tag{4.6}$$

Proof. In $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$, we have $(-1)^{\ell_F(e^\lambda)}\iota^F(\boldsymbol{\tau}^F_{e^\lambda}) = q^{\ell_F(e^\lambda)}(\boldsymbol{\tau}^F_{e^{-\lambda}})^{-1}$. Lemma 2.11 for the Hecke algebra of M_F then gives the first statement. Use Lemma 2.9 for the result about F-positivity.

By an argument similar to the one in the proof of Proposition 4.2 (in the setting of the root system corresponding to M_F), the element

$$\theta_F(\lambda) := q^{(\ell_F(\nu)-\ell_F(\mu))/2}\boldsymbol{\tau}^F_{e^\mu}(\boldsymbol{\tau}^F_{e^\nu})^{-1} \in \tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$$

does not depend on the choice of $\lambda, \nu \in \mathrm{X}_*(\mathrm{T})$ such that $\lambda = \mu - \nu$ and $\langle \mu, \alpha \rangle \leq 0$, $\langle \nu, \alpha \rangle \leq 0$ for all $\alpha \in \Phi^+_F$.
Choose $\mu, \nu \in \mathscr{C}^+(F)$ such that $\lambda = \mu - \nu$. Then $j_F(q^{\ell_F(\lambda)/2}\theta_F(\lambda)) = q^{(\ell_F(\lambda)+\ell_F(\nu)-\ell_F(\mu))/2}\boldsymbol{\tau}_{e^\mu}(\boldsymbol{\tau}_{e^\nu})^{-1}$ because μ and ν are in particular F-positive. By (2.6), we therefore have $\mathcal{B}^+_F(\lambda) = j_F(q^{\ell_F(\lambda)/2}\theta_F(\lambda))$. Now choose $\mu = 0$ and $\nu = -\lambda$. We have $q^{\ell_F(\lambda)/2}\theta_F(\lambda) = (-1)^{\ell_F(e^\lambda)}\iota^F(\boldsymbol{\tau}^F_{e^\lambda})$ and therefore $j_F(q^{\ell_F(\lambda)/2}\theta_F(\lambda)) = j_F^+((-1)^{\ell_F(e^\lambda)}\iota^F(\boldsymbol{\tau}^F_{e^\lambda}))$.

□

4.2. **Satake isomorphism.** Let χ be a character of $\tilde{\mathfrak{H}}$ with values in k and F_χ the associated standard facet as in 3.4.

Lemma 4.8. *We have a morphism of k-algebras*

$$\begin{array}{ccc} \bar{\chi} \otimes_{k[\mathrm{T}^0/\mathrm{T}^1]} k[\tilde{\mathrm{X}}_*^+(\mathrm{T})] & \longrightarrow & \mathrm{Hom}_{\tilde{\mathrm{H}}}(\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}, \chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}) \\ 1 \otimes \lambda & \longmapsto & (1 \otimes 1 \mapsto 1 \otimes \mathcal{B}^+_{F_\chi}(\lambda)) \end{array} \tag{4.7}$$

Proof. We have to check that, for $\lambda \in \tilde{\mathrm{X}}_*^+(\mathrm{T})$, the element $1 \otimes \mathcal{B}^+_{F_\chi}(\lambda)$ is an eigenvector for the right action of $\tilde{\mathfrak{H}}$ and the character χ. Recall that the finite Hecke algebra $\tilde{\mathfrak{H}}$ is generated by all $\boldsymbol{\tau}_t$, $t \in \mathrm{T}^0/\mathrm{T}^1$ and $\boldsymbol{\tau}_{n_\alpha}$ for $\alpha \in \Pi$.

- First note that for $t \in \mathrm{T}^0/\mathrm{T}^1$, we have $\mathcal{B}^+_F(t + \lambda) = \boldsymbol{\tau}_t\mathcal{B}^+_F(\lambda)$. Therefore $\boldsymbol{\tau}_t$ acts on $1 \otimes \mathcal{B}^+_{F_\chi}(\lambda)$ by multiplication by $\chi(\boldsymbol{\tau}_t)$ and $\epsilon_{\bar{\chi}}$ acts by 1.
- Let $\alpha \notin \Pi_{\bar{\chi}}$. We have $\chi(\boldsymbol{\tau}_{n_\alpha}) = 0$. By the quadratic relations (2.10), we have $\epsilon_{\bar{\chi}}\boldsymbol{\tau}^*_{n_\alpha} = \bar{\chi}(h_\alpha(-1))\,\epsilon_{\bar{\chi}}\boldsymbol{\tau}_{n_\alpha}$ in $\tilde{\mathrm{H}}$. Now, since $\Pi_\chi \subseteq \Pi_{\bar{\chi}}$, Proposition 4.6 b') implies that $\boldsymbol{\tau}_{n_\alpha}$ acts by 0 on $1 \otimes \mathcal{B}^+_{F_\chi}(\lambda)$.

- Let $\alpha \in \Pi_{\bar{\chi}} - \Pi_{\chi}$. We have $\chi(\boldsymbol{\tau}_{n_\alpha}) = -1$ and by the quadratic relations (2.10), $\epsilon_{\bar{\chi}}\boldsymbol{\tau}^*_{n_\alpha} = \epsilon_{\bar{\chi}}(\boldsymbol{\tau}_{n_\alpha}+1)$ in $\tilde{\mathrm{H}}$, which by proposition 4.6 b'), acts by 0 on $1\otimes\mathcal{B}^+_{F_\chi}(\lambda)$.
- Let $\alpha \in \Pi_\chi$. We have $\chi(\boldsymbol{\tau}_{n_\alpha}) = 0$ and by proposition 4.6 a'), $\boldsymbol{\tau}_{n_\alpha}$ acts by 0 on $1\otimes\mathcal{B}^+_{F_\chi}(\lambda)$.

We have proved that (4.7) is a well defined map. It is a morphism of k-algebras by Remark 4.3i. □

Proposition 4.9. *The map (4.7) is an isomorphism of k-algebras.*

Proof. The proof relies on the following observation: a basis for $\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}$ is given by all $1\otimes\boldsymbol{\tau}_{\tilde{d}}$ for $d\in\mathcal{D}$ (Proposition 3.4). Recall that $\mathcal{D}$ contains the set of all e^μ for $\mu\in\mathrm{X}^+_*(\mathrm{T})$. By Lemma 2.11 (and using the braid relations (2.7) together with (2.3)), $1\otimes\mathcal{B}^+_{F_\chi}(\tilde{\mu})$ is a sum of $1\otimes\boldsymbol{\tau}_{\widetilde{e^\mu}}$ and of elements in $\oplus_{d<e^\mu}k\otimes\boldsymbol{\tau}_{\tilde{d}}$.

We first deduce from this the injectivity of (4.7) because a basis for $\bar{\chi}\otimes_{k[\mathrm{T}^0/\mathrm{T}^1]}k[\tilde{\mathrm{X}}^+_*(\mathrm{T})]$ is given by the set of all $1\otimes\widetilde{e^\mu}$ for $\mu\in\mathrm{X}^+_*(\mathrm{T})$.

Now we prove the surjectivity. Denote, for $\mu\in\mathrm{X}^+_*(\mathrm{T})$, by $\tilde{\mathrm{H}}[\mu]$ the subspace of the functions in $\tilde{\mathrm{H}}$ with support in $\mathrm{K}\widehat{e^\mu}\mathrm{K}$. Then $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi,\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}})$ decomposes into the direct sum of all subspaces $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi,\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}[\mu])$ for $\mu\in\mathrm{X}^+_*(\mathrm{T})$ and after Corollary 3.14 and its proof, each of the spaces $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi,\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}[\mu])$ is at most one dimensional.

Let $\mu\in\mathrm{X}^+_*(\mathrm{T})$. By Lemma 2.4ii and the observation at the beginning of this proof, the image of $1\otimes\mathcal{B}_{F_\chi}(\tilde{\mu})$ by (4.7) decomposes in the direct sum of all $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi,\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}[\lambda])$ for $e^\lambda\leq e^\mu$ and it has a non zero component in $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi,\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}[\mu])$. We conclude by induction on $\ell(e^\mu)$ that $\mathrm{Hom}_{\tilde{\mathfrak{H}}}(\chi,\chi\otimes_{\tilde{\mathfrak{H}}}\tilde{\mathrm{H}}[\mu])$ is contained in the image of (4.7) for all $\mu\in\mathrm{X}^+_*(\mathrm{T})$. □

Remark 4.10. Recall that given $\lambda\in\mathrm{X}^+_*(\mathrm{T})$, a lift for $e^\lambda\in\mathrm{W}$ is given by $\lambda(\varpi^{-1})\in\mathrm{T}$ (see 2.2.4). More precisely, the map $\lambda\mapsto\lambda(\varpi^{-1})\bmod\mathrm{T}^1$ is a splitting for the exact sequence of abelian groups

$$0\longrightarrow\mathrm{T}^0/\mathrm{T}^1\longrightarrow\tilde{\mathrm{X}}_*(\mathrm{T})\longrightarrow\mathrm{X}_*(\mathrm{T})\longrightarrow 0$$

and it respects the actions of $\mathfrak{W}$.

By abuse of notation, we identify below the element $\lambda(\varpi^{-1})\in N_{\mathrm{G}}(\mathrm{T})$ with image in $\tilde{\mathrm{X}}^+_*(\mathrm{T})\subset\tilde{\mathrm{W}}$.

Let (ρ,V) be the weight corresponding to the character χ of $\tilde{\mathfrak{H}}$. As in 3.1.4, we fix a basis v for $\rho^{\tilde{\mathrm{I}}}$. Composing (4.7) with the inverse of (3.3) gives the following.

Theorem 4.11. *We have an isomorphism*

$$\bar{\chi}\otimes_{k[\mathrm{T}^0/\mathrm{T}^1]}k[\tilde{\mathrm{X}}^+_*(\mathrm{T})]\xrightarrow{\sim}\mathcal{H}(\mathrm{G},\rho)\tag{4.8}$$

carrying, for $\lambda\in\mathrm{X}^+_*(\mathrm{T})$, the element $1\otimes\lambda(\varpi^{-1})$ onto the G*-equivariant map determined by*

$$\begin{array}{rccc}\mathcal{T}_\lambda: & \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho & \longrightarrow & \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho\\ & \mathbf{1}_{\mathrm{K},v} & \longmapsto & \mathbf{1}_{\mathrm{K},v}\mathcal{B}_{F_\chi}(\lambda(\varpi^{-1})).\end{array}\tag{4.9}$$

The map $\lambda \in \mathrm{X}_*^+(\mathrm{T}) \to \lambda(\varpi^{-1}) \bmod \mathrm{T}^1$ yields an isomorphism

$$k[\mathrm{X}_*^+(\mathrm{T})] \simeq \bar{\chi} \otimes_{k[\mathrm{T}^0/\mathrm{T}^1]} k[\tilde{\mathrm{X}}_*^+(\mathrm{T})]$$

which we compose with (4.8) to obtain the isomorphism of k-algebras

$$\begin{array}{rccc} \mathcal{T}: & k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{\sim} & \mathcal{H}(\mathrm{G},\rho) \\ & \lambda & \longmapsto & \mathcal{T}_\lambda \end{array} \tag{4.10}$$

The next section is devoted to proving that, in the case where the derived subgroup of **G** is simply connected, this map is an inverse to the Satake isomorphism constructed in [19].

5. Explicit computation of the mod p modified Bernstein maps

5.1. **Support of the modified Bernstein functions.**

5.1.1. *Preliminary lemmas.*

Lemma 5.1. *Let* $\mathbf{1} : \mathrm{T}^0/\mathrm{T}^1 \to k^\times$ *be the trivial character of* $\mathrm{T}^0/\mathrm{T}^1$ *and* $\epsilon_{\mathbf{1}} \in \tilde{\mathrm{H}}$ *the corresponding idempotent. For any* $w \in \mathrm{W}$*, we have in* $\tilde{\mathrm{H}}$ *the following equality:*

$$(-1)^{\ell(w)}\iota(\epsilon_{\mathbf{1}}\boldsymbol{\tau}_{\tilde{w}}) = \sum_{v\in \mathrm{W}, v\leq w} \epsilon_{\mathbf{1}}\boldsymbol{\tau}_{\tilde{v}}. \tag{5.1}$$

Proof. We consider in this proof the field k as the residue field of an algebraic closure $\overline{\mathbb{Q}}_p$ of the field of p-adic numbers $\mathbb{Q}_p$. Let $\overline{\mathbb{Z}}_p$ be the ring of integers of $\overline{\mathbb{Q}}_p$ and $r : \overline{\mathbb{Z}}_p \to k$ the reduction. The ring $\overline{\mathbb{Z}}_p$ satisfies the hypotheses of 2.4.3. In this proof we identify q with its image $q.1_{\overline{\mathbb{Z}}_p}$ in $\overline{\mathbb{Z}}_p$. We work in the Hecke algebra $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}_p$ in which we prove that

$$(-1)^{\ell(w)}\iota(\epsilon_{\mathbf{1}}\boldsymbol{\tau}_{\tilde{w}}) \ \in \sum_{v\in \mathrm{W}, v\leq w} (1-q)^{\ell(w)-\ell(v)}\epsilon_{\mathbf{1}}\boldsymbol{\tau}_{\tilde{v}} + q\,(\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}_p). \tag{5.2}$$

It is enough to consider the case $w \in \mathrm{W}_{aff}$ and we proceed by induction on $\ell(w)$. Let $w \in \mathrm{W}_{aff}$ and $s \in \mathrm{S}_{aff}$ such that with $\ell(sw) = \ell(w)+1$. Applying [15, Lemma 4.3], we see that the set of the $v \in \mathrm{W}$ such that $v \leq sw$ is the disjoint union of

$$\{v \in \mathrm{W}, v \leq sv, w\} \text{ and } \{v \in \mathrm{W}, sv \leq v, w\}.$$

Noticing that $\epsilon_1\iota(\boldsymbol{\tau}_{\tilde{s}}) = -\epsilon_{\mathbf{1}}(\boldsymbol{\tau}_{\tilde{s}} + 1 - q)$, we have, by induction,

$$(-1)^{\ell(\widetilde{sw})}\epsilon_{\mathbf{1}}\iota(\boldsymbol{\tau}_{\widetilde{sw}}) = (-1)^{\ell(w)}\epsilon_{\mathbf{1}}(\boldsymbol{\tau}_{\tilde{s}}+1-q)\iota(\boldsymbol{\tau}_{\tilde{w}}) \in \epsilon_{\mathbf{1}}(\boldsymbol{\tau}_{\tilde{s}}+1-q) \sum_{v\in \mathrm{W}, v\leq w} (1-q)^{\ell(w)-\ell(v)}\boldsymbol{\tau}_{\tilde{v}} + q(\tilde{\mathrm{H}}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\overline{\mathbb{Z}}_p)$$

and $(\boldsymbol{\tau}_{\tilde{s}} + 1 - q) \sum_{v\in \tilde{\mathrm{W}}, v\leq w} (1-q)^{\ell(w)-\ell(v)}\epsilon_{\mathbf{1}}\boldsymbol{\tau}_{\tilde{v}}$ is successively equal to

$$\begin{aligned}
&\epsilon_{\mathbf{1}}(\boldsymbol{\tau}_{\tilde{s}}+1-q)(\sum_{v\leq sv,w}(1-q)^{\ell(w)-\ell(v)}\boldsymbol{\tau}_{\tilde{v}}+\sum_{sv\leq v\leq w}(1-q)^{\ell(w)-\ell(v)}\boldsymbol{\tau}_{\tilde{v}})\\
&=\sum_{v\leq sv,w}(1-q)^{\ell(w)-\ell(v)}\boldsymbol{\tau}_{\tilde{s}\tilde{v}}+\sum_{v\leq sv,w}(1-q)^{\ell(w)-\ell(v)+1}\boldsymbol{\tau}_{\tilde{v}}+\sum_{sv\leq v\leq w}q(1-q)^{\ell(w)-\ell(v)}\boldsymbol{\tau}_{\tilde{s}\tilde{v}}\\
&\in\sum_{sv\leq v,w}(1-q)^{\ell(sw)-\ell(v)}\boldsymbol{\tau}_{\tilde{v}}+\sum_{v\leq sv,w}(1-q)^{\ell(sw)-\ell(v)}\boldsymbol{\tau}_{\tilde{v}}+\sum_{sv\leq v\leq w}q(1-q)^{\ell(w)-\ell(v)}\boldsymbol{\tau}_{\tilde{s}\tilde{v}}
\end{aligned}$$

which proves the claim. Applying the reduction $r:\overline{\mathbb{Z}}_p\to k$, we get (5.1) in $\tilde{\mathrm{H}}$.

□

Lemma 5.2. *Suppose that the derived subgroup of* $\mathbf{G}$ *is simply connected. Let* $\xi:\mathrm{T}^0/\mathrm{T}^1\to k^\times$ *be a character that is trivial on* T_α *for all* $\alpha\in\Pi$ *(see definition in 2.2.5). Then there exists a character* $\boldsymbol{\alpha}:\mathrm{G}\to k^\times$ *that coincides with* ξ^{-1} *on* T^0*, such that* $\boldsymbol{\alpha}(\mu(\varpi))=1$ *for all* $\mu\in\mathrm{X}_*(\mathrm{T})$*. It satisfies the following equality in* $\tilde{\mathrm{H}}$*, for* $\lambda\in\mathrm{X}_*^+(\mathrm{T})$*:*

$$\epsilon_\xi\mathcal{B}_{x_0}^+(\lambda(\varpi^{-1}))=\epsilon_\xi(-1)^{\ell(e^\lambda)}\iota(\boldsymbol{\tau}_{\lambda(\varpi^{-1})})=\sum_{v\in\mathrm{W},v\leq e^\lambda}\epsilon_\xi\boldsymbol{\alpha}(\tilde{v})\boldsymbol{\tau}_{\tilde{v}}.$$

Remark 5.3. For $v\in\mathrm{W}$ with chosen lift $\tilde{v}\in\tilde{\mathrm{W}}$, the value of $\boldsymbol{\alpha}(\hat{\tilde{v}})$ does not depend on the choice of $\hat{\tilde{v}}$ lifting $\tilde{v}$ and we denote it by $\boldsymbol{\alpha}(\tilde{v})$ above. Furthermore, $\epsilon_\xi\boldsymbol{\alpha}(\tilde{v})\boldsymbol{\tau}_{\tilde{v}}$ does not depend on the choice of the lift $\tilde{v}$.

Proof. Define a character $\chi:\tilde{\mathfrak{H}}\to k^\times$ by $\bar{\chi}:=\xi$ and $\Pi_\chi=\Pi_{\bar{\chi}}=\Pi$ and consider the associated weight as in 3.4. By the proof of [20, Proposition 5.1] (see also the remark following Definition 2.4 *loc.cit* and Remark 3.16), this weight is a character $\mathrm{K}\to k^\times$ and by [1, Corollary 3.4], it extends uniquely to a character $\boldsymbol{\alpha}$ of G satisfying $\boldsymbol{\alpha}(\mu(\varpi))=1$ for all $\mu\in\mathrm{X}_*(\mathrm{T})$. Note that $\boldsymbol{\alpha}$ coincides with $\bar{\chi}^{-1}$ on $\mathrm{T}^0/\mathrm{T}^1$.

Fact. *The* k*-linear map* $\Psi:\tilde{\mathrm{H}}\to\tilde{\mathrm{H}}$, $\boldsymbol{\tau}_g\mapsto\boldsymbol{\alpha}(g)\,\boldsymbol{\tau}_g$ *is an isomorphism of* k*-algebras preserving the support of the functions. It sends* $\epsilon_{\mathbf{1}}$ *onto* ϵ_ξ *and commutes with the involution* ι.

Proof of the fact. The image of $\boldsymbol{\tau}_g=\mathbf{1}_{\tilde{\mathrm{I}}g\tilde{\mathrm{I}}}$ is independent of the choice of a representative in $\tilde{\mathrm{I}}g\tilde{\mathrm{I}}$. The image of $\epsilon_{\mathbf{1}}$ is clearly ϵ_ξ. One easily checks that Ψ respects the product. Now in order to check that Ψ commutes with the involution ι, it is enough to show that Ψ and $\iota\Psi\iota$ coincide on elements of the form $\boldsymbol{\tau}_u$ with $u\in\tilde{\mathrm{W}}$ such that $\ell(u)=0$ and $u=n_A$ for $A\in\Pi_{aff}$. For the former elements, the claim is clear since ι fixes such $\boldsymbol{\tau}_u$ when $\ell(u)=0$. Now let $A=(\alpha,r)\in\Pi_{aff}$. We consider the morphism $\phi_\alpha:\mathrm{SL}_2(\mathfrak{F})\to\mathrm{G}_\alpha$ as in 2.2.5. Since $\mathfrak{F}$ is infinite, the restriction of $\boldsymbol{\alpha}$ to the image of ϕ_α is trivial. Now by Remark 2.10, we have $\iota\Psi\iota(\boldsymbol{\tau}_{n_A})=\boldsymbol{\tau}_{n_A}$.

□

We deduce from this (and using (4.6)) that

$$\Psi(\epsilon_{\mathbf{1}}\mathcal{B}_{x_0}^+(\lambda(\varpi^{-1})))=\epsilon_\xi(-1)^{\ell(e^\lambda)}\Psi(\iota(\boldsymbol{\tau}_{\lambda(\varpi^{-1})}))=\epsilon_\xi\mathcal{B}_{x_0}^+(\lambda(\varpi^{-1})).$$

Conclude using Lemma 5.1.

□

5.1.2. For $\chi : \tilde{\mathfrak{H}} \to k^\times$, consider the associated facet F_χ as in 3.4 and M_χ the corresponding standard Levi subgroup as in 4.1.4.

Proposition 5.4. *There is a character* $\boldsymbol{\alpha}_\chi : \mathrm{M}_{F_\chi} \to k^\times$ *such that*
a) $\boldsymbol{\alpha}_\chi$ *coincides with* $\bar\chi^{-1}$ *on* T^0 *and satisfies* $\boldsymbol{\alpha}_\chi(\mu(\varpi)) = 1$ *for all* $\mu \in \mathrm{X}_*(\mathrm{T})$,
b) in $\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}$ *we have, for* $\lambda \in \mathrm{X}_*^+(\mathrm{T})$,

$$1 \otimes \mathcal{B}_{F_\chi}(\lambda(\varpi^{-1})) = \sum_{d \in \mathrm{W}_{F_\chi} \cap \mathcal{D}_{F_\chi},\, d \underset{F}{\leq} e^\lambda} \boldsymbol{\alpha}_\chi(\tilde d) \otimes \boldsymbol{\tau}_{\tilde d} \tag{5.3}$$

where $\tilde d$ *denotes any lift in* $\tilde{\mathrm{W}}_{F_\chi}$ *for* $d \in \mathrm{W}_{F_\chi}$.

Proof. In this proof we write F instead of F_χ. Let $\lambda \in \mathrm{X}_*^+(\mathrm{T})$. Consider the restriction to $\mathrm{M}_F \cap \mathrm{K}$ of the weight ρ associated to χ and the corresponding restriction of χ to the finite Hecke algebra $\tilde{\mathfrak{H}}_{x_F}(\mathrm{M}_F)$ (see Remark 3.5 for the definition of this subalgebra of $\tilde{\mathrm{H}}(\mathrm{M}_F)$ attached to the compact open subgroup $\mathrm{M}_F \cap \mathrm{K}$ of M_F). It satisfies the hypotheses of Lemma 5.2, where M_F plays the role of G. Note that under our hypothesis for G, the derived subgroup of M_F is equally simply connected. Therefore, there exists a character $\boldsymbol{\alpha}_\chi : \mathrm{M}_F \to k^\times$ that coincides with $\bar\chi^{-1}$ on T^0, such that $\boldsymbol{\alpha}_\chi(\mu(\varpi)) = 1$ for all $\mu \in \mathrm{X}_*(\mathrm{T})$ and satisfying the following equality in $\tilde{\mathrm{H}}(\mathrm{M}_F)$ (see (4.6) applied to the Levi M_F)

$$\epsilon_{\bar\chi}\, (-1)^{\ell_F(e^\lambda)} \iota^F(\boldsymbol{\tau}^F_{\lambda(\varpi^{-1})}) = \sum_{v \in \mathrm{W}_F, v \underset{F}{\leq} e^\lambda} \epsilon_{\bar\chi} \boldsymbol{\alpha}_\chi(\tilde v) \boldsymbol{\tau}^F_{\tilde v}.$$

Now applying (4.5) to the facet F, we have in $\tilde{\mathrm{H}}$,

$$\epsilon_{\bar\chi} \mathcal{B}^+_F(\lambda(\varpi^{-1})) = \sum_{v \in \mathrm{W}_F, v \underset{F}{\leq} e^\lambda} \epsilon_{\bar\chi} \boldsymbol{\alpha}_\chi(\tilde v) \boldsymbol{\tau}_{\tilde v}.$$

Before projecting this relation into $\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}$, recall that $\chi(\boldsymbol{\tau}_{\tilde w}) = 0$ for all $w \in \mathfrak{W}_F$. By definition of $\mathcal{D}_F$ (see also Remark 3.3), we therefore have, in $\chi \otimes_{\tilde{\mathfrak{H}}} \tilde{\mathrm{H}}$,

$$1 \otimes \mathcal{B}^+_F(\lambda(\varpi^{-1})) = 1 \otimes \epsilon_{\bar\chi} \mathcal{B}^+_F(\lambda(\varpi^{-1})) = \sum_{d \in \mathrm{W}_F \cap \mathcal{D}_F,\, d \underset{F}{\leq} e^\lambda} \boldsymbol{\alpha}_\chi(\tilde d) \otimes \boldsymbol{\tau}_{\tilde d}.$$

□

5.2. **An inverse to the mod p Satake transform of** [19]**.** Let (ρ, V) be a weight, $\chi : \tilde{\mathfrak{H}} \to k$ the corresponding character and F_χ the facet defined as in 3.4. Consider the isomorphism

$$\mathcal{S} : \mathcal{H}(\mathrm{G}, \rho) \xrightarrow{\sim} k[\mathrm{X}_*^+(\mathrm{T})] \tag{5.4}$$

constructed in [19] (see Remark 2.5). For $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, denote by f_λ the function in $\mathcal{H}(\mathrm{G}, \rho)$ with support in $\mathrm{K}\lambda(\varpi^{-1})\mathrm{K}$ and value at $\lambda(\varpi^{-1})$ equal to the k-linear projection $\mathrm{V} \to \mathrm{V}$ defined by

[19, (2.8)]. Note that this projection coincides with the identity on $\mathrm{V}^{\tilde{\mathrm{I}}}$ (step 3 of the proof of Theorem 1.2 *loc.cit*). Any function in $\mathcal{H}(\mathrm{G},\rho)$ with support in $\mathrm{K}\lambda(\varpi^{-1})\mathrm{K}$ is a k-multiple of f_λ. The element $T_{f_\lambda}(\mathbf{1}_{\mathrm{K},v})$ defined by $g \mapsto f_\lambda(g)v$ (see notation in Remark 3.7) is the unique element in $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}$ with support in $\mathrm{K}\lambda(\varpi^{-1})\mathrm{K}$ and value v at $\lambda(\varpi^{-1})$ which is an eigenvector for the action of $\tilde{\mathfrak{H}}$ and the character χ (see Corollary 3.14 and Remark 3.15).

Recall that the isomorphism $\mathcal{T}: k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\sim} \mathcal{H}(\mathrm{G},\rho)$ was defined in (4.10) and that both $\mathcal{S}$ and $\mathcal{T}$ are defined with no further condition on the derived subgroup of $\mathbf{G}$.

Theorem 5.5. *Suppose that the derived subgroup of* $\mathbf{G}$ *is simply connected.*

i. *For* $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ *we have*

$$\mathcal{T}_\lambda = \sum_{\mu \underset{F_\chi}{\preceq} \lambda} f_\mu. \tag{5.5}$$

ii. *The map* $\mathcal{T}$ *is an inverse for* $\mathcal{S}$*.*

Remark 5.6. In particular, the matrix coefficients of $\mathcal{T}$ in the bases $\{\lambda\}_{\lambda\in\mathrm{X}_*^+(\mathrm{T})}$ and $\{f_\lambda\}_{\lambda\in\mathrm{X}_*^+(\mathrm{T})}$ depend only on the facet F_χ, and not on χ itself.

Proof. In this proof, we write F for F_χ. For i, we have to show that $\mathcal{T}_\lambda$ has support the set of all double cosets $\mathrm{K}\mu(\varpi^{-1})\mathrm{K}$ for $\mu \underset{F}{\preceq} \lambda$ and that for $v \in \rho^{\tilde{\mathrm{I}}}$ and such a μ, the value of $\mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v})$ at $\mu(\varpi^{-1})$ is v. By (5.3), we have

$$\mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}) = \sum_{d\in \mathrm{W}_F\cap\mathcal{D}_F,\, d\underset{F}{\leq} e^\lambda} \boldsymbol{\alpha}_\chi(\tilde{d})\mathbf{1}_{\mathrm{K},v}\boldsymbol{\tau}_{\tilde{d}}.$$

By Lemma 2.9ii, this element has support in the expected set, and using furthermore Lemma 3.2iii, we see that any $d \in \mathrm{W}_F \cap \mathcal{D}_F$ satisfying $d \underset{F}{\leq} e^\lambda$ lies in $\mathcal{D}$. Therefore, by Fact 1, $\mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v})$ has value $\boldsymbol{\alpha}_\chi(\tilde{d})v$ at $\hat{\tilde{d}}$ for all $d \in \mathrm{W}_F \cap \mathcal{D}_F$, $d \underset{F}{\leq} e^\lambda$. Now recall that for any $\mu \in \mathrm{X}_*^+(\mathrm{T})$, the element e^μ is in $\mathcal{D}$ and therefore in $\mathrm{W}_F \cap \mathcal{D}_F$ (Remark 2.3), that $\mu(\varpi^{-1})$ is a lift in G for e^μ and that $\boldsymbol{\alpha}_\chi(\mu(\varpi^{-1})) = 1$. It proves that $\mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v})$ has value v at $\mu(\varpi^{-1})$ for $\mu \underset{F}{\preceq} \lambda$. We have proved the formula (5.5).

Finally, let $\lambda \in \mathrm{X}_*^+(\mathrm{T})$. Under the hypothesis that the derived subgroup of $\mathbf{G}$ is simply connected, $\sum_{\mu \underset{F}{\preceq} \lambda} \mathcal{S}(f_\mu)$ is equal to the element λ seen in $k[\mathrm{X}_*^+(\mathrm{T})]$ by [20, Proposition 5.1]. It proves ii.

□

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