Marks

[42] **1.** **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

(a) Evaluate $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$ or determine that this limit does not exist.

$$= \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$$

$$= \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h)$$

Answer: 6

(b) Evaluate $\lim_{x\to\infty}(\sqrt{4x^2+x}-2x)$ or determine that this limit does not exist.

$$= \lim_{x\to\infty}\left(\sqrt{4x^2+x}-2x\right)\frac{\sqrt{4x^2+x}+2x}{\sqrt{4x^2+x}+2x}$$

$$= \lim_{x\to\infty}\frac{4x^2+x-4x^2}{\sqrt{4x^2+x}+2x}$$

$$= \lim_{x\to\infty}\frac{x}{\sqrt{4x^2+x}+2x} = \lim_{x\to\infty}\frac{1}{\sqrt{4+\frac{1}{x}}+2} = \frac{1}{\sqrt{4+0}+2}$$

Answer: $\frac{1}{4}$

(c) Find all values of the constant c that make the function f continuous everywhere, or determine that no such value exists:

$$f(x) = \begin{cases} \dfrac{\sin(4x)}{x} & \text{if } x \neq 0, \\ c & \text{if } x = 0 \end{cases}$$

Need
$c = \lim_{x\to 0} f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin(4x)}{x}$$

$$= \lim_{x\to 0}\frac{\sin(4x)}{\frac{1}{4}(4x)}$$

$$= 4\lim_{x\to 0}\frac{\sin(4x)}{4x} = 4\cdot 1$$

Answer: $c = 4$

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(d) Find the derivative of $(t^3 + 2t)e^t$.

Answer $(t^3 + 3t^2 + 2t + 2)\,e^t$

Use Product Rule

If $f(t) = (t^3 + 2t)\,e^t$

then $f'(t) = (3t^2 + 2)\,e^t + (t^3 + 2t)\,e^t$

(e) Find the derivative of $y = \dfrac{\sin x}{x^4}$.

Answer $y' = \dfrac{\cos x}{x^4} - \dfrac{4\sin x}{x^5}$

Use Quotient Rule

$$y' = \frac{(\cos x)(x^4) - (\sin x)(4x^3)}{(x^4)^2}$$

(f) Find $f'(x)$, if $f(x) = e^{\cos x}$.

Answer $f'(x) = -(\sin x)\,e^{\cos x}$

Use Chain Rule

$$f'(x) = e^{\cos x} \cdot \frac{d}{dx}(\cos x)$$

(g) Find the slope of the tangent line to the curve $\sqrt{x} + 3\sqrt{y} = 5$ at the point $(4, 1)$.

Answer $-\frac{1}{6}$

Use implicit differentiation

$x^{1/2} + 3y^{1/2} = 5$

$\frac{1}{2}x^{-1/2} + \frac{3}{2}y^{-1/2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x^{-1/2}}{3y^{-1/2}} = -\frac{\sqrt{y}}{3\sqrt{x}} = -\frac{\sqrt{1}}{3\sqrt{4}} \quad \text{at } (x,y) = (4,1)$$

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(h) Find y' if $y = \sin^{-1}(x^3)$. [*Note:* Another notation for $\sin^{-1}$ is arcsin.]

Answer $y' = \dfrac{3x^2}{\sqrt{1-x^6}}$

$$y' = \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3)$$

(i) Find $f'(x)$ if $f(x) = x^{\sin x}$.

Use logarithmic differentiation

Answer $f'(x) = x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x} \right)$

$$\ln f(x) = \sin x \ln x$$

$$\frac{f'(x)}{f(x)} = \cos x \ln x + \sin x \cdot \frac{1}{x}$$

(j) Use a linear approximation to estimate $(1.999)^4$.

Answer 15.968

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(x) = x^4, \quad f'(x) = 4x^3$$

$$x = 1.999, \quad a = 2$$

$$(1.999)^4 \approx 2^4 + 4 \cdot 2^3(1.999 - 2) = 16 - 32(0.001)$$

(k) Find the first three nonzero terms in the Maclaurin series for $f(x) = x^4 \sin(x^2)$.

Answer $x^6 - \frac{1}{6}x^{10} + \frac{1}{120}x^{14} - \ldots$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

$$\sin(x^2) = (x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \ldots$$

$$x^4 \sin(x^2) = x^4\left[x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \ldots \right]$$

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(l) Find the absolute maximum *value* of $f(x) = x^{2/3}$ on the interval $[-1, 2]$.

Answer $2^{2/3}$

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$

$f(x) = \sqrt[3]{x^2} = (\sqrt[3]{x})^2$ is continuous everywhere

0 is a critical number ($f'(0)$ does not exist)

$f(-1) = 1$, $f(0) = 0$, $f(2) = 2^{2/3}$ the largest of these is $2^{2/3} = \sqrt[3]{4} > 1$

(m) Newton's Method is used to approximate a solution of the equation $x + \ln x = 0$, starting with the initial approximation $x_1 = 1$. Find x_2.

Answer $x_2 = \frac{1}{2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = x + \ln x \quad (x > 0)$

$f'(x) = 1 + \frac{1}{x}$

$$x_2 = x_1 - \frac{x_1 + \ln x_1}{1 + \frac{1}{x_1}} = 1 - \frac{1 + \ln 1}{1 + \frac{1}{1}}$$

(n) A particle is moving with velocity function $v(t) = \cos t - \sin t$ and initial displacement $s(0) = 0$. Find the displacement at any time t.

Answer $s(t) = \sin t + \cos t - 1$

$v(t) = s'(t) = \cos t - \sin t$

$s(t) = \sin t + \cos t + C$

$s(0) = \underbrace{\sin 0}_{0} + \underbrace{\cos 0}_{1} + C = 0$

$C = -1$

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Full-Solution Problems. In questions 2–6, justify your answers and **show all your work**. If a box is provided, write your final answer there. Simplification of answers is not required unless explicitly requested.

[10] **2.** A bacteria culture grows with constant relative growth rate. After 2 days there are 40,000 bacteria and after 7 days the count is 4 billion $= 4 \cdot 10^9$.

(a) Write a differential equation satisfied by the bacteria population at any time t.

Answer $\frac{dP}{dt} = kP$

(b) Find the initial population, expressed as an integer.

Answer 400

$P(t) = P(0)e^{kt}$

At $t = 2$

$4 \cdot 10^4 = P(0)e^{2k}$

At $t = 7$

$4 \cdot 10^9 = P(0)e^{7k}$

$e^{5k} = \frac{4 \cdot 10^9}{4 \cdot 10^4} = 10^5$

$e^k = 10$, $k = \ln 10$ $\left(= \frac{1}{5} \ln 10^5 \right)$

$P(0) = 4 \cdot 10^4 e^{-2k}$ $\left(\text{or } 4 \cdot 10^7 e^{-7k} \right)$

$= 4 \cdot 10^4 e^{-2 \ln 10}$

$= 4 \cdot 10^4\, 10^{-2}$

(c) Find the population after t days.

Answer $400\, e^{(\ln 10)t}$

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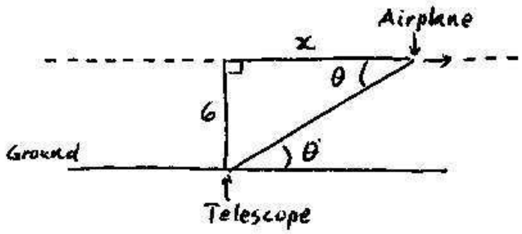
[10] **3.** An airplane flies horizontally at an altitude of 6 km and passes directly over a tracking telescope on the ground. When the angle of elevation (i.e. the angle at the telescope measured upwards from the horizontal to the airplane) is $\pi/6$, this angle is decreasing at a rate of 40 rad/min. How fast is the airplane travelling at that time? (somewhat unrealistically,)

Answer
960 km/min



x = horizontal distance (in km) from point directly above telescope

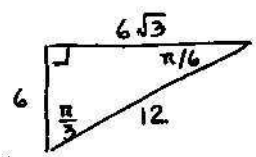
Find $\frac{dx}{dt}$ = speed of airplane

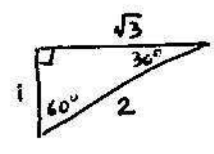
$$\frac{6}{x} = \tan\theta$$

$$-\frac{6}{x^2}\frac{dx}{dt} = \sec^2\theta\,\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = -\frac{x^2\sec^2\theta}{6}\,\frac{d\theta}{dt}$$

When $\theta = \pi/6$





$x = 6\sqrt{3}$

$\sec\theta = \frac{2}{\sqrt{3}}$

$\frac{d\theta}{dt} = -40$

$$\frac{dx}{dt} = -\frac{(6\sqrt{3})^2\left(\frac{2}{\sqrt{3}}\right)^2(-40)}{6} = +6\cdot 4\cdot 40$$

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[12] **4.** Let $f(x) = x^{5/3} + \frac{5}{2}x^{2/3}$.

(a) (1 mark) Find the domain of $f(x)$. $\left(= \sqrt[3]{x^5} + \frac{5}{2}\sqrt[3]{x^2} \right)$

$$-\infty < x < \infty$$

(b) (4 marks) Determine intervals where $f(x)$ is increasing or decreasing and the x- and y-coordinates of all local maxima or minima (if any).

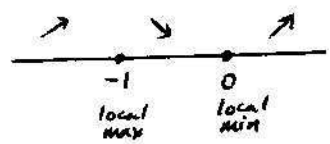
$$f'(x) = \frac{5}{3}x^{2/3} + \frac{5}{3}x^{-1/3} = \frac{5}{3}\frac{x+1}{\sqrt[3]{x}}$$

Critical numbers -1, 0

Interval	$x+1$	$\sqrt[3]{x}$	f'	f
$(-\infty,-1)$	$-$	$-$	$+$	increasing
$(-1,0)$	$+$	$-$	$-$	decreasing
$(0,\infty)$	$+$	$+$	$+$	increasing

local maximum $x=-1$, $y=\frac{3}{2}$

local minimum $x=0$, $y=0$

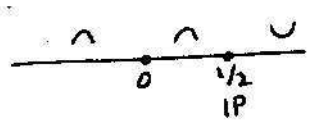


(c) (3 marks) Determine intervals where $f(x)$ is concave upwards or downwards, and the x-coordinates of all inflection points (if any).

$$f''(x) = \frac{10}{9}x^{-1/3} - \frac{5}{9}x^{-4/3} = \frac{5}{9}\frac{2x-1}{\sqrt[3]{x^4}}$$

Interval	$2x-1$	$\sqrt[3]{x^4}$	f''	Concavity
$(-\infty,0)$	$-$	$+$	$-$	downward
$(0,\frac{1}{2})$	$-$	$+$	$-$	downward
$(\frac{1}{2},\infty)$	$+$	$+$	$+$	upward

inflection point when $x=\frac{1}{2}$



Question 4 continues on the next page...

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Question 4 continued

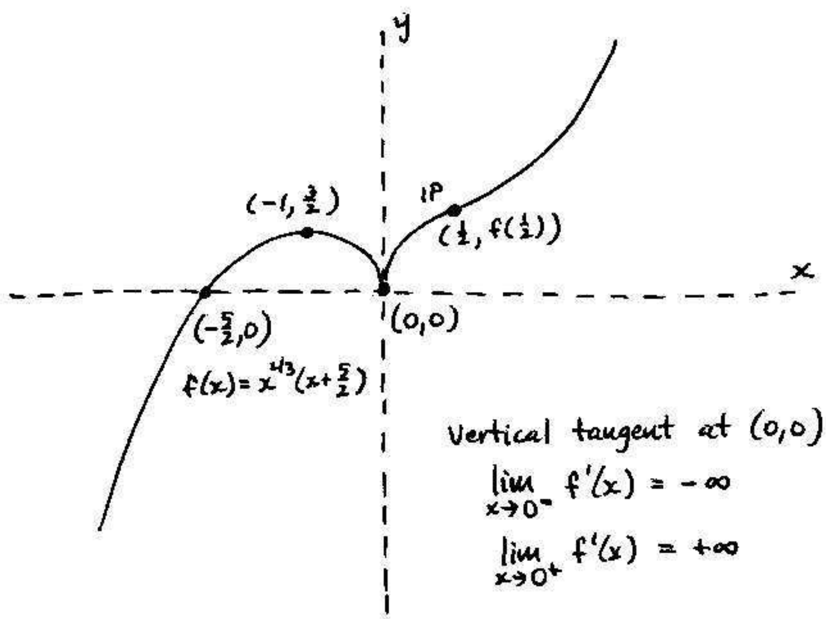
(d) (2 marks) Find and verify the equations of any asymptotes (horizontal, vertical or slant), or else determine that there are no asymptotes.

No vertical asymptotes ($f(x)$ is finite for all finite x)

No horizontal asymptotes ($\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$)

No slant asymptotes ($\lim_{x\to-\infty} \frac{f(x)}{x} = +\infty$, $\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty$)

(e) (4 marks) Sketch the graph of $y = f(x)$, showing the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above and also all x-intercepts (if any).



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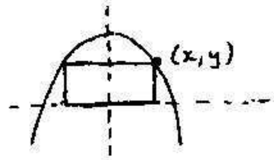
[10] 5. Find (with justification) the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 15 - x^2$.



Answer

width $= 2\sqrt{5}$, height $= 10$

Area of rectangle $= 2xy$

$f(x) = 2x(15 - x^2) = 30x - 2x^3$

$0 \le x \le \sqrt{15}$

$f'(x) = 30 - 6x^2 = 6(5 - x^2)$

$= 0$ if and only if $x = \pm\sqrt{5}$

$\sqrt{5}$ is in the domain ($5 < 15 \Rightarrow \sqrt{5} < \sqrt{15}$)

$-\sqrt{5}$ is not in the domain

Justification of absolute maximum attained when $x = \sqrt{5}$

Closed Interval Method

f (a polynomial) is continuous on $[0, \sqrt{15}]$

$f(0) = 0$, $f(\sqrt{5})\ (= 20\sqrt{5}) > 0$, $f(\sqrt{15}) = 0$

The greatest number of these three is $f(\sqrt{5})$, the only positive one

or First Derivative Test for Absolute Extreme Values

f (a polynomial) is continuous on $[0, \sqrt{15}]$

$f'(x) > 0$ for $0 \le x < \sqrt{5}$

$f'(x) < 0$ for $\sqrt{5} < x \le \sqrt{15}$

Dimensions of rectangle

$2x = 2\sqrt{5}$

$y = 15 - (\sqrt{5})^2$

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[4] **6.** Use the *definition of the derivative* to find $f'(x)$, if

$$f(x) = \sqrt{x+1}.$$

You may not use derivative formulas such as the Power Rule or the Chain Rule to answer this question.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}} \qquad (x > -1)$$

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[4] **7.** Determine what degree Maclaurin polynomial for $\ln(1-x)$ that should be used to approximate $\ln(1.1)$, so that the approximation is guaranteed to be accurate to within 10^{-9}.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{where } c \text{ is between } a \text{ and } x$$

$$x = -0.1, \quad a = 0, \quad n = \text{degree of Maclaurin polynomial}$$

$$f(x) = \ln(1-x)$$

$$f'(x) = -\frac{1}{1-x}$$

$$f''(x) = -\frac{1}{(1-x)^2} \qquad f^{(n+1)}(x) = -\frac{n!}{(1-x)^{n+1}}$$

$$f^{(3)}(x) = -\frac{(1)(2)}{(1-x)^3}$$

$$f^{(4)}(x) = -\frac{(1)(2)(3)}{(1-x)^4}$$

$$|R_n(-0.1)| = \frac{1}{(n+1)!}\left|-\frac{n!}{(1-c)^{n+1}}\right|\left|(-0.1)^{n+1}\right|$$

$$= \frac{1}{n+1}\frac{1}{(1-c)^{n+1}}\left(\frac{1}{10}\right)^{n+1} \quad \text{where } -\tfrac{1}{10} < c < 0$$

This positive value increases as c increases from $-\frac{1}{10}$ to 0

so for an upper bound take $c = 0$

$$|R_n(-0.1)| \le \frac{1}{n+1}\left(\frac{1}{10}\right)^{n+1}$$

To guarantee accuracy, require

$$\frac{1}{n+1}\left(\frac{1}{10}\right)^{n+1} < 10^{-9}$$

$$\frac{1}{n+1}\,10^{-(n+1)} < 10^{-9}$$

$$n \ge 8$$

($n \le 7$ will not guarantee accuracy: $\frac{1}{8}\cdot 10^{-8} \not< 10^{-9}$)

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[8] **8.**

(a) (4 marks) Prove that $x + \ln|x| = 0$ has at least one solution in the open interval $(-1, 1)$.

$f(x) = x + \ln|x|$ is continuous on $[-1,0)$ and on $(0,1]$ but is discontinuous at 0

$f(1) = 1$, $f(a) = a + \ln a < 0$ for any $a > 0$ sufficiently close to 0, since $\lim_{x\to 0^+} \ln|x| = -\infty$

e.g. $a = \frac{1}{2}$ or $a = e^{-1}$ etc.

Choosing $0 < a < 1$ so that $f(a) < 0$

f is continuous on $[a, 1]$

By the Intermediate Value Theorem $f(x) = 0$ has at least one solution in $(a, 1)$ which is therefore also in $(-1, 1)$

(b) (4 marks) Prove that $x + \ln|x| = 0$ has exactly one solution in the open interval $(-1, 1)$.

$f'(x) = 1 + \frac{1}{x}$, $x \neq 0$

$f'(x) > 0$ for $0 < x < 1$

$f(x) = 0$ can have no more than one solution in $(0, 1)$

Justification

f is increasing

or use MVT and proof by contradiction

$f(x) = 0$ cannot have any solutions in $(-1, 0)$

Justification

$x < 0$, $\ln|x| < 0 \Rightarrow x + \ln|x| < 0$ for $-1 < x < 0$

or $f'(x) < 0$, f is decreasing in $(-1, 0)$ and $f(-1) = -1 < 0$

$\Rightarrow f(x) \leq -1$ for $-1 < x < 0$.

The End