Master-slave synchronization and the Lorenz equations

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(Received 30 January 1997; accepted for publication 6 June 1997)

Since the seminal remark by Pecora and Carroll [Phys. Rev. Lett. **64**, 821 (1990)] that one can synchronize chaotic systems, the main example in the related literature has been the Lorenz equations. Yet this literature contains a mixture of true and false, and of justified and unsubstantiated claims about the synchronization properties of the Lorenz equations. In this note we clarify some of the confusion. [S1054-1500(97)00803-3]

Pecora and Carroll[1,2] have recently introduced a novel notion of synchronization for (possibly chaotic) dynamical systems. Roughly speaking, this notion is that two identical systems can be coupled in such a way that the solution of one always converges to the solution of the other, independently of the initial conditions. However, although one system responds to the other, the reciprocal does not occur. Thus this phenomenon is also called MASTER-SLAVE SYNCHRONIZATION (Ref. 3) in order to differentiate it from other, better known, phenomena, such as the collective phase locking of populations of coupled oscillators. Several applications of master-slave synchronization include the control of chaos and chaotic signal masking (see, e.g. Refs. 4 and 5). The example that is most often used in the literature on the subject, starting with Ref. 1, is the Lorenz equations[6] [see Eq. (3)]. In fact, it is widely cited that this example leads us to observe the synchronization of chaotic systems, even though there is still no proof that the Lorenz equations have a strange attractor for any parameter values. In any event, the synchronizability of the Lorenz equations is interesting whether it is truly chaotic or not, because of the complicated solutions it generates. This short note is motivated by the fact that previous remarks concerning the synchronizability of the Lorenz equations are a strange mixture of true and false statements, with several true statements appearing without justification.

We first recall what we need from Refs. 1–3 about master-slave synchronization. Consider a dynamical system with equations of motion

$$\begin{aligned} u' &= f(u,v), \\ v' &= g(u,v), \end{aligned} \tag{1}$$

where u and v stand for the vectors of dependent variables, and the prime means a time derivative in the case of differential equations, or the next value in the case of maps. Now choose some initial conditions $(u(0),v(0))$, and let $(u(t),v(t))$ be the corresponding solution, where t is the continuous or discrete independent variable. We call the non-autonomous equation

$$V' = g(u(t),V) \tag{2}$$

the reduced equation corresponding to Eq. (1) driven by u. If for all choices of $(u(0),v(0))$ and of $V(0)$, the solution $V(t)$ of Eq. (2) converges towards $v(t)$ as t goes to infinity, we say that u is a *synchronizing vector* for our dynamical system (or sometimes a *synchronizing coordinate* if the dimension of u is one). Notice then that the above non-autonomous equation also makes sense for any function, $u(t)$, not necessarily obtained by solving the original Eq. (1) (but assumed to be regular enough to insure existence and unicity of the solutions of Eq. (2) when it is a differential equation). If for all such $u(t)$, and for any two initial conditions $V_1(0)$ and $V_2(0)$ of the non-autonomous equation, the corresponding solutions $V_1(t)$ and $V_2(t)$ of (2) converge to each other as t goes to infinity, we say that u is an *absolutely synchronizing vector (or coordinate)* for our dynamical system.

It caused a certain amount of surprise when Pecora and Carroll[1] reported the numerical observation that x and y are each synchronizing coordinates in the Lorenz equations (with positive parameter values),

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - xz - y, \\ \dot{z} &= xy - bz, \end{aligned} \tag{3}$$

even for parameter values such that numerical simulations indicate the existence of a strange attractor with sensitive dependence upon initial conditions. In the same paper, Pecora and Carroll also reported that z is not a synchronizing coordinate, and they discriminated between synchronizing and non-synchronizing coordinates by the following criterion which gives an obvious necessary condition: *if the non-autonomous system (2) corresponding to (1) is driven by a synchronizing vector, then that system can have no positive Lyapunov exponent along orbits of (1).* It is in general hard to get provably accurate approximations to Lyapunov exponents, so that this criterion is not much used to actually prove

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results. We now report on five provable facts about the synchronization properties of the Lorenz equations.

(I) The reduced equation corresponding to Eq. (3) driven by x reads

$$\begin{aligned}\dot{Y}&=rx(t)-x(t)Z-Y,\\ \dot{Z}&=x(t)Y-bZ.\end{aligned}\tag{4}$$

If Δ_x stands for the square of the distance between two solutions (Y_1,Z_1) and (Y_2, Z_2) of Eq. (4), then we have

$$\dot{\Delta}_x=-2(Y_1-Y_2)^2-2b(Z_1-Z_2)^2,\tag{5}$$

which is strictly negative as long as the two solutions are distinct, for any choice of $x(t)$. Hence, as noticed in Ref. 7, Δ_x can be identified as a Lyapunov function, which proves that x is a synchronizing coordinate for Eq. (3). Moreover, it tells us that x is an absolutely synchronizing coordinate (as stated in Ref. 3).

(II) The reduced equation corresponding to Eq. (3) driven by y reads

$$\begin{aligned}\dot{X}&=\sigma\ (y(t)-X),\\ \dot{Z}&=Xy(t)-bZ.\end{aligned}\tag{6}$$

With (X_1,Z_1) and (X_2,Z_2) standing for two solutions of Eq. (6), we write $(\delta_X,\delta_Z)=(X_2-X_1,Z_2-Z_1)$ to obtain:

$$\begin{aligned}\dot{\delta}_X&=-\sigma\delta_X,\\ \dot{\delta}_Z&=\delta_X y(t)-b\delta_Z.\end{aligned}\tag{7}$$

It is evident that δ_X decays exponentially, and that

$$\delta_Z(t)=\delta_Z(0)e^{-bt}+\delta_X(0)e^{-bt}\int_0^t e^{bs}y(s)e^{-\sigma s}ds.\tag{8}$$

As stated in Ref. 3, this shows that y is not an absolutely synchronizing coordinate but that it is nevertheless a synchronizing coordinate for Eq. (3), since all orbits of Eq. (3) are bounded.[6]

(III) We give two simple proofs that z cannot be synchronizing for Eq. (3). The first proof is based on the remark that the reduced equation corresponding to Eq. (3) driven by z reads

$$\begin{aligned}\dot{X}&=\sigma\ (Y-X),\\ \dot{Y}&=rX-Xz(t)-Y,\end{aligned}\tag{9}$$

which is a homogeneous, linear system. Given $z(t)$, this implies that for every corresponding solution $S(t)=(X(t),Y(t))$ of Eq. (9) and every real number k, $(kX(t), kY(t))$ is also a solution of Eq. (9). As a consequence, for any solution $S(t)$ which does not converge to the origin, one can find infinitely many solutions which do not converge to $S(t)$. In particular, this holds true if $(X(t),Y(t))=(x(t),y(t))$, where $(x(t),y(t),z(t))$ is the chosen solution of Eq. (3). Furthermore, by choosing k close to 1, we can find solutions arbitrarily close to $S(t)$ which do not converge to $S(t)$. Thus the Lorenz equations driven by z are not even locally synchronizing.

The second proof, already stated in Ref. 3, is based on the remark that the Lorenz equations are invariant under a symmetry. By making the transformation $x\to -x$, $y\to -y$, $z\to z$, we obtain the same vector field; in other words, the Lorenz vector field is invariant under the axial symmetry about the z axis. Therefore for every solution $L(t)=(x(t),y(t),z(t))$ of Eq. (3), there is also a solution $L'(t)=(-x(t),-y(t),z(t))$. In particular, simply knowing the z-coordinate is not sufficient to differentiate between these two solutions. Furthermore, these two solutions do not always converge to each other (this can only occur if they both converge to the origin). These two facts imply that z cannot be a synchronizing coordinate.

(IV) Some time ago, it was proposed to replace Eq. (2), for the case of continuous time, by a discrete time updating of the forcing variable U at each time step τ, for τ in some range depending on the system considered;[9] the so called *finite time step method* which does improve performance in several cases (see also Ref. 8). In other words, one prescribes $U(t_0+n\tau)\stackrel{\text{def}}{=}u(t_0+n\tau)$ for $n=0,1,2,\ldots$ and allows (U,V) to evolve freely according to Eq. (1) in the time intervals in between. The authors of Ref. 9 applied this method to the Lorenz equations to see if it allowed the z-coordinate to be a synchronizing coordinate for τ and the parameters of Eq. (3) in some range. The numerical results presented in Ref. 9, including negative Lyapunov exponents for the driven system, were rather convincing in showing that z is indeed a synchronizing coordinate for the Lorenz system when using the finite step method. However, because of the symmetry argument given above, synchronization by driving with z is impossible even with the finite step method (again we use the fact that for every solution $L(t)=(x(t),y(t),z(t))$ of Eq. (3), there is also a solution $L'(t)=(-x(t),-y(t),z(t))$, so that the numerical results documented in Ref. 9 and their interpretation (see also Ref. 8) are misleading. With $X(t)$ standing for a typical solution of the slave system, and $\|\cdot\|$ standing for the sup norm, we list here what might happen if we try to synchronize Eq. (3) with z using the step method, but cannot decide which outcome(s) hold(s) true:

(a) For almost all initial conditions, $\|X(t)-L(t)\|$ or $\|X(t)-L'(t)\|$ converges to zero (the analog of synchronization, but with two basins),
(b) (a) is false but $\min(\|X(t)-L(t)\|, \|X(t)-L'(t)\|)$ converges to zero [then the slave solution forever switches between neighborhoods of $L(t)$ and $L'(t)$: a kind of generalized synchronization],
(c) $\min(\|X(t)-L(t)\|, \|X(t)-L'(t)\|)$ does not converge to zero (no synchronization).

Numerical experiments suggested outcome (c) in most cases, but $\min(\|X(t)-L(t)\|, \|X(t)-L'(t)\|)$ appears to remain "small" for sufficiently large t. In fact, what happens may depend on the parameter values and/or on $L(t)$.

(V) Another question about Eq. (9) which has been raised in the literature is the nature of its Lyapunov spectrum. It has been reported in several places that along orbits of Eq. (3), the greatest Lyapunov exponent of Eq. (9) with

$z(t)$ a solution of Eq. (3) is positive. However, this statement is wrong. In fact, answering this question is not trivial because we do not know enough about the Lorenz equations. What can be said is that when there is a Lyapunov spectrum for an orbit $(X(t),Y(t))$ of Eq. (9) not converging to zero, with $(X(t),Y(t))=(x(t),y(t))$ and $(x(t),y(t),z(t))$ a solution of Eq. (3), then it must consist of one negative and one zero Lyapunov exponent. This follows from the combined facts that Eq. (9) is linear and that the trace $-(1+\sigma)$ of the matrix of Eq. (9) is negative. More precisely:

(a) The linearity tells us that if $(X_1(0),Y_1(0))=k(X(0),Y(0))$, for some real k, then $(X_1(t),Y_1(t))=k(X(t),Y(t))$ for all t. This forces one of the exponents to be zero (provided both exponents exist) since these orbits do not converge to zero by hypothesis, and are bounded as already proven in Ref. 6.

(b) The trace condition tells us that, when defined, the sum of the two exponents is $-(1+\sigma)$ (hence negative) so that, using (a), the second exponent must be negative and equal to $-(1+\sigma)$.

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