Heat kernel estimates for boundary trace of reflected diffusions on uniform domains*

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December 13, 2023

Abstract

We study boundary trace process of a reflected diffusion for uniform domains. We obtain stable-like heat kernel estimates for the boundary trace process of a reflected diffusion for uniform domains when the diffusion on the underlying ambient space satisfies sub-Gaussian heat kernel estimates. Our arguments rely on new results of independent interest such as sharp estimates and doubling properties of the harmonic measure, continuous extension of Naïm kernel to the topological boundary, and the Doob-Naïm formula for the Dirichlet energy of boundary trace process.

Keywords: Boundary trace process, reflected diffusion, Doob-Naïm formula, Harmonic measure, uniform domains, capacity density condition, Naïm kernel, Heat kernel, symmetric stable process.

Mathematical Subject Classification (2020): Primary 60J60, 60J76, 31E05, 31C25, 35K08; Secondary: 31B25, 35J08, 60J45

Contents

*This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

†Research partially supported by JSPS KAKENHI Grant Number JP22H01128.

‡Research partially supported by NSERC and the Canada research chairs program.

1 Introduction

A classical theorem of Spitzer implies that the trace process of the reflected Brownian motion on the $(n+1)$-dimensional upper half-space on its boundary is the n-dimensional Cauchy process [Spi, Mol]. Molchanov and Ostrowski discovered that one can realize all symmetric stable processes on $\mathbb{R}^n$ as a trace process on the boundary of a reflected diffusion on the $(n+1)$-dimensional upper half-space [MO]. This was later revisited in a celebrated work to analyze the fractional Laplace operator and is now known as the *Caferelli-Silvestre extension* [CS]. Caferelli and Silvestre demonstrated that properties of non-local operators could be understood using corresponding properties of the associated local operators [CS, §5]. The local and non-local operators in [CS] are the generators of the diffusion in the upper half-space and the boundary trace process respectively in [MO]. Our work aims to extend this idea to understand the behavior of boundary trace process (a jump process) using that of the associated diffusion process.

In light of the work of Molchanov and Ostrowski mentioned above, the following natural question arises: *does the boundary trace process of a reflected diffusion behave like a symmetric stable process in other settings?* The goal of this work is to answer the above question affirmatively by obtaining stable-like heat kernel estimates for the boundary trace process of reflected diffusion in a broad class of examples. The generator of the boundary trace process is typically a non-local (or integro-differential) operator.

Therefore from an analytic viewpoint, our work shows that the fundamental solution of the 'heat equation' corresponding to this non-local operator on the boundary behaves like that of the fractional Laplacian on Euclidean spaces. We note that stable-like estimate of the heat kernel for jump process has been extensively studied for the past two decades [BL, BGK09, CK03, CK08, CKW, GHL14, GHH23, GHH23+, Mal, MS19].

Our setting is a metric measure space equipped with a m-symmetric diffusion process, where m is a Radon measure with full support. Equivalently, we consider a metric space $(\mathcal{X}, d)$ equipped with a strongly local, regular, Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$. We call $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ the metric measure space with a strongly local, regular Dirichlet form or MMD space for short. We refer to [FOT, CF] for the theory of Dirichlet forms.

We consider symmetric diffusion admitting **sub-Gaussian heat kernel estimates**. Our setting includes examples with Gaussian heat kernel estimates [Gri, Sal, Stu] such as Brownian motion in Euclidean space or manifolds with non-negative Ricci curvature, diffusion generated by degenerate elliptic operators [FKS] and uniformly elliptic operators in divergence-form in $\mathbb{R}^n$ [Mos], diffusion on connected nilpotent Lie groups associated with a left invariant Riemannian metric or with sub-Laplacians of the form $\Delta = \sum_{i=1}^{k} X_i^2$, where $\{X_i : 1 \leqslant i \leqslant k\}$ is a family of left-invariant vector fields satisfying the Hörmander's condition [VSC], weighted Euclidean spaces and manifolds [GrS]. Another significant class of examples arise from diffusion on fractals such as the Sierpiński gasket, Sierpiński carpet, and their variants [Bar98, BB89, BB92, BB99, BP, BH, FHK, Kum].

Given an MMD space as above, we consider **uniform domains** satisfying a **capacity density condition**. Uniform domains were introduced independently by Martio and Sarvas [MS] and Jones [Jon]. This class includes Lipschitz domains, and more generally non-tangentially accesible (NTA) domains introduced by Jerison and Kenig [JK]. Due to similar definition, we note that uniform domains are also referred as *1-sided NTA domains* [AHMT1, HMM]. Uniform domains are relevant in various contexts such as extension property [BS, Jon, HerK], Gromov hyperbolicity [BHK], boundary Harnack principle [Aik01], geometric function theory [MS, GH, Geh], and heat kernel estimates [GyS, CKKW, Mur23+]. One reason for the importance of uniform domains is its close connection to Gromov hyperbolic spaces [BHK]. Another reason is the abundance of uniform domains. In fact, by [Raj, Theorem 1.1] every bounded domain is arbitrarily close to a uniform domain in a large class of metric spaces.

The NTA domains introduced by Jerison and Kenig are examples of uniform domains satisfying the capacity density condition. The capacity density condition guarantees that every boundary point is regular for the associated diffusion and can be viewed as a stronger version of Wiener's test of regularity. Uniform domains satisfying the capacity density condition provide a fruitful setting to study various aspects of the harmonic measure [Anc86, AH, AHMT1, AHMT2, CDMT]. For Brownian motion on the Euclidean space, the capacity density condition of a domain Ω is expressed by the estimate:

$$\mathrm{Cap}_{B(\xi,2r)}(B(\xi,r)) \lesssim \mathrm{Cap}_{B(\xi,2r)}(B(\xi,r)\backslash\Omega), \quad \text{for all } \xi \in \partial\Omega, 0 < r \lesssim \mathrm{diam}(\Omega),$$

where $\mathrm{Cap}_{B(\xi,2r)}(K)$ denotes the capacity between the sets K and $B(\xi,2r)^c$. The fact that uniform domains satisfying the capacity density condition satisfy good properties on harmonic measure was recognized by Aikawa and Hirata [AH]. As we will see later, estimates on harmonic measure play an important role in our work.

Let us examine the results of Molchanov-Ostrowski, Cafferelli-Silvestre [MO, CS] in further detail to provide context. For $\alpha \in (0, 2)$, we recall that the symmetric α-stable operator is generated by the *fractional Laplace operator* $(-\Delta)^{-\alpha/2}$ on $\mathbb{R}^n$,

$$(-\Delta)^{\alpha/2} f(x) := c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} \, dy,$$

where $c_{n,\alpha} \in (0, \infty)$ is a normalizing constant. Writing $\mathbb{R}^{n+1} = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}\}$ as $\mathbb{R}^n \times \mathbb{R}$, we consider the Dirichlet form

$$\mathcal{E}(u, u) := \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\nabla u|^2(x, y) |y|^{1-\alpha} \, dy \, dx$$

on $L^2(\mathbb{R}^{n+1}, |y|^{1-\alpha} \, dy \, dx)$. The corresponding diffusion is generated by the degenerate elliptic operator

$$L_\alpha u := \Delta_x u + \frac{1 - \alpha}{y} u_y + u_{yy}.$$

Gaussian heat kernel estimates for diffusion generated by such degenerate elliptic operators follow from results of [FKS, Sal, Gri]. To compute the Dirichlet form corresponding to the trace process on the boundary, we consider the Dirichlet boundary value problem

$$L_\alpha u = 0 \text{ on } \mathbb{R}^n \times (0, \infty), \quad u(x, 0) = f(x), \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a prescribed boundary value on a suitable function space. Then by [CS, §3.2], the Dirichlet energy of the solution u to (1.1) can be expressed in terms of the boundary data f as

$$\int_{\mathbb{R}^n} \int_{(0,\infty)} |\nabla u|^2(x, y) |y|^{1-\alpha} \, dy \, dx = \int_{\mathbb{R}^n} f(\xi) (-\Delta)^{\alpha/2} f(\xi) \, d\xi. \tag{1.2}$$

The equality (1.2) implies that the boundary trace process of the reflected diffusion generated by L_α is the symmetric α-stable stable process. We refer to [Kwa] for new results in this direction.

An earlier example of an expression such as (1.2) that relates a local operator on a domain to a non-local operator on its boundary is due to J. Douglas [Dou]. The *Douglas formula* states that the harmonic function u on the unit disk $B(0, 1)$ in $\mathbb{R}^2$ with boundary value regarded as a function $f : [0, 2\pi) \to \mathbb{R}$ has Dirichlet energy given by

$$\int_{B(0,1)} |\nabla u|^2(x) \, dx = \frac{1}{8\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(f(\eta) - f(\xi))^2}{\sin^2((\eta - \xi)/2)} \, d\eta \, d\xi. \tag{1.3}$$

The right hand side above can be viewed as the Dirichlet form of the boundary trace process corresponding to the reflected Brownian motion on the unit disk. This was later extended to smooth domains in $\mathbb{R}^2$ by Osborn [Osb]. More generally, if u is harmonic in a smooth domain $\Omega \subset \mathbb{R}^n$ with boundary value $f : \partial\Omega \to \mathbb{R}$, then the Dirichlet energy is given by

$$\int_\Omega |\nabla u|^2(x)\, dx = \int_{\partial\Omega}\int_{\partial\Omega} (f(\eta) - f(\xi))^2 \frac{\partial^2 g_\Omega(\xi,\eta)}{\partial \vec{n}_\xi \partial \vec{n}_\eta}\, d\sigma(\xi)\, d\sigma(\eta), \tag{1.4}$$

where σ is the surface measure $\partial\Omega$, $g_\Omega(\cdot,\cdot)$ is the Green function on Ω, $\frac{\partial}{\partial \vec{n}_\xi}, \frac{\partial}{\partial \vec{n}_\eta}$ denote the inward pointing normal derivatives at ξ, η respectively [CF, (5.8.4)].

J. Doob found a remarkable extension of (1.2), (1.3), (1.4) to domains that are not necessarily smooth. Doob's result is under an abstract potential theoretic setting of (locally Euclidean) Green spaces in the sense of Brelot and Choquet [BC, Doo]. The boundary conditions of the harmonic function are prescribed on the *Martin boundary* $\partial_M\Omega$ of the domain Ω. To describe Doob's result, we recall the **Naïm kernel** defined by

$$\Theta^\Omega_{x_0}(\xi,\eta) = \lim_{x\to\xi}\lim_{y\to\eta} \frac{g_\Omega(x,y)}{g_\Omega(x_0,x)g_\Omega(x_0,y)}, \quad \text{for } \xi,\eta \in \partial_M\Omega, \xi \neq \eta,$$

where the limits are with respect to the *fine topology*, $x_0 \in \Omega$ is an arbitrary base point, and $g_\Omega(\cdot,\cdot)$ is the Green function on Ω as before. The existence of the above limits in the setting of Green spaces follows from the fundamental work of L. Naïm [Naï]. The **Doob-Naïm formula** states that

$$\int_\Omega |\nabla u|^2(x)\, dx = \int_{\partial_M\Omega}\int_{\partial_M\Omega} (f(\xi) - f(\eta))^2 \Theta^U_{x_0}(\xi,\eta)\, d\omega^\Omega_{x_0}(\xi)\, d\omega^\Omega_{x_0}(\eta), \tag{1.5}$$

if u is harmonic in a domain Ω with fine boundary value $f : \partial_M\Omega \to \mathbb{R}$ and $\omega^\Omega_{x_0}$ is the **harmonic measure** of the corresponding diffusion started at x_0 [Doo, Theorem 9.2]. Fukushima gave an alternate proof of the Doob-Naïm formula for Green spaces [Fuk]. There are versions of Doob-Naïm formula for Markov chains on a countable state space due to M. Silverstein [Sil, Theorem 3.5] and for random walks on transient trees in [BGPW, Theorem 6.4].

Even though these earlier approaches in [Doo, Fuk, Sil, BGPW] do not apply to our setting, we obtain a version of the Doob-Naïm formula with the Martin boundary $\partial_M\Omega$ replaced with the *topological* boundary $\partial\Omega$ (see Theorem 5.12). Using a variant of Moser's oscillation lemma [Mos] and the *boundary Harnack principle*, we show that the Naïm kernel on a uniform domain U given

$$\Theta^U_{x_0}(x,y) = \frac{g_U(x,y)}{g_U(x_0,x)g_U(x_0,y)} \text{ on } (U\backslash\{x_0\}) \times (U\backslash\{x_0\})\backslash(U\backslash\{x_0\})_{\mathrm{diag}}$$

extends to a jointly continuous function on $\left((\overline{U}\backslash\{x_0\}) \times (\overline{U}\backslash\{x_0\})\right)\backslash(\overline{U}\backslash\{x_0\})_{\mathrm{diag}}$ up to the *topological* boundary (see Proposition 3.15). Our proof of the Doob-Naïm formula relies on the boundary Harnack principle unlike earlier approaches. Other important ingredients in our proof of the Doob-Naïm formula is the vanishing of energy measure on the boundary of a uniform domain recently shown in [Mur23+, Theorem 2.9] and estimates on the harmonic measure discussed below in (1.7).

Our version of the Doob-Naïm formula states that the Dirichlet form corresponding to the boundary trace process for the reflected diffusion on a uniform domain U satisfying the capacity density condition is given by the *Doob–Naïm energy*

$$\int_{\partial U}\int_{\partial U}(f(\xi)-f(\eta))^2\Theta^U_{x_0}(\xi,\eta)\,d\omega^U_{x_0}(\xi)\,d\omega^U_{x_0}(\eta). \tag{1.6}$$

Equivalently, the Doob-Naïm energy of a function f above is the Dirichlet energy $\mathcal{E}^{\mathrm{ref}}(u,u)$, where $\mathcal{E}^{\mathrm{ref}}(\cdot,\cdot)$ is the Dirichlet energy corresponding to the reflected diffusion on $\overline{U}$, u is harmonic with respect to the generator of diffusion in U with boundary condition f on ∂U.

The expression (1.6) suggests that the Doob–Naïm energy should be viewed as a quadratic form corresponding to a self-adjoint non-local operator with respect to a reference measure that is mutually absolutely continuous with respect to harmonic measure $\omega^{\Omega}_{x_0}$. In other words, the Doob–Naïm formula is an expression for the kernel of an integro-differential (non-local) operator on the boundary associated to an elliptic (local) operator in the domain. Due to (1.4), this reference measure on the boundary is usually taken to be the surface measure on the boundary for reflected Brownian motion in smooth settings [Hsu]. In the smooth case, this integro-differential operator can be identified with the Dirichlet-to-Neumann (or voltage-to-current) map [Hsu]. However, in general, even on smooth domains for uniformly elliptic operators, the harmonic measure might differ significantly from the surface measure, possibly being singular [CFK]. It is worth mentioning that our results on the stable-like heat kernel estimates for boundary trace process also apply to situations when the harmonic measure is singular with respect to the surface measure.

From a probabilistic viewpoint, the choice of this reference measure on the boundary is equivalent to choosing a time parametrization for the trace process due to the Revuz correspondence [FOT, Theoren 5.1.7(i)]. More precisely, a suitable reference measure on the boundary determines a positive continuous additive functional supported on the boundary which can be considered as the *boundary local time* of the reflected diffusion. Heuristically, the boundary local time is a continuous, non-increasing process which 'measures' the time spent by a reflection diffusion on the boundary. If τ_t denotes the right continuous inverse of the boundary local time of the reflected diffusion X^{ref}_t, then the trace process on the boundary is given by $t \mapsto X^{\mathrm{ref}}_{\tau_t}$ and is a Markov process on the boundary that is symmetric with respect to the chosen reference measure on the boundary.

The above considerations naturally lead us to the study of harmonic measure and its estimates. Aikawa and Hirata obtain doubling properties and estimates of the harmonic measure for Brownian motion on uniform domains under the capacity density condition [AH, Lemmas 3.5 and 3.6]. These estimates and doubling properties generalize similar results obtained by Jerison and Kenig for NTA domains [JK, Lemma 4.8] and Dahlberg for Lipschitz domains [Dah, Lemma 1]. The key estimate on the harmonic measure $\omega^U_{x_0}(\cdot)$ for the diffusion started at x_0 that generalize the estimates for Brownian motion in [Dah, JK, AH] is given by

$$\omega^U_{x_0}(B(\xi,r)\cap\partial U)\asymp\frac{g_U(x_0,\xi_r)}{g_U(\xi'_r,\xi_r)}\asymp g_U(x_0,\xi_r)\,\mathrm{Cap}_{B(\xi,2r)}(B(\xi,r)) \tag{1.7}$$

for all $x_0 \in U, \xi \in \partial U$ such that $r < d(x_0, \xi)/A$ for a suitably chosen constant $A \in (1, \infty)$, where $\xi_r, \xi'_r \in U$ are chosen so that $\mathrm{dist}(\xi_r, U^c) \asymp r, \mathrm{dist}(\xi'_r, U^c) \asymp r, d(\xi, \xi_r) \asymp r, d(\xi_r, \xi'_r) \asymp r$. We refer to Theorem 4.6 for a precise statement for our estimate on harmonic measure. While our lower bound on the harmonic measure follows the same line of reasoning as [AH], our argument for the upper bound provides a new proof avoiding the delicate iteration argument (box argument) in [AH].

Next, we discuss our choice of the reference measure, denoted as μ, on the boundary ∂U. This measure corresponds either to the symmetric measure for the trace process or, equivalently, to the Revuz measure corresponding to the boundary's local time. As mentioned earlier, the Doob-Naïm formula suggests that the reference measure to be chosen mutually absolutely continuous with respect to the harmonic measure $\omega_{x_0}^U(\cdot)$. Therefore if U is bounded, we choose the reference measure μ on the boundary ∂U as the harmonic measure $\omega_{x_0}^U$ where the base point is sufficiently far away from the boundary such that $\mathrm{dist}(x_0, U^c) \gtrsim \mathrm{diam}(U)$ which guarantees good doubling properties at many scales due to (1.7). If U is bounded, then by the elliptic Harnack inequality the harmonic measure with different base points sufficiently far away from the boundary are comparable in the sense that they are mutually absolutely continuous with Radon-Nikodym derivative uniformly bounded above and below.

In cases where U is unbounded, a canonical measure exists on the boundary, unique up to a multiplicative constant. This measure is constructed from rescaled limits of harmonic measures along a sequence of base points that tend toward infinity. The consideration of such measures dates back to Kenig and Toro [KT, Corollary 3.2], who first studied this measure in the context of non-tangentially accessible (NTA) domains within Euclidean spaces. In our setting, the existence of such a measure follows from the boundary Harnack principle (Proposition 4.15). Following [BTZ, Lemma 3.5], we call such a measure on unbounded uniform domains as the *elliptic measure at infinity*.

Next, we describe how our choice of the reference measure above leads to the boundary trace process. We show that our choice of the reference measure above defines a positive continuous additive functional in the strict sense whose support is the topological boundary (Lemma 5.4 and Proposition 5.7) which can be thought of as the *boundary local time*. This in turn defines the trace process $\check{X}_t^{\mathrm{ref}}$ as $\check{X}_t^{\mathrm{ref}} := X_{\tau_t}^{\mathrm{ref}}$, where τ_t is the right continuous inverse of the positive continuous additive functional mentioned above and X_t^{ref} denotes the reflected diffusion on the uniform domain. We show that this boundary trace process admits a continuous heat kernel and obtain matching upper and lower bounds.

Before describing the stable-like heat kernel bounds of the boundary trace process, we recall a few properties of symmetric α-stable process on $\mathbb{R}^n$. For any $\alpha \in (0, 2), n \in \mathbb{N}$, the symmetric stable-process on $\mathbb{R}^n$ generated by the fractional Laplace operator $(-\Delta)^{\alpha/2}$ satisfies bounds on jump kernel $J(\cdot, \cdot)$, expected exit time from ball $\mathbb{E}_x[\tau_{B(x,r)}]$, and the heat kernel $p_t(\cdot, \cdot)$ that can be conveniently expressed in terms of the Lebesgue measure m, Euclidean distance $d(\cdot, \cdot)$ and the space-time scaling function $\phi(r) = r^\alpha$. There bounds are given by

$$J(x, y) \asymp \frac{1}{m(B(x, d(x, y)))\phi(d(x, y))}, \quad \mathbb{E}_x[\tau_{B(x,r)}] \asymp \phi(r),$$

and

$$p_t(x,y) \asymp \frac{1}{m(B(x,\phi^{-1}t))} \wedge \frac{t}{m(B(x,d(x,y)))\phi(d(x,y))}.$$

for all $x, y \in \mathbb{R}^n, t, r > 0$.

Unlike the symmetric stable process, the space-time scaling function of the boundary trace process depends both on the starting point in ∂U and distance; that is, $\Phi : \partial U \times (0,\infty) \to (0,\infty)$ instead of $\phi : (0,\infty) \to (0,\infty)$ as above. Except for this change, the bounds on the jump kernel, exit times and the heat kernel are exactly same as above. Next, we describe the space-time scaling function $\Phi(\cdot,\cdot)$ that governs the behavior the boundary trace process. If U is bounded, then the scaling function satisfies the two-sided bound

$$\Phi(\xi, r) \asymp g_U(\xi_r, x_0), \quad \text{for all } \xi \in \partial U, 0 < r < \operatorname{diam}(U)/A,$$

where $A > 1, x_0 \in U$ is the 'central' base point chosen as before for the reference measure, $\xi_r \in U$ is chosen so that $d(\xi, \xi_r) = r$, $\operatorname{dist}(\xi_r, U^c) \asymp r$ and $g_U(\cdot,\cdot)$ denotes the Green function of the diffusion process killed upon exiting U. In the case when U is unbounded, the scaling function satisfies the two-sided bound,

$$\Phi(\xi, r) \asymp h^U_{x_0}(\xi_r), \quad \text{for all } \xi \in \partial U, r > 0,$$

where ξ_r, x_0 is as above and $h^U_{x_0}(\cdot)$ is the unique positive harmonic function on U with Dirichlet (zero) boundary condition on ∂U normalized so that $h^U_{x_0}(x_0) = 1$. The existence and uniqueness of such a harmonic function is a well-known consequence of the boundary Harnack principle.

In order to prove the stable-like heat kernel bounds, it suffices to obtain stable-like bounds on the jump kernel and exit time. This follows from results of [CKW, GHH23, GHH23+] as shown in Theorem 2.32. The desired jump kernel bound is an easy consequence of the Doob-Naïm formula while the proof of exit time bound requires the heat kernel estimate on reflected diffusion obtained in [Mur23+, Theorem 2.8].

To illustrate the generality of our results, we list a few examples of diffusion and domain that have stable-like behavior of the boundary trace process of the corresponding reflected diffusion. Our results on stable-like heat kernel bounds for the boundary trace process applies to Brownian motion on Lipschitz and more generally non-tangentially accessible domains in $\mathbb{R}^n$. In particular, this class includes non-smooth domains such as the von Koch snowflake domain. More generally, reflected Brownian motion could be replaced a reflected diffusion generated by a uniformly elliptic operator or degenerate elliptic operators corresponding to A_2 weights [Mos, FKS]. Another class of examples include NTA domains in Heisenberg group equipped with Carnot-Carthéodory distance and the diffusion generated by the corresponding left-invariant sub-Laplacian satisfying the Hörmander condition as mentioned before [VSC]. Specific examples of NTA domains in this setting are given in [CG, CGN, Gre]. Our results on the heat kernel of boundary trace process also applies to the complement of the outer square boundary and the domain formed by removing the bottom line of the Sierpiński carpet. They are uniform domains (see [Lie22, Proposition 4.4] and[CQ, Proposition 2.4]) satisfying the capacity density condition for the Brownian motion on the Sierpiński carpet constructed in [BB92].

To summarize, the following are our main contributions:

(i) Two-sided estimates on the harmonic measure and the associated elliptic measure at infinity that are sharp up to multiplicative constants (Theorem 4.6 and Proposition 4.15).

(ii) The calculation of the Dirichlet form for the boundary trace process given by Doob-Naïm formula. Equivalently, this is an expression for the non-local operator on the boundary associated with a local (diffusion) operator on the domain. In particular, we show that the boundary trace process is a pure jump process (Theorem 5.12).

(iii) Heat kernel estimate for the boundary trace process that are similar to that of the symmetric stable process on Euclidean space (Theorem 5.19).

2 Preliminaries

2.1 Doubling metric space and doubling measures

Throughout this paper, we consider a metric space $(\mathcal{X}, d)$ in which $B(x,r) := B_d(x,r) := \{y \in \mathcal{X} \mid d(x,y) < r\}$ is relatively compact (i.e., has compact closure) for any $(x,r) \in X \times (0,\infty)$, and a Radon measure m on X with full support, i.e., a Borel measure m on X which is finite on any compact subset of X and strictly positive on any non-empty open subset of X. Such a triple $(\mathcal{X}, d, m)$ is referred to as a *metric measure space*. We set $\operatorname{diam}(A) := \sup_{x,y\in A} d(x,y)$ for $A \subset X$ ($\sup \varnothing := 0$).

In much of this work, we will be in the setting for a doubling metric space equipped with a doubling measure.

Definition 2.1. A metric d on $\mathcal{X}$ is said to be a doubling metric (or equivalently, $(\mathcal{X}, d)$ is a doubling metric space), if there exists $N \in \mathbb{N}$ such that every ball $B(x,R)$ can be covered by N balls of radii $R/2$ for all $x \in \mathcal{X}, R > 0$.

Next, we recall the closely related notion of doubling measures on subsets of $\mathcal{X}$.

Definition 2.2. Let $(\mathcal{X}, d)$ be a metric space and let $V \subset \mathcal{X}$. We say that a Borel measure m is *doubling on* V if $m(V) \neq 0$ and there exists $D_0 \geqslant 1$ such that

$$m(B(x,2r) \cap V) \leqslant D_0 m(B(x,r) \cap V), \quad \text{for all } x \in V \text{ and all } r > 0.$$

We say that a non-zero Borel measure m on $\mathcal{X}$ is *doubling*, if m is doubling on $\mathcal{X}$.

The basic relationship between these notions is that if there is a (non-zero) doubing measure on a metric space $(\mathcal{X}, d)$, then $(\mathcal{X}, d)$ is a doubling metric space. Conversely, every complete doubling metric space admits a doubling measure [Hei, Chapter 13]. By iterating the doubling condition, it is easy to see that for all m is a doubling measure on $(\mathcal{X}, d)$, then there exist $C \in (1,\infty), \beta \in (0,\infty)$ such that

$$\frac{m(B(y,R))}{m(B(x,r))} \leqslant C \left(\frac{d(x,y) + R}{r} \right)^{\beta} \quad \text{for all } x, y \in \mathcal{X}, 0 < r \leqslant R. \tag{2.1}$$

We recall the closely related reverse volume doubling condition. To this end, we recall the relevant definition.

Definition 2.3. We say that a metric space $(\mathcal{X}, d)$ is *uniformly perfect* if there exists $K_0 \in (1, \infty)$ such that for all $x \in \mathcal{X}, r > 0$ such that $B(x, r) \neq \mathcal{X}$, we have

$$B(x, r) \backslash B(x, K_0^{-1} r) \neq \varnothing.$$

We record the following result for later use.

Lemma 2.4 ([Hei, Exercise 13.1])**.** *Let m be a doubling measure on a uniformly perfect metric space $(\mathcal{X}, d)$. Then the measure m satisfies the following* reverse volume doubling *condition: there exist $C \in (1, \infty), \alpha \in (0, \infty)$ such that for all $x \in \mathcal{X}, 0 < r < R <$* $\operatorname{diam}(X, d)$ *such that*

$$\frac{m(B(x,R))}{m(B(x,r))} \geqslant C^{-1} \left(\frac{R}{r}\right)^{\alpha}. \tag{2.2}$$

2.2 Uniform domains

Let $U \subset \mathcal{X}$ be an open set. A *curve in* U is a continuous function $\gamma : [a, b] \to U$ such that $\gamma(0) = x, \gamma(b) = y$. We sometimes identify γ with it its image $\gamma([a, b])$, so that $\gamma \subset U$. The *length* of a curve $\gamma : [a, b] \to \mathcal{X}$ is

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) : a \leqslant t_0 < t_1 \ldots < t_n \leqslant b \right\}.$$

A metric space is a *length space* if $d(x, y)$ is equal to the infimum of the length of curves joining x and y. Let $U \subset \mathcal{X}$ be a connected open subset. We define the *intrinsic distance* d_U by

$$d_U(x, y) = \inf \left\{ L(\gamma) : \gamma : [0,1] \to U \text{ continuous}, \gamma(0) = x, \gamma(1) = y \right\}.$$

Definition 2.5. Let $c_U \in (0,1), C_U \in (1, \infty)$. A connected, non-empty, proper open set $U \subsetneq X$ is said to be a **length** (c_U, C_U)**-uniform domain** if for every pair of points $x, y \in U$, there exists a curve γ in U from x to y such that its length $\ell(\gamma) \leqslant C_U d(x, y)$ and for all $z \in \gamma$,

$$\delta_U(z) \geqslant c_U \min\left(\ell(\gamma_{x,z}), \ell(\gamma_{z,y})\right),$$

where $\gamma_{x,z}, \gamma_{z,y}$ are subcurves of γ from x and z and from z to y respectively and $\delta_U(z) = \operatorname{dist}(z, U^c)$. Such a curve γ is called a *length* (c_U, C_U)*-uniform curve.*

A connected, non-empty, proper open set $U \subsetneq X$ is said to be a (c_U, C_U)**-uniform domain** if for every pair of points $x, y \in U$, there exists a curve γ in U from x to y such that its diameter $\operatorname{diam}(\gamma) \leqslant C_U d(x, y)$ and for all $z \in \gamma$,

$$\delta_U(z) \geqslant c_U \min\left(d(x, z), d(y, z)\right).$$

Such a curve γ is called a (c_U, C_U)*-uniform curve.*

There are different definitions of uniform domains in the literature [Mar, Väi]. The above definition of uniform domain was introduced in [Mur23+] because of the advantage that this notion of uniform domain is preserved under a quasisymmetric change of metric in the underlying space. Furthermore, this definition also allows us to consider metric spaces that does not have non-constant rectifiable curves. We note that our definition of length uniform domain is what is usually called a uniform domain.

The following is a variant of [GyS, Proposition 3.20].

Lemma 2.6. *Let $(\mathcal{X}, d)$ be a complete, locally compact, length metric space and let $U \subset \mathcal{X}$ be an open, (c_U, C_U)-uniform domain for some $c_U \in (0,1), C_U \in (1,\infty)$. For any $\xi \in \partial U, r \in (0, \operatorname{diam}(U)/4)$, there exists $\xi_r \in U$ such that*

$$d(\xi, \xi_r) = r, \quad \delta_U(\xi_r) > \frac{c_U r}{2}. \tag{2.3}$$

Proof. Since $r < \operatorname{diam}(U)/4$, there exists a point $y \in U$ such that $d(\xi, y) > 2r$. By considering a (c_U, C_U)-uniform curve γ from a point $x \in B(\xi, r/2) \cap U$ to y and the continuity of $d(\xi, \cdot)$ along γ, there exists $\xi_r \in \gamma$ such that $d(\xi, \xi_r) = r$ and

$$\begin{aligned}\delta_U(\xi_r) &\geqslant c_U \min\left(d_U(x, \xi_r), d_U(y, \xi_r)\right) \geqslant c_U \min\left(d(x, \xi_r), d(y, \xi_r)\right) \\ &\geqslant c_U \min\left(d(\xi, \xi_r) - d(\xi, x), d(\xi, y) - d(\xi, \xi_r)\right) > \frac{c_U r}{2}.\end{aligned}$$

□

2.3 Dirichlet form and symmetric Hunt process

Let $(\mathcal{E}, \mathcal{F})$ be a *symmetric Dirichlet form* on $L^2(\mathcal{X}, m)$; that is, $\mathcal{F}$ is a dense linear subspace of $L^2(\mathcal{X}, m)$, and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a non-negative definite symmetric bilinear form which is *closed* ($\mathcal{F}$ is a Hilbert space under the inner product $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(\mathcal{X}, m)}$) and *Markovian* ($f^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(f^+ \wedge 1, f^+ \wedge 1) \leqslant \mathcal{E}(f, f)$ for any $f \in \mathcal{F}$). Recall that $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_c(\mathcal{X})$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(C_c(\mathcal{X}), \|\cdot\|_{\sup})$, and that $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with $\operatorname{supp}_m[f]$, $\operatorname{supp}_m[g]$ compact and $\operatorname{supp}_m[f - a\mathbb{1}_{\mathcal{X}}] \cap \operatorname{supp}_m[g] = \emptyset$ for some $a \in \mathbb{R}$. Here $C_c(\mathcal{X})$ denotes the space of $\mathbb{R}$-valued continuous functions on $\mathcal{X}$ with compact support, and for a Borel measurable function $f : \mathcal{X} \to [-\infty, \infty]$ or an m-equivalence class f of such functions, $\operatorname{supp}_m[f]$ denotes the support of the measure $|f|\,dm$, i.e., the smallest closed subset F of $\mathcal{X}$ with $\int_{\mathcal{X}\setminus F} |f|\,dm = 0$, which exists since $\mathcal{X}$ has a countable open base for its topology; note that $\operatorname{supp}_m[f]$ coincides with the closure of $\mathcal{X}\setminus f^{-1}(0)$ in $\mathcal{X}$ if f is continuous. The pair $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ of a metric measure space $(\mathcal{X}, d, m)$ and a strongly local, regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is termed a *metric measure Dirichlet space*, or an *MMD space* in abbreviation.

Associated with a Dirichlet form is a *strongly continuous contraction semigroup* $(T_t)_{t>0}$; that is, a family of symmetric bounded linear operators $T_t : L^2(\mathcal{X}, m) \to L^2(\mathcal{X}, m)$ such that

$$T_{t+s} f = T_t(T_s f), \quad \|T_t f\|_2 \leqslant \|f\|_2, \quad \lim_{t \downarrow 0} \|T_t f - f\|_2 = 0,$$

for all $t, s > 0$, $f \in L^2(\mathcal{X}, m)$. In this case, we can express $(\mathcal{E}, \mathcal{F})$ in terms of the semigroup as

$$\mathcal{F} = \{f \in L^2(X, m) : \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle < \infty\}, \quad \mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle, \tag{2.4}$$

for all $f \in \mathcal{F}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(X, m)$ [FOT, Theorem 1.3.1 and Lemmas 1.3.3 and 1.3.4]. It is known that P_t restricted to $L^2(X, m) \cap L^\infty(X, m)$ extends to a linear contraction on $L^\infty(X, m)$ [CF, pp. 5 and 6]. If $P_t 1 = 1$ (m a.e.) for all $t > 0$, we say that the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *conservative*.

According to a fundamental theorem of Fukushima, the MMD space corresponds to a symmetric Markov processes on $\mathcal{X}$ with continuous sample paths [FOT, Theorem 7.2.1 and 7.2.2]. We refer to [FOT, CF] for details of the theory of symmetric Dirichlet forms.

Recall that a Hunt process $X = \{X_t, t \geqslant 0; \mathbb{P}^x, x \in \mathcal{X}\}$ on a locally compact separable metric space $\mathcal{X}$ is a strong Markov process that is right continuous and quasi-left continuous on the one-point compactification $\mathcal{X}_\partial := \mathcal{X} \cup \{\partial\}$ of $\mathcal{X}$. A set $C \subset \mathcal{X}_\partial$ is said to be *nearly Borel measurable* if for any probability measure μ on X there are Borel sets A_1, A_2 such that $A_1 \subset C \subset A_2$ and

$$\mathbb{P}^\mu(\text{there is some } t \geqslant 0 \text{ such that } X_t \in A_2 \backslash A_1) = 0.$$

Let m be a Radon measure with full support on X. A Hunt process X is said to be *m-symmetric* if the transition semigroup is symmetric on $L^2(\mathcal{X}, m)$. For an m-symmetric Hunt process X on $\mathcal{X}$, a set $\mathcal{N} \subset \mathcal{X}$ is said to be *properly exceptional* for $\mathcal{X}$ if $\mathcal{N}$ is nearly Borel measurable, $m(\mathcal{N}) = 0$ and

$$\mathbb{P}^x(X_t \in \mathcal{X}_\partial \backslash \mathcal{N} \text{ and } X_{t-} \in \mathcal{X}_\partial \backslash \mathcal{N} \text{ for all } t > 0) = 1 \quad \text{for every } x \in \mathcal{X} \backslash \mathcal{N}.$$

The transition semigroup of the process X is a version of the strongly continuous semigroup $\{T_t; t \geqslant 0\}$ on $L^2(\mathcal{X}, m)$ corresponding to $(\mathcal{E}, \mathcal{F})$, see [FOT, Theorem 7.2.1]. Furthermore, for any non-negative Borel measurable $f \in L^2(\mathcal{X}, m)$ and $t > 0$,

$$P_t f(x) := \mathbb{E}_x[f(X_t)]$$

is a quasi-continuous version of $T_t f$ on $\mathcal{X}$. The Hunt process X associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is unique in the following sense (see [FOT, Theorem 4.2.8]): if X' is another Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$, then there is a common properly exceptional set outside which these two Hunt processes have the same transition functions.

We recall the definition of energy measure. Note that $fg \in \mathcal{F}$ for any $f, g \in \mathcal{F} \cap L^\infty(\mathcal{X}, m)$ by [FOT, Theorem 1.4.2-(ii)] and that $\{(-n) \vee (f \wedge n)\}_{n=1}^\infty \subset \mathcal{F}$ and $\lim_{n \to \infty} (-n) \vee (f \wedge n) = f$ in norm in $(\mathcal{F}, \mathcal{E}_1)$ by [FOT, Theorem 1.4.2-(iii)].

Definition 2.7. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. The *energy measure* $\Gamma(f, f)$ of $f \in \mathcal{F}$ associated with $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is defined, first for $f \in \mathcal{F} \cap L^\infty(\mathcal{X}, m)$ as the unique ($[0, \infty]$-valued) Borel measure on $\mathcal{X}$ such that

$$\int_{\mathcal{X}} g \, d\Gamma(f, f) = \mathcal{E}(f, fg) - \frac{1}{2} \mathcal{E}(f^2, g) \qquad \text{for all } g \in \mathcal{F} \cap C_c(\mathcal{X}), \tag{2.5}$$

and then by $\Gamma(f,f)(A) := \lim_{n\to\infty} \Gamma\big((-n) \vee (f \wedge n), (-n) \vee (f \wedge n)\big)(A)$ for each Borel subset A of X for general $f \in \mathcal{F}$.

Definition 2.8 (Local Dirichlet space and its energy measure)**.** For an open set $D \subset \mathcal{X}$ of an MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, we define the local Dirichlet space $\mathcal{F}_{\mathrm{loc}}(D)$ as

$$\mathcal{F}_{\mathrm{loc}}(D) := \left\{ f \,\middle|\, \begin{array}{l} f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable} \\ \text{functions on } D \text{ such that } f\mathbb{1}_V = f^{\#}\mathbb{1}_V \ m\text{-a.e. for some} \\ f^{\#} \in \mathcal{F} \text{ for each relatively compact open subset } V \text{ of } D \end{array} \right\} \tag{2.6}$$

and the energy measure $\Gamma_D(f,f)$ of $f \in \mathcal{F}_{\mathrm{loc}}(D)$ associated with $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is defined as the unique Borel measure on D such that $\Gamma_D(f,f)(A) = \Gamma(f^{\#}, f^{\#})(A)$ for any relatively compact Borel subset A of D and any $V, f^{\#}$ as in (2.6) with $A \subset V$; note that $\Gamma(f^{\#}, f^{\#})(A)$ is independent of a particular choice of such $V, f^{\#}$.

For $U \subset \mathcal{X}$, we define

$$\mathcal{F}(U) := \{f \in \mathcal{F}_{\mathrm{loc}}(U) : \int_U f^2\, dm + \int_U \Gamma_U(f,f) < \infty\}, \tag{2.7}$$

and the bilinear form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ as

$$\mathcal{E}^{\mathrm{ref}}(f,f) = \int_U \Gamma_U(f,f), \quad \text{for all } f \in \mathcal{F}(U). \tag{2.8}$$

The form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ need not be a regular Dirichlet form on $L^2(\overline{U}, m)$ in general. Nevertheless, Theorem 2.12 provides a sufficient condition for $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ to be a regular Dirichlet form on $L^2(\overline{U}, m|_{\overline{U}})$.

We recall the definition of extended Dirichlet space.

Definition 2.9 (Extended Dirichlet space)**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $\mathcal{F}_e$ denote its extended Dirichlet space. Recall that the *extended Dirichlet space* $\mathcal{F}_e$ of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is defined as the space of m-equivalence classes of functions $f \colon X \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n = f$ m-a.e. on X for some $\mathcal{E}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, that the limit $\mathcal{E}(f,f) := \lim_{n\to\infty} \mathcal{E}(f_n, f_n) \in \mathbb{R}$ exists, is independent of a choice of such $(f_n)_{n\in\mathbb{N}}$ for each $f \in \mathcal{F}_e$ and defines an extension of $\mathcal{E}$ to $\mathcal{F}_e \times \mathcal{F}_e$, and that $\mathcal{F} = \mathcal{F}_e \cap L^2(X, m)$; see [CF, Definition 1.1.4 and Theorem 1.1.5].

Every function in the extended Dirichlet space admits a quasi continuous version [FOT, Theorem 2.1.7]. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *transient* if there exists a bounded $g \in L^1(\mathcal{X}; m)$ that is strictly positive on $\mathcal{X}$ so that

$$\int_{\mathcal{X}} |u(x)| g(x) m(dx) \leqslant \mathcal{E}(u,u)^{1/2} \quad \text{for every } u \in \mathcal{F}.$$

If $(\mathcal{E}, \mathcal{F})$ is transient, then $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space [FOT, Theorem 1.5.3]. Denote by $\{T_t; t \geqslant 0\}$ the semigroup on $L^2(\mathcal{X}; m)$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. By [CF, Lemma 2.1.4(ii)] or [FOT, p. 40], $(\mathcal{E}, \mathcal{F})$ is transient if and only if there is some $g \in L^1(\mathcal{X}; m)$ that is strictly positive on $\mathcal{X}$ and satisfies

$$\int_{\mathcal{X}} gGg\, dm < \infty, \quad \text{where } Gg = \textstyle\int_0^\infty T_t g\, dt. \tag{2.9}$$

Definition 2.10 (Part process)**.** Let D be an open subset of $\mathcal{X}$. The part process X^D of X killed upon exiting D is a Hunt process on D; that is,

$$X_t^D := \begin{cases} X_t & \text{if } t > \tau_D, \\ \partial & \text{if } t \geqslant \tau_D, \end{cases}$$

where the life time of the part process is $\tau_D := \inf\{t > 0 : X_t \notin D\}$.

The associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ of the part process X^D is regular. Here $m|_D$ is the measure m restricted to the open set D, and

$$\mathcal{F}^0(D) = \left\{ f \in \mathcal{F} \;\middle|\; \begin{array}{l} f \text{ admits a quasi-continuous modification } \widetilde{f} \text{ such that} \\ \widetilde{f} = 0 \ \mathcal{E}\text{-q.e. on } D^c \end{array} \right\} \tag{2.10}$$

and $\mathcal{E}^D = \mathcal{E}$ on $\mathcal{F}^0(D)$ [CF, Exercise 3.3.7 and Theorem 3.3.9]. We denote the extended Dirichlet space corresponding to part process by $(\mathcal{F}^0(D))_e$. By [CF, Theorem 3.4.9] $(\mathcal{F}^0(D))_e$ can be alternately described in terms of the extended Dirichlet space $\mathcal{F}_e$ of $(\mathcal{E}, \mathcal{F})$ as

$$(\mathcal{F}^0(D))_e = \{f \in \mathcal{F}_e : \widetilde{f} = 0 \ \mathcal{E} - \text{q.e. on } \mathcal{X} \backslash D\}. \tag{2.11}$$

2.4 Sub-Gaussian heat kernel estimates

Let $\Psi : (0, \infty) \to (0, \infty)$ be a continuous increasing bijection of $(0, \infty)$ onto itself, such that for all $0 < r \leqslant R$,

$$C^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leqslant \frac{\Psi(R)}{\Psi(r)} \leqslant C \left(\frac{R}{r}\right)^{\beta_2}, \tag{2.12}$$

for some constants $1 < \beta_1 < \beta_2$ and $C > 1$. If necessary, we extend Ψ by setting $\Psi(\infty) = \infty$. Such a function Ψ is said to be a **scale function**. For Ψ satisfying (2.12), we define

$$\widetilde{\Psi}(s) = \sup_{r>0} \left(\frac{s}{r} - \frac{1}{\Psi(r)} \right). \tag{2.13}$$

Definition 2.11 (HKE(Ψ))**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space, and let $\{P_t\}_{t>0}$ denote its associated Markov semigroup. A family $\{p_t\}_{t>0}$ of non-negative Borel measurable functions on $\mathcal{X} \times \mathcal{X}$ is called the *heat kernel* of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, if p_t is the integral kernel of the operator P_t for any $t > 0$, that is, for any $t > 0$ and for any $f \in L^2(\mathcal{X}, m)$,

$$P_t f(x) = \int_X p_t(x, y) f(y)\, dm(y) \qquad \text{for } m\text{-almost all } x \in \mathcal{X}.$$

We say that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the **heat kernel estimates** HKE(Ψ), if there exist

$C_1, c_1, c_2, c_3, \delta \in (0,\infty)$ and a heat kernel $\{p_t\}_{t>0}$ such that for any $t > 0$,

$$p_t(x,y) \leqslant \frac{C_1}{m\big(B(x,\Psi^{-1}(t))\big)} \exp\left(-c_1 t \widetilde{\Psi}\left(c_2 \frac{d(x,y)}{t}\right)\right) \qquad \text{for } m\text{-almost all } x,y \in \mathcal{X}, \tag{2.14}$$

$$p_t(x,y) \geqslant \frac{c_3}{m\big(B(x,\Psi^{-1}(t))\big)} \qquad \text{for } m\text{-almost all } x,y \in \mathcal{X} \text{ with } d(x,y) \leqslant \delta\Psi^{-1}(t), \tag{2.15}$$

where Φ is as defined in (2.13).

We recall the following heat kernel estimate for reflected diffusions obtained in [Mur23+, Theorem 2.8].

Theorem 2.12 ([Mur23+, Theorem 2.8])**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the heat kernel estimate HKE(Ψ) for some scale function Ψ and let m be a doubling measure. Then for any uniform domain U, the bi-linear form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ is a strongly-local regular Dirichlet form on $L^2(\overline{U}, m)$. Moreover, the corresponding MMD space $(\overline{U}, d, m, \mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ satisfies the heat kernel estimate HKE(Ψ).*

Let $C_0(\mathcal{X})$ denote the space of all continuous functions vanishing at infinity. We recall that the sub-Gaussian heat kernel estimates implies the strong Feller property. The general theory of Dirichlet forms [FOT, Theorems 7.2.1 and 7.2.2] only guarantees the existence of diffusion process starting outside a properly exceptional set as recalled in §2.3. Nevertheless under sub-Gaussian heat kernel bounds, the Feller and strong Feller property allows us to define the diffusion starting from *every* point $x \in \mathcal{X}$ as we recall below.

Proposition 2.13 ([Lie15, Proposition 3.2])**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the heat kernel estimate HKE(Ψ) for some scale function Ψ and let m be a doubling measure. Then there exists a* continuous *heat kernel $(t,x,y) \mapsto p_t(x,y)$ corresponding to $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$. The Markovian transition function $(P_t)_{t\geqslant 0}$ on $\mathcal{X}$, defined by $P_t(x,dy) = p_t(x,y)\,m(dy), t > 0, x \in \mathcal{X}$, has the **Feller property** $P_t C_0(\mathcal{X}) \subset C_0(\mathcal{X})$ for all $t \geqslant 0$ and $\lim_{t\downarrow 0} \|P_t f - f\|_\infty = 0$ for any $f \in C_0(\mathcal{X})$, and the **strong Feller property**, i.e. $P_t f$ is continuous for any bounded Borel measurable function $f : \mathcal{X} \to \mathbb{R}$. In particular, there exists a diffusion process $((X_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in\mathcal{X}}, (\mathcal{F}_t)_{t\geqslant 0})$ whose transition densities are given by the continuous heat kernel.*

Due to Theorem 2.12 and Proposition 2.13, we often impose the following assumption.

Assumption 2.14. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the heat kernel estimate HKE(Ψ) for some scale function Ψ and let m be a doubling measure. We assume that the corresponding diffusion X be defined from every starting point in $\mathcal{X}$ with a continuous heat kernel, so that the transition function of X satisfies both the Feller and strong Feller properties as given in Proposition 2.13. Furthermore, for any uniform domain U in $\mathcal{X}$, we assume that the reflected diffusion X^{ref} corresponding to the MMD space $(\overline{U}, d, m\big|_{\overline{U}}, \mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ is also defined from every starting point in $\overline{U}$ with a continuous heat kernel, so that the transition function of X satisfies both the Feller and strong Feller properties as given in Proposition 2.13.

2.5 Harmonic functions and the elliptic Harnack inequality

We recall the definition of harmonic functions and the elliptic Harnack inequality.

Definition 2.15. Let D be an open subset of $\mathcal{X}$. We say a function $h \in \mathcal{F}_{\text{loc}}(D)$ is harmonic in D if

$$\mathcal{E}(h, v) = 0 \quad \text{for every } v \in C_c(D) \cap \mathcal{F}. \tag{2.16}$$

Here by the strong locality of $(\mathcal{E}, \mathcal{F})$, we can unambiguously define $\mathcal{E}(h, v) = \mathcal{E}(h^{\#}, v)$ where $h^{\#} \in \mathcal{F}$ and $h = h^{\#}$ m-a.e. in $\operatorname{supp}(v)$.

Definition 2.16. We say that an MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *elliptic Harnack inequality* (abbreviated as EHI), if there exist $C > 1$ and $\delta \in (0, 1)$ such that for all $x \in \mathcal{X}$, $r > 0$ and for any $h \in \mathcal{F}_{\text{loc}}(B(x, r))$ that is non-negative on $B(x, r)$ and harmonic on $B(x, r)$, we have

$$\operatorname*{ess\,sup}_{B(x,\delta r)} h \leqslant C \operatorname*{ess\,inf}_{B(x,\delta r)} h. \tag{EHI}$$

There is a close relationship between the heat kernel bounds HKE(Ψ) and the elliptic Harnack inequality EHI as we recall below.

Remark 2.17. If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is an MMD space that satisfies HKE(Ψ), then it satisfies the elliptic Harnack inequality. Conversely, if an MMD space satisfies the elliptic Harnack inequality, there is a suitable reparametrization of space and time by a quasisymmetric change of metric $\widetilde{d}$ and a smooth measure with full quasi support $\widetilde{m}$ such that the time-changed MMD space with respect to the new metric $\widetilde{d}$ satisfies HKE(Ψ) for some scale function Ψ [BM18, BCM].

We are often interested in harmonic functions with zero (or Dirichlet) boundary condition on some part of the boundary as defined below.

Definition 2.18 (Dirichlet boundary condition)**.** Let $V \subset U$ be open subsets of $\mathcal{X}$. Set

$$\begin{aligned}\mathcal{F}^0_{\text{loc}}(U, V) =& \{f \in L^2_{\text{loc}}(V, m) : \forall \text{ open } A \subset V \text{ relatively compact in } \overline{U} \text{ with} \\ & \operatorname{dist}(A, U \backslash V) > 0, \exists f^{\sharp} \in \mathcal{F}^0(U) : f^{\sharp} = f \; m\text{-a.e. on } A\}.\end{aligned}$$

We say that a function $u : V \to \mathbb{R}$ satisfies *Dirichlet boundary condition on the boundary* $\partial U \cap \overline{V}$ if $u \in \mathcal{F}^0_{\text{loc}}(U, V)$. Note that we always have $\mathcal{F}^0_{\text{loc}}(U, V) \subset \mathcal{F}_{\text{loc}}(V)$.

The following lemma shows that harmonicity and Dirichlet boundary condition are preserved under uniform convergence.

Lemma 2.19. *(a) Let $U \subset \mathcal{X}$ be open and let $h_n \in \mathcal{F}_{\text{loc}}(U), n \geqslant 1$ be a sequence of locally bounded harmonic functions such that h_n converges to h uniformly on compact subsets of U. Then $h \in \mathcal{F}_{\text{loc}}(U)$ and h is harmonic in U.*

(b) Let $V \subset U \subset \mathcal{X}$ be such that U and V are open subsets of $\mathcal{X}$ and let $h_n \in \mathcal{F}^0_{\text{loc}}(U, V), n \geqslant 1$ be a sequence of bounded harmonic functions in V such that for any $A \subset V$ relatively compact in $\overline{U}$ with $\operatorname{dist}(A, U \backslash V) > 0$, h_n converges to h uniformly on A. Then $h \in \mathcal{F}^0_{\text{loc}}(U, V)$ and is harmonic in V.

Proof. (a) Let V be relatively compact open subset of U. Since $(\mathcal{X}, d)$ is locally compact there is a compact neighborhood of $\overline{V}$, say W such that $\overline{V} \subset W \subset U$. Since $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, there exists a $\phi \in C_c(U)$ such that $0 \leqslant \phi \leqslant 1$ and $\phi\big|_{\overline{V}} \equiv 1$ and $\phi\big|_{W^c} \equiv 0$. Since h_i is locally bounded and $\mathrm{supp}[\phi]$ is compact, by [FOT, Theorem 1.4.2(ii)] we obtain $h_i\phi \in \mathcal{F}$. Since $h_n \to h$ uniformly on compact subsets of U, we have that ϕh_n converges to ϕh in $L^2(m)$. We claim that $\phi h_n, n \in \mathbb{N}$ is an $\mathcal{E}_1$-Cauchy sequence that converges to $\phi h \in \mathcal{F}$. To see this note that

$$\begin{aligned}\mathcal{E}(\phi(h_i - h_j), \phi(h_i - h_j)) &= \int_W (h_i - h_j)^2 \, d\Gamma(\phi, \phi) + \mathcal{E}(h_i - h_j, \phi^2(h_i - h_j)) \\ &= \int_W (h_i - h_j)^2 \, d\Gamma(\phi, \phi) \quad (\text{since } h_i - h_j \text{ is harmonic on } U)\end{aligned} \tag{2.17}$$

Since h_i converges uniformly on W, we obtain that ϕh_i is a $\mathcal{E}_1$-Cauchy sequence whose limit is ϕh_i. By (2.17) and $\lim_{i\to\infty} \phi h_i = h$ for m-almost everywhere on V, we conclude that $h \in \mathcal{F}_{\mathrm{loc}}(U)$.

Let $\psi \in C_c(U) \cap \mathcal{F}$. Let V be a relatively compact open subset containing $\mathrm{supp}[\psi]$. Then choosing ϕ as above, by strong locality and harmonicity of h_i we obtain

$$\mathcal{E}(h, \psi) = \mathcal{E}(\phi h, \psi) = \lim_{i\to\infty} \mathcal{E}(\phi h_i, \psi) = \lim_{i\to\infty} \mathcal{E}(h_i, \psi) = 0.$$

Therefore h is harmonic in U.

(b) Let $A \subset V$ be open such that A is relatively compact in $\overline{U}$ with $\mathrm{dist}(A, U\backslash V) > 0$. Since $\mathcal{X}$ is locally compact, there exists a neighborhood W of $\overline{A}$ such that $\overline{W}$ is compact and satisfies $\mathrm{dist}(\overline{W}, U\backslash V) > 0$. Therefore, there exists $\phi \in C_c(\mathcal{X}) \cap \mathcal{F}$ such that ϕ is $[0,1]$-valued, $\phi\big|_W \equiv 1$ and $\mathrm{supp}[\phi] \cap (U\backslash V) = \varnothing$. Let $\widehat{h}_i \in \mathcal{F}^0(U)$ be such that $h_i = \widehat{h}_i$ m-almost everywhere on A for all $i \in \mathbb{N}$. By replacing $\widehat{h}_i$ with $(-M_i \vee \widehat{h}_i) \wedge M_i$, where $M_i = \sup_A |h_i|$, we may assume that $\widehat{h}_i \in L^\infty \cap \mathcal{F}^0(U)$. Therefore $\phi\widehat{h}_i \in \mathcal{F}^0(U)$ is such that it admits a quasi continuous modification which vanishes quasi-everywhere on V^c for all $i \in \mathbb{N}$. Therefore $\phi\widehat{h}_i \in \mathcal{F}^0(V)$ for all $i \in \mathbb{N}$. Using the harmonicity of h_i in V and the same argument as used in (2.17), we conclude that the sequence $\phi\widehat{h}_i \in \mathcal{F}^0(V)$ is $\mathcal{E}_1$-Cauchy and converges to $\phi h \in \mathcal{F}^0(V)$. Since $\phi h = h$ for m-almost every A, we conclude that $h \in \mathcal{F}^0(U, V)$. The assertion that h is harmonic in V follows from (a). □

Remark 2.20. The argument used in the proof of Lemma 2.19 implies the following facts.

(a) If $h_n \in \mathcal{F}_{\mathrm{loc}}(U), n \geqslant 1$ is a sequence of locally bounded, harmonic functions such that h_n converges to h uniformly on compact subsets of U, then for any $\phi \in C_c(U) \cap \mathcal{F}$, the sequence $\phi h_n \in \mathcal{F}, n \in \mathbb{N}$ is $\mathcal{E}_1$-Cauchy and converges to $\phi h \in \mathcal{F}$.

(b) Let $h_n \in \mathcal{F}^0_{\mathrm{loc}}(U,V), n \geqslant 1$ be a sequence of bounded harmonic functions in V satisfying the assumption of Lemma 2.19(b) and let $h \in \mathcal{F}^0_{\mathrm{loc}}(U,V)$ be the limit. Let us extend h_n, h to by setting $h_n\big|_{U^c} = h\big|_{U^c} \equiv 0$ for all $n \in \mathbb{N}$. Then for any $\phi \in C_c(\mathcal{X}) \cap \mathcal{F}$ such that $\mathrm{supp}(\phi) \subset V \cup U^c, \mathrm{dist}(\mathrm{supp}(\phi), U \backslash V) > 0$, we have $h_n\phi \in \mathcal{F}$ for all $n \in \mathbb{N}$ and converges in $\mathcal{E}_1$-norm to $h\phi \in \mathcal{F}$.

Harnack inequality is often used along a chain of balls. We recall the definition of Harnack chain – see [JK, Section 3]. For a ball $B = B(x,r)$, we use the notation $M^{-1}B$ to denote the ball $B(x, M^{-1}r)$.

Definition 2.21 (Harnack chain)**.** Let $D \subsetneq \mathcal{X}$ be a connected open set. For $x, y \in U$, an M**-Harnack chain from** x **to** y in U is a sequence of balls $B_1, B_2, \ldots, B_n$ each contained in U such that $x \in M^{-1}B_1, y \in M^{-1}B_n$, and $M^{-1}B_i \cap M^{-1}B_{i+1} \neq \varnothing$, for $i = 1, 2, \ldots, n-1$. The number n of balls in a Harnack chain is called the *length* of the Harnack chain. For a domain D write $N_D(x, y; M)$ for the length of the shortest M-Harnack chain in D from x to y.

Let $K \geqslant 1$. We say that $(\mathcal{X}, d)$ is K**-relatively ball connected** if for any $\epsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $x_0 \in \mathcal{X}, x, y \in B(x_0, r)$, we have

$$N_{B(x_0,Kr)}(x, y; \epsilon^{-1}) \leqslant N(\epsilon). \tag{2.18}$$

Remark 2.22. Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies the elliptic Harnack inequality with constants C_H and δ. If u is a positive continuous harmonic function on a domain D, then

$$C_H^{-N_D(x_1,x_2;\delta^{-1})} u(x_1) \leqslant u(x_2) \leqslant C_H^{N_D(x_1,x_2;\delta^{-1})} u(x_1). \tag{2.19}$$

for all $x_1,\ x_2 \in D$.

The following lemma lists some useful estimates on length of Harnack chains.

Lemma 2.23. *(a) ([BCM, Theorem 5.4]) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space satisfying the elliptic Harnack inequality and such that $(\mathcal{X}, d)$ satisfies the metric doubling property. Then there exists $K > 1$ such that $(\mathcal{X}, d)$ is K-relatively ball connected.*

(b) Let $(\mathcal{X}, d)$ be a locally compact, separable space that satisfies the metric doubling property. Let $U \subsetneq \mathcal{X}$ be a (c_U, C_U)-uniform domain in $(\mathcal{X}, d)$. Then for each $M > 1$ there exists $C \in (0, \infty)$, depending only on c_U, C_U and M, such that for all $x, y \in U$

$$N_U(x, y; M) \leqslant C \log\left(\frac{d(x,y)}{\min(\delta_U(x), \delta_U(y))} + 1\right) + C. \tag{2.20}$$

Proof. The conclusion in (a) is contained in [BCM, Theorem 5.4].

Let γ be (c_U, C_U)-uniform curve between $x, y \in U$. Without loss of generality, we may assume $\delta_U(x) \leqslant \delta_U(y)$. Since

$$\delta_U(z) \geqslant \max\left(c_U \min(d(x,z), d(y,z)), \delta_U(x) - d(x,z), \delta_U(y) - d(y,z)\right) \text{ for any } z \in \gamma,$$

we have

$$\delta_U(z) \geqslant c_U \delta_U(x)/2. \tag{2.21}$$

If $d(x,y) \leqslant 4\delta_U(x)$, we choose a maximal $M^{-1}c_U\delta_U(x)/2$ subset of γ. Observing that $\gamma \subset B(x, 2C_U d(x,y)) \subset B(x, 8C_U\delta_U(x))$ and using the metric doubling property we obtain the desired upper bound.

For $i \in \mathbb{N}$, choose $z_i \in \gamma$ such that $d(x, z_i) = 2^{-i}d(x,y)$ and such that z_{i+1} lies on the subcurve from x to z_i. Note that

$$d(z_i, z_{i+1}) \leqslant 2^{-i+1}d(x,y), \quad \delta_U(z_i) \geqslant c_U 2^{-i} d(x,y) \quad \text{for all } i \geqslant 1.$$

First we show that

$$N_U(z_i, z_{i+1}; M) \lesssim 1 \quad \text{for all } i \geqslant 1.$$

To see this, we choose a maximal $M^{-1}c_U^2 2^{-i-2}d(x,y)$ subset N_i of a (c_U, C_U)-uniform curve γ_i from z_i to z_{i+1}. Since the balls $\{B(n, M^{-1}c_U^2 2^{-i-2}d(x,y)) : n \in N_i\}$ cover γ_i and $\operatorname{diam}(\gamma_i) \leqslant C_U 2^{-i+1}d(x,y)$, and are contained in U by (2.21), the metric doubling property [Hei, Exercise 10.17] implies that

$$N_U(z_i, z_{i+1}; M) \lesssim \# N_i \lesssim 1 \quad \text{for all } i \geqslant 1. \tag{2.22}$$

Let $k \in \mathbb{N}$ be the smallest number such that $z_{k+1} \in B(x, M^{-1}\delta_U(x))$, so that $k \asymp 1 + \log\left(\frac{d(x,y)}{\delta_U(x)} + 1\right)$. By joining M-Harnack chains of length $N_U(z_i, z_{i+1}; M)$ from z_i to z_{i+1} successively and using the ball $B(x, M^{-1}\delta_U(x))$, we obtain a M-Harnack chain from x to z_1 they yields the estimate

$$N_U(x, z_1; M) \leqslant 1 + \sum_{i=1}^{k} N_U(z_i, z_{i+1}; M) \lesssim \log\left(\frac{d(x,y)}{\delta_U(x)} + 1\right) + 1. \tag{2.23}$$

Similarly for $i \in \mathbb{N}$, choose $w_i \in \gamma$ such that $d(y, w_i) = 2^{-i}d(x,y)$ and such that w_{i+1} lies on the subcurve from w_i to y. Similar to (2.23), we obtain

$$N_U(y, w_1; M) \leqslant 1 + \sum_{i=1}^{k} N_U(w_i, w_{i+1}; M) \lesssim \log\left(\frac{d(x,y)}{\delta_U(y)} + 1\right) + 1. \tag{2.24}$$

Since $\delta_U(z_1) \wedge \delta_U(w_1) \geqslant c_U d(x,y)/2$ and $d(z_1, w_1) \leqslant 2d(x,y)$, by the same argument as (2.22), we have

$$N_U(z_1, w_1; M) \lesssim 1. \tag{2.25}$$

By (2.23), (2.24) and (2.25), we conclude (2.20). □

We record a few more consequences of Harnack chaining.

Lemma 2.24. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space satisfying the elliptic Harnack inequality and let $U \subset \mathcal{X}$ be a uniform domain. There exist $A_0, A_1, C_1 \in (1, \infty)$ and $\gamma \in (0, \infty)$ such that for any $\xi \in \partial U, 0 < r < R < \operatorname{diam}(U, d/)/A_1$ and for any function*

$h : U \cap B(\xi, A_0 R) \to (0,\infty)$ *non-negative, continuous and harmonic on* $U \cap B(\xi, A_0R)$ *then we have*

$$C_1^{-1}\left(\frac{r}{R}\right)^{\gamma} h(\xi_r) \leqslant h(\xi_R) \leqslant C_1\left(\frac{R}{r}\right)^{\gamma} h(\xi_r), \quad \text{where } \xi_R, \xi_r \text{ are as given in Lemma 2.6.} \tag{2.26}$$

Furthermore if ξ_R, ξ_R' *are two points that satisfy the conclusion of Lemma 2.6, that is*

$$d(\xi,\xi_R) = d(\xi,\xi_R') = R, \quad \delta_U(\xi_R) \wedge \delta_U(\xi_R') > \frac{c_U r}{2},$$

then

$$C_1^{-1} h(\xi_R') \leqslant h(\xi_R) \leqslant C_1 h(\xi_R'). \tag{2.27}$$

Proof. Let $\delta \in (0,1)$ denote the constant in EHI. By Lemma 2.23(b), for any $\xi \in \partial U, 0 < r < R$, we have $N_U(\xi_r, \xi_R; \delta^{-1}) \leqslant C_1$, where C_1 only depends on δ and the constants associated to the uniformity of U. By Lemma 2.6 and the proof of Lemma 2.23(b), there exist $A_0, A_1 \in (1,\infty)$ depending only on δ and the constants associated to the uniformity of U such that for all $\xi \in \partial U, 0 < r < R < \operatorname{diam}(U,d)$

$$N_U(\xi_r,\xi_R;\delta^{-1}) \leqslant N_{U\cap B(\xi,A_0R)}(\xi_r,\xi_R;\delta^{-1}) \leqslant C_1\left(1+\log(R/r)\right). \tag{2.28}$$

The estimate (2.26) now follows from (2.28) and Remark 2.22. The estimate (2.27) also follows from the same argument. □

2.6 Trace Dirichlet form

Given an MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and $A \subset \mathcal{X}$, we define its 1-capacity as

$$\operatorname{Cap}_1(A) := \inf\left\{\mathcal{E}_1(f,f) \,\middle|\, f \in \mathcal{F},\ f \geqslant 1\ m\text{-a.e. on a neighborhood of } A\right\}, \tag{2.29}$$

where $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(\mathcal{X},m)}$ as defined before.

Definition 2.25 (Smooth measures)**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. A Radon measure μ on $\mathcal{X}$, i.e., a Borel measure μ on $\mathcal{X}$ which is finite on any compact subset of $\mathcal{X}$, is said to be *smooth* if μ charges no set of zero capacity (that is, $\mu(A) = 0$ for any Borel subset A of $\mathcal{X}$ with $\operatorname{Cap}_1(A) = 0$).

For example, the energy measure $\Gamma(f,f)$ of $f \in \mathcal{F}_e$ is smooth by [FOT, Lemma 3.2.4]. An essential feature of a smooth Radon measure μ on $\mathcal{X}$ is that the μ-equivalence class of each $f \in \mathcal{F}_e$ is canonically determined by considering a *quasi-continuous* m-version of f, which exists by [FOT, Theorem 2.1.7] and is unique *q.e.* (i.e., up to sets of capacity zero) by [FOT, Lemma 2.1.4]; see [FOT, Section 2.1] and [CF, Sections 1.2, 1.3 and 2.3] for the definition and basic properties of quasi-continuous functions with respect to a regular symmetric Dirichlet form. In what follows, we always consider a quasi-continuous m-version of $f \in \mathcal{F}_e$.

An increasing sequence $\{F_k; k \geqslant 1\}$ of closed subsets of an MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is said to be a *nest* if $\bigcup_{k \geqslant 1} \mathcal{F}_{F_k}$ is $\sqrt{\mathcal{E}_1}$-dense in $\mathcal{F}$, where

$$\mathcal{F}_{F_k} := \{f \in \mathcal{F} \mid f = 0 \text{ } m\text{-a.e. on } \mathcal{X} \backslash F_k\}.$$

Recall that $D \subset \mathcal{X}$ is *quasi-open* if there exists a nest $\{F_n\}$ such that $D \cap F_n$ is an open subset of F_n in the relative topology for each $n \in \mathbb{N}$. The complement in $\mathcal{X}$ of a quasi-open set is called *quasi-closed*. We recall the definition of a quasi-support of a smooth measure [CF, Definition 3.3.4].

Definition 2.26 (Quasi-support)**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let μ be a smooth Radon measure on $\mathcal{X}$. A set $F \subset \mathcal{X}$ is said to be the *quasi-support* of μ if it satisfies:

(a) F is quasi-closed and $\mu(\mathcal{X} \backslash F) = 0$.

(b) If $\widetilde{F}$ is another set with property (a), then $\mathrm{Cap}_1(F \backslash \widetilde{F}) = 0$.

The quasi-support of a smooth measure is unique up to q.e. equivalence; that is, if F_1 and F_2 are two quasi-supports of a smooth Radon measure μ, then $\mathrm{Cap}_1(F_1 \Delta F_2) = 0$.

The quasi-support can be described more explicitly in terms of the corresponding positive continuous additive functional (PCAF) which we recall below. Consider a m-symmetric *Hunt process* $X = \{\Omega, \mathcal{M}, X_t, t \geqslant 0, \mathbb{P}_x\}$, where $\mathcal{N}$ is a properly exceptional set for the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ and $(\Omega, \mathcal{M}, \mathbb{P}_x)$. For any measure ν on X, we denote by $\mathbb{P}_\nu$ the measure $\mathbb{P}_\nu(A) = \int_X \mathbb{P}_x(A)\, d\nu(x)$. Any function f on M is extended to $\mathcal{X}_\partial = \mathcal{X} \cup \{\Delta\}$ by setting $f(\Delta) = 0$, where Δ denotes the cemetery state. The set $\mathcal{X}_\partial$ as a topological space is the one point compactification of $\mathcal{X}$. Let $(\mathcal{M}_t)_{0 \leqslant t \leqslant \infty}$ denote the minimum augmented admissible filtration on Ω.

A collection of random variables $A := \{A_s : \Omega \to \mathbb{R}_+ | s \in \mathbb{R}_+\}$, is called a *positive continuous additive functional* (for short, a PCAF), if it satisfies the following conditions:

(i) $A_t(\cdot)$ is $(\mathcal{M}_t)$-measurable,

(ii) there exist a set $\Lambda \in \mathcal{M}_\infty$ and an exceptional set $\mathcal{N} \subset M$ for X such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in M \backslash \mathcal{N}$ and $\theta_t \Lambda \subset \Lambda$ for all $t > 0$, where θ_t denotes the shift map on Ω.

(iii) For any $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is continuous, non-negative with $A_0(\omega) = 0$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geqslant \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for any $s, t \geqslant 0$. Here $\zeta(\cdot)$ denotes the life time of the process.

The sets Λ and $\mathcal{N}$ are referred to as a *defining set* and *exceptional set* of the PCAF A_t respectively. If $\mathcal{N}$ can be taken to the empty set, then we say that A_t is a PCAF in the *strict sense*.

A measure ν is called the *Revuz measure* of the PCAF A, if and only if for any non-negative Borel functions h and f,

$$\mathbb{E}^{h \cdot \mu} \left(\int_0^t f(X_s(\omega))\, dA_s(\omega) \right) = \int_0^t \langle f \cdot \nu, P_s h \rangle\, ds, \tag{2.30}$$

where P_s denotes the Markov semigroup corresponding to the Hunt process. By [FOT, Theorem 5.1.4], the Revuz measure ν is uniquely determined by A. Conversely, given a smooth Radon measure μ, there exists a PCAF A whose Revuz measure is μ [FOT, Theorem 5.1.4]. Let us consider a PCAF A whose Revuz measure is a smooth Radon measure μ and let N denote the exceptional set of the PCAF A. Define the *support* of A as

$$F := \{x \in \mathcal{X}\backslash N : \mathbb{P}_x(R = 0) = 1\}, \quad R(\omega) := \inf\{t > 0 : A_t(\omega) > 0\}. \tag{2.31}$$

Then F is nearly Borel measurable, finely closed, quasi closed and is a quasi support of the smooth measure μ [CF, p.175, Theorem 5.2.1(i)]. The right continuous inverse of such a PCAF A is defined by

$$\tau_t(\omega) := \begin{cases} \inf\{s > 0 : A_s(\omega) > t\} & \text{if } \lim_{r\uparrow\zeta(\omega)} A_r(\omega) < t \\ \infty & \text{otherwise.} \end{cases}$$

By [CF, Theorem 5.2.1(ii)], the process

$$\check{X}_t(\omega) := X_{\tau_t(\omega)}(\omega), t \geqslant 0, \quad \check{\zeta}(\omega) = \lim_{r\uparrow\zeta(\omega)} A_r(\omega) \tag{2.32}$$

is a μ-symmetric Markov process on the support F of A as given in (2.31).

We recall the definition of 0-order hitting distribution.

Definition 2.27 (Harmonic measure; hitting distribution)**.** Let F be nearly Borel measurable and quasi-closed subset of $\mathcal{X}$. Let $\sigma_F := \inf\{t > 0 : X_t \in F\}$ be the first hitting time of the set F. The *hitting distribution* H_F is defined as

$$H_F(x, A) := \mathbb{P}_x[X_{\sigma_F} \in A, \sigma_F < \infty], \quad \text{for all } x \in \mathcal{X}.$$

For any function $u \in \mathcal{F}_e$, we define

$$H_F\widetilde{u}(x) := \mathbb{E}_x[\widetilde{u}(X_{\sigma_F})\mathbb{1}_{\{\sigma_F<\infty\}}], \tag{2.33}$$

where $\widetilde{u}$ is a quasi-continuous version of u. Finally, if $U \subset \mathcal{X}$ is an open set, we define the *harmonic measure* $\omega_x^U(A)$ as

$$\omega_x^U(A) := H_{\mathcal{X}\backslash U}(x, A). \tag{2.34}$$

We recall some basic properties of the harmonic measure.

Lemma 2.28. *Let the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and the corresponding diffusion satisfy Assumption 2.14. Let D be a non-empty open set and let $U \subset \mathcal{X}$ be a uniform domain.*

(a) ([Lie15, Lemma 3.2]) For any $x \in D$, the measure ω_x^D charges no set of zero capacity and is supported on ∂D.

(b) ([Lie15, Lemma 3.2]) For any bounded Borel function $f : \partial D \to \mathbb{R}$, the map $h : D \to \mathbb{R}$ defined by

$$h(x) := \int_{\partial U} f(y)\, \omega_x^D(dy)$$

belongs to $\mathcal{F}_{\mathrm{loc}}(D)$ and is continuous and $\mathcal{E}$-harmonic in D.

(c) For any $x, y \in D$, we have $\omega_x^D \ll \omega_y^D$.

(d) For any $x \in U$, ∂U is a quasi support of ω_x^U with respect to Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ on $L^2(\overline{U}, m|_{\overline{U}})$.

Proof. (a,b) That ω_x^D is supported on ∂D follows from that property that (X_t) has continuous sample paths. The remaining properties are proved in [Lie15, Lemma 3.2]. Although [Lie15, Lemma 3.2] assumes that D is a relatively compact open subset of $\mathcal{X}$, the proofs presented there work for an arbitrary open subset.
(c) Let $A \subset \partial D$ be a Borel subset such that $\omega_y^D(A) = 0$. By (b), the function $h_A(z) := \int_{\partial D} \mathbb{1}_A(\xi)\omega_z^D(d\xi) = \omega_z^D(A)$ on D is continuous, non-negative, $\mathcal{E}$-harmonic in D and belongs to $\mathcal{F}_{\mathrm{loc}}(D)$. Since $h_A(y) = \omega_y^D(A) = 0$, by the elliptic Harnack inequality we conclude that h_A is identically zero on D. In particular $\omega_x^D(A) = 0$ and hence $\omega_x^D \ll \omega_y^D$.
(d) Let $x \in U$. For $y \in \overline{U}$, we define the 1-order hitting distribution for the reflected Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ as

$$H^1_{\partial U}(y, B) := \mathbb{E}_y^{\mathrm{ref}}\left[e^{-\sigma_{\partial U}}\mathbb{1}_B(X^{\mathrm{ref}}_{\sigma_{\partial U}})\mathbb{1}_{\{\sigma_{\partial U} < \infty\}}\right], \quad \sigma_{\partial U} = \inf\{t > 0 : X_t^{\mathrm{ref}} \in \partial U\}$$

for any Borel set $B \subset \partial U$. Then by (c), we conclude that the first order hitting measure $H^1_{\partial U}(y, \cdot)$ is absolutely continuous with respect to ω_x^U for all $y \in U$. Since $m(\partial U) = 0$, by [FOT, Exercise 4.6.1] we conclude that ∂U is a quasi support of ω_x^U for all $x \in U$. ☐

Definition 2.29 (Time-changed Dirichlet form)**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. If μ is a smooth Radon measure, it defines a time change of the process whose associated Dirichlet form is called the trace Dirichlet form and denoted by $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ (see [FOT, Section 6.2] and [CF, Section 5.2]). Let μ is a smooth measure with quasi support F that is finely closed and nearly Borel measurable. Let

$$\mathcal{F}^\mu = \{f\big|_F : f \in \mathcal{F}_e, f\big|_F \in L^2(F, \mu), \text{ and } f \text{ is quasi-continuous}\},$$

where we identify functions that coincide q.e. on F. By [CF, Theorem 3.3.5], two functions in $\mathcal{F}^\mu$ agree q.e. on F if and only if they agree μ-a.e. We define a quadratic form by setting

$$\mathcal{E}^\mu(u\big|_F, v\big|_F) = \mathcal{E}(H_F u, H_F v) \tag{2.35}$$

for all quasi-continuous functions $u, v \in \mathcal{F}_e$

In probabilistic terms, $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is a regular Dirichlet form corresponding to the time-changed process $\check{X}_t$ defined in (2.32) where the positive continuous additive functional $(A_t)_{t \geqslant 0}$ of (X_t) has Revuz measure μ; see [FOT, Section 6.2] and [CF, Theorem 5.2.2] for details.

2.7 Stable-like heat kernel bounds

We recall a generalization of scale function considered in §2.4 from [BCM, Defintion 7.2] (see also [BM18, Definition 5.4]). Let $(\mathcal{X}, d)$ be a metric space.

Definition 2.30. We say that a function $\Phi : \mathcal{X} \times (0,\infty) \to (0,\infty)$ on a metric space $(\mathcal{X}, d)$ is a *regular scale function* if $\Phi(x,\cdot) : (0,\infty) \to (0,\infty)$ is an increasing homeomorphism for all $x \in \mathcal{X}$, and there exist $C_1, \beta_1, \beta_2 > 0$ such that, for all $x, y \in \mathcal{X}$, $0 < s \leqslant r \leqslant \operatorname{diam}(\mathcal{X}, d)$ we have, writing $d(x,y) = R$,

$$C_1^{-1}\left(\frac{r}{R \vee r}\right)^{\beta_2}\left(\frac{R \vee r}{s}\right)^{\beta_1} \leqslant \frac{\Phi(x,r)}{\Phi(y,s)} \leqslant C_1\left(\frac{r}{R \vee r}\right)^{\beta_1}\left(\frac{R \vee r}{s}\right)^{\beta_2}. \tag{2.36}$$

The definition in [BCM, Defintion 7.2] does not state that $\Phi(x,\cdot) : (0,\infty) \to (0,\infty)$ is a homeomorphism but this condition can be achieved by replacing Φ with a comparable function if necessary as we will see in the proof of Lemma 5.2.

Definition 2.31. Let $(\mathcal{X}, d)$ be a metric space with a Radon measure m equipped with full support. Let $\Phi : \mathcal{X} \times (0,\infty) \to (0,\infty)$ be a regular scale function. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(\mathcal{X}, m)$.

(a) (Jump kernel bound) We say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ satisfies $\mathrm{J}(\Phi)$ if there exist a symmetric function $J : (\mathcal{X} \times \mathcal{X}) \backslash \mathcal{X}_{\mathrm{diag}} : (0,\infty)$ and $C \in (1,\infty)$ such that

$$\frac{C^{-1}}{m(B(x,d(x,y)))\Phi(x,d(x,y))} \leqslant J(x,y) \leqslant \frac{C}{m(B(x,d(x,y)))\Phi(x,d(x,y))},$$

for all $(x,y) \in (\mathcal{X} \times \mathcal{X}) \backslash \mathcal{X}_{\mathrm{diag}}$, and for all $u \in \mathcal{F}$, we have

$$\mathcal{E}(u,u) = \int_{\mathcal{X}} \int_{\mathcal{X}} (u(x) - u(y))^2 J(x,y)\, m(dx)\, m(dy).$$

(b) (Exit time bound) We say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ satisfies the exit time lower bound $\mathrm{E}(\Phi)_{\geqslant}$, if there exist $C, A \in (1,\infty)$ for all $x \in \mathcal{X}, 0 < r < \operatorname{diam}(\mathcal{X}, d)/A$ the corresponding Hunt process satisfies

$$\mathbb{E}_x[\tau_{B(x,r)}] \geqslant C^{-1}\Phi(x,r). \tag{2.37}$$

We denote the corresponding upper bound and the two-sided bound by $\mathrm{E}(\Phi)_{\leqslant}$ and $\mathrm{E}(\Phi)$ respectively.

(c) (Stable-like heat kernel bound) We say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ satisfies the stable-like heat kernel bound $\mathrm{SHK}(\Phi)$ if there exists a heat kernel $p_t(x,y)$ of the semigroup (P_t) associated with $(\mathcal{E}, \mathcal{F})$, and $C_1, A_1 \in (1,\infty)$ such that

$$p_t(x,y) \leqslant C_1\left(\frac{1}{m(B(x,\Phi^{-1}(x,t)))} \wedge \frac{t}{m(B(x,d(x,y)))\Phi(x,d(x,y))}\right),$$
$$p_t(x,y) \geqslant C_1^{-1}\left(\frac{1}{m(B(x,\Phi^{-1}(x,t)))} \wedge \frac{t}{m(B(x,d(x,y)))\Phi(x,d(x,y))}\right),$$

for all $x, y \in \mathcal{X}$ and for all $0 < t < A^{-1}\Phi(x, \operatorname{diam}(\mathcal{X}))$, where we use the convention that $\Phi(x, \operatorname{diam}(\mathcal{X})) = \infty$ if $\operatorname{diam}(\mathcal{X}) = \infty$ and $\Phi^{-1}(x,\cdot)$ denotes the inverse of the homeomorphism $\Phi(x,\cdot) : (0,\infty) \to (0,\infty)$.

The following result plays a key role in our proof of heat kernel estimates for the boundary trace process. It characterizes stable-like heat kernel estimates using a combination of the jump kernel bound and exit time lower bound stated above. If $\mathcal{X}$ is unbounded then this characterization is essentially contained in [CKW]. It is a slight modification of the equivalence between (1) and (2) in [CKW, Theorem 1.15]. If $\mathcal{X}$ is bounded, we argue using results in [GHH23]. In Theorem 2.32, we assume that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is of *pure jump type*. That is, there exist a Radon measure on $(\mathcal{X} \times \mathcal{X})\backslash\mathcal{X}_{\mathrm{diag}}$ such that

$$\mathcal{E}(f,g) = \int_{(\mathcal{X}\times\mathcal{X})\backslash\mathcal{X}_{\mathrm{diag}}} (\widetilde{f}(x) - \widetilde{f}(y))(\widetilde{g}(x) - \widetilde{g}(y))\, J(dx, dy),$$

for all $f, g \in \mathcal{F}$, where $\widetilde{f}, \widetilde{g}$ denote quasi-continuous versions of f, g respectively.

Theorem 2.32. *Let $(\mathcal{X}, d)$ be a uniformly perfect metric space and let m be a doubling measure on $(\mathcal{X}, d)$. Let $\Phi : \mathcal{X} \times (0, \infty) \to (0, \infty)$ be a regular scale function. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(\mathcal{X}, m)$ of pure jump type. Then the following properties of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ on the metric space $(\mathcal{X}, d)$ are equivalent:*

(1) Stable-like heat kernel bound SHK(Φ).

(2) Jump kernel bound J(Φ) *and exit time lower bound* $\mathrm{E}(\Phi)_{\geqslant}$.

Furthermore, one the above conditions implies that the strongly continuous semigroup corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ admits a continuous heat kernel.

Proof. We note that uniform perfectness implies the reverse volume doubling property by Lemma 2.4. By a quasisymmetric change of metric as given in [BM18, Proposition 5.2] and [BM18, (5.7), Proof of Lemma 5.7], it suffices to consider the case $\Phi(x, r) = r^\beta$ for all $x \in \mathcal{X}, r > 0$, where $\beta > 0$ (see also [Kig12] where this kind of metric change first appeared). Therefore we will assume without loss of generality that $\Phi(x, r) = r^\beta$ for all $x \in \mathcal{X}, r > 0$, for some $\beta > 0$.

The result (1) implies (2) follows from the same argument as the proof of (1) implies (2) in [CKW, Theorem 1.15] regardless of whether or not $\mathcal{X}$ is bounded.

For the converse implication (2) implies (1), the proof splits into two cases depending on whether or not $\mathcal{X}$ is bounded.

Case 1: $\mathcal{X}$ is unbounded. By [CKW, Theorem 1.15], it suffices to show the exit time upper bound $\mathrm{E}(\Phi)_{\leqslant}$. The exit time upper bound $\mathrm{E}(\Phi)_{\leqslant}$ follows from the Faber-Krahn inequality shown in [CKW, §4.1] along with [CKW, Lemma 4.14].

Case 2: $\mathcal{X}$ is bounded. The exit time upper bound $\mathrm{E}(\Phi)_{\leqslant}$ stated in the unbounded case also holds in the bounded case with almost the same proof. Since the proof of the Faber-Krahn inequality relies on the reverse volume doubling estimate, the statement of the Faber-Krahn inequality has to be modified so that it holds of all balls of radii $0 < r < c\,\mathrm{diam}(\mathcal{X})$, where $c > 0$ as given in [GHH23, Definition 2.4].

Once the on-diagonal upper bound in the conclusion of [CKW, Theorem 4.25] is obtained, then the two-sided bounds on Jump kernel J(Φ) and exit time E(Φ) implies the stable-like heat kernel bound SHK(Φ) by arguments in [CKW, Chapter 5] with minor modifications to take into account that $\mathcal{X}$ is bounded. Therefore it is enough to prove the on-diagonal upper bound.

In order to show the on-diagonal bound, by [GHH23, Theorems 2.10 and 2.12], it suffices to show the condition (Gcap) in [GHH23, Definition 2.3]. The condition (Gcap) in turn follows from [GHH23+, Proposition 13.4 and Lemma 13.5] or [GHH23+, Theorem 14.1] along with the two-sided exit time bound E(Φ).

The assertion on the continuity of heat kernel follows from [CKW, Lemma 5.6]. □

Remark 2.33. If $\mathcal{X}$ is unbounded, the on-diagonal upper bound in the proof of (2) implies (1) above follows from [CKW, Theorem 4.25]. However, the proof there doesn't directly generalize to the case when $\mathcal{X}$ is bounded. This is because [CKW, Proof of Theorem 4.25] relies on [CKW, Proposition 4.23] which in turn uses [CKW, Proposition 4.18] on a sequence of radii going to infinity. However, the generalization of [CKW, Proposition 4.18] which relies on Faber-Krahn inequality requires the radii to satisfy $r < c\,\mathrm{diam}(\mathcal{X})$ for some $c > 0$, which seems insufficient for the argument in [CKW, Proof of Proposition 4.23].

3 Green function, Martin kernel, and Naïm kernel

3.1 Properties of Green function

We recall the notion of transient Dirichlet forms. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $(T_t)_{t \geqslant 0}$ be the strongly continuous semigroup corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$. The semigroup extends $(T_t)_{t \geqslant 0}$ uniquely to a contraction operator on $L^1(\mathcal{X}, m)$ from $L^1(\mathcal{X}, m) \cap L^2(\mathcal{X}, m)$ [FOT, p. 33]. For any non-negative function in $L^1(\mathcal{X}, m)$, we define Green operator as

$$Gf := \lim_{N \to \infty} \int_0^N T_s f \, ds. \tag{3.1}$$

We say that the Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is **transient** if there exists a bounded m-integrable function strictly positive m-a.e. on $\mathcal{X}$ such that $\int_{\mathcal{X}} |u| g \, dm \leqslant \sqrt{\mathcal{E}(u, u)}$ for all $u \in \mathcal{F}$.

The elliptic Harnack inequality implies the existence of Green function as shown in [BCM, Theorem 4.4] which we recall below.

Proposition 3.1. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space satisfying the elliptic Harnack inequality. Let $(X_t)_{t \geqslant 0}$ be the associated diffusion process. Let D be a non-empty open subset of $\mathcal{X}$ such that the associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ of the part process X^D is transient. There exists a non-negative $\mathcal{B}(D \times D)$-measurable function $g_D(x, y)$ on $D \times D$ and a Borel properly exceptional set $\mathcal{N}$ of X such that*

(i) (Symmetry) $g_D(x, y) = g_D(y, x)$ for all $(x, y) \in D \times D$;

(ii) *(Continuity)* $g_D(x,y)$ *is* $[0,\infty)$*-valued and jointly continuous in* $(x,y) \in D \times D \backslash D_{\mathrm{diag}}$*;*

(iii) *(Occupation density formula) For any* $f \in \mathcal{B}_+(D)$*;*

$$\mathbb{E}_x \int_0^{\tau_D} f(X_s) ds = \int_D g_D(x,y) f(y)\, m(dy) \quad \text{for every } x \in D \backslash \mathcal{N}; \tag{3.2}$$

(iv) *(Harmonicity) For any fixed* $y \in D$*, the function* $x \mapsto g_D(x,y)$ *is in* $\mathcal{F}_{\mathrm{loc}}(D \backslash \{y\})$ *and harmonic in* $D \backslash \{y\}$*. For any open subset* U *of* D *with* $y \notin \overline{U}$*,* $\mathbb{E}_x\left[\left|g(X^D_{\tau_U}, y)\right|\right] < \infty$ *and* $g_D(x,y) = \mathbb{E}_x[g(X^D_{\tau_U}, y)]$ *for* $\mathcal{E}$*-q.e.* $x \in D$*, where we adopt the convention that* $g_D(x, \partial) = g_D(\partial, x) = 0$ *for all* $x \in D$*.*

(v) *(Maximum principles) If* $x_0 \in V \Subset D$*, then*

$$\inf_{\overline{V} \backslash \{x_0\}} g_D(x_0, \cdot) = \inf_{\partial V} g_D(x_0, \cdot), \qquad \sup_{D \backslash V} g_D(x_0, \cdot) = \sup_{\partial V} g_D(x_0, \cdot). \tag{3.3}$$

We call $g_D(x,y)$ *the Green function of* $(\mathcal{E}, \mathcal{F})$ *in* D*.*

Proof. All parts except (iv) follows from [BCM, Theorem 4.4].

The claims that $x \mapsto g_D(x,y)$ belongs to $\mathcal{F}_{\mathrm{loc}}(D \backslash \{y\})$ and is harmonic in $D \backslash \{y\}$ follows from [BCM, Remark 2.7(ii), Proposition 2.9(iii), Theorem 4.4]. The remaining claims in (iv) are contained in [BCM, Theorem 4.4]. □

Definition 3.2. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and D be a non-empty open subset of $\mathcal{X}$ satisfying the assumption of Proposition 3.1. For a non-negative Borel measure function $f : D \to [0,\infty)$, we define

$$G_D f(x) := \begin{cases} \int_D g_D(x,y)\, m(dy) & \text{if } x \in D, \\ 0 & \text{if } x \notin D. \end{cases}$$

By [FOT, Theorem 4.2.6], for any non-negative measurable function $f : U \to [0,\infty)$ with $\int_D f G_D f \, dm < \infty$, then $G_D f$ is a quasi-continuous version of the Green operator defined in (3.1) for the Dirichlet form corresponding to the part process $(\mathcal{E}^D, \mathcal{F}^0(D))$ and $G_D f \in (\mathcal{F}^0(D))_e$.

The existence of Green function is closely related to the following absolute continuity condition (AC) whose definition we recall below [CF, Definition A.2.16].

Definition 3.3. Let X be a m-symmetric Markov process on $\mathcal{X}$ and let $\{P_t : t \geqslant 0\}$ be the corresponding transition semigroup. We say that X satisfies the absolute continuity (AC) any $t > 0$ the measure $P_t(x, \cdot)$ is absolutely continuous with respect to m.

We obtain that the exceptional sets in Proposition 3.1 can be taken to be the empty set if we the diffusion process in Proposition 2.13.

Lemma 3.4. *Let* $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ *be an MMD space and* X *be the corresponding diffusion process that satisfies Assumption 2.14.*

(a) *For any open set $D \subset \mathcal{X}$ then the corresponding part process X^D defined from every starting point on D satisfies the strong Feller property and has a continuous heat kernel.*

(b) *Assume that the part Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on D is transient. Let $g_D^p : D \times D \to [0, \infty]$, be defined as*

$$g_D^p(x,y) := \int_0^\infty p_t^D(x,y)\, dt, \quad \text{for all } x, y \in D, \tag{3.4}$$

where $p_t^D(\cdot,\cdot)$ is the continuous heat kernel of X^D as given in (a). Then g_D^p is continuous on $(D \times D)\backslash D_{\text{diag}}$. We have the occupation density formula

$$\mathbb{E}_x \int_0^{\tau_D} f(X_s) ds = \int_D g_D^p(x,y) f(y)\, m(dy) \quad \text{for all } x \in D, f \in \mathcal{B}_+(D). \tag{3.5}$$

If $g_D(\cdot,\cdot)$ denote the Green function in Proposition 3.1, then

$$g_D^p(x,y) = g_D(x,y) \quad \text{for all } (x,y) \in (D \times D)\backslash D_{\text{diag}}. \tag{3.6}$$

For any open subset U of D with $y \notin \overline{U}$, $\mathbb{E}_x\left[\left|g(X_{\tau_U}^D, y)\right|\right] < \infty$ and $g_D(x,y) = \mathbb{E}_x[g(X_{\tau_U}^D, y)]$ for all $x \in D$, where we adopt the convention that $g_D(x,\partial) = g_D(\partial, x) = 0$ for all $x \in D$. Furthermore for each $x \in D$, the function $g_D^p(x,\cdot)$ is X^D-excessive.

Proof. (a) For the process X whose transition function is both Feller and strong Feller, [Chu, p. 69, Section 1, Proof of Theorem] shows that the part process X^D has the semigroup strong Feller property (as a process on D).

Since X satisfies (AC) so does X^D. Let P_t^D denote the transition semigroup of X^D which satisfies (AC) and let Q_t^D denote the transition semigroup defined by the continuous heat kernel of associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ which exists due to [BGK12, Theorem 3.1].

Let f be a bounded continuous function on D. Then for any $s, t > 0$ and any $x \in D$, by $P_t^D f = Q_t^D f$ a.e. and (AC) of P_s we obtain

$$P_t^D(P_s^D f)(x) = (P_{t+s}^D f)(x) = P_s^D(P_t^D f)(x) = P_s^D(Q_t^D f)(x),$$

and letting $s \downarrow 0$ yields

$$(P_t^D f)(x) = (Q_t^D f)(x)$$

by dominated convergence theorem, since $(P_s^D f)(y) \to f(y)$ as $s \downarrow 0$ for any $y \in D$ by the continuity of f, right continuity of sample paths, and $P_s^D(Q_t^D f)(x) \to (Q_t^D f)(x)$ as $s \downarrow 0$ by the continuity of $Q_t^D f$. The continuity of $Q_t^D f$ can be easily verified using HKE(Ψ). Thus

$$P_t^D(x, dy) = Q_t^D(x, dy) \quad \text{for all } t > 0, x \in D. \tag{3.7}$$

(b) The occupation density formula (3.5) follows from Fubini's theorem as

$$\mathbb{E}_x \int_0^{\tau_D} f(X_s)ds = \int_0^\infty \int_D f(y)p_t^D(x,y)f(y)\,m(dy)\,dt = \int_D f(y)g_D^P(x,y)f(y)\,m(dy).$$

By the transience of X^D, we have

$$g_D^p(x,y) < \infty, \qquad \text{for } m \times m\text{-a.e. } (x,y). \tag{3.8}$$

By the heat kernel estimate HKE(Ψ), the function

$$(x,y) \mapsto \int_\delta^\infty p_t^D(x,y)\,dt$$

converges uniformly on compact subsets of $(D \times D)\backslash D_{\text{diag}}$ as $\delta \downarrow 0$. Therefore it suffices to show that for each $\delta > 0, (x_0, y_0) \in (D \times D)\backslash D_{\text{diag}}$, the function $(x,y) \mapsto \int_\delta^\infty p_t^D(x,y)\,dt$ is continuous at (x_0,y_0). In order to establish to continuity of $(x,y) \mapsto \int_\delta^\infty p_t^D(x,y)\,dt$ is continuous at $(x_0,y_0) \in (D \times D)\backslash D_{\text{diag}}, \delta > 0$, by the parabolic Harnack inequality [BGK12, Theorem 3.1], we can choose disjoint open neighborhoods B_1 and B_2 of x_0, y_0 and constants $C_1, C_2 > 0$ such that

$$\sup_{(x,y)\in B_1 \times B_2} p_t^D(x,y) \leqslant C_1 \inf_{(x,y)\in B_1\times B_2} p^D_{C_2^{-1}t}(x,y) \leqslant C_1 p^D_{C_2^{-1}t}(x',y') \quad \text{for all } t \geqslant \delta,$$

where $(x',y') \in B_1 \times B_2$ is chosen using (3.8) such that $g_D^p(x',y') < \infty$. Combining the above estimate with the transience of X^D, and the dominated convergence theorem, we conclude that $(x,y) \mapsto \int_\delta^\infty p_t^D(x,y)\,dt$ is continuous at (x_0,y_0).

The equality (3.6) follows from the continuity of g_D^p, g_D along with (3.5) and (3.2). The claim $g_D(x,y) = \mathbb{E}_x[g(X^D_{\tau_U}, y)]$ for all $x \in D$ follows Proposition 3.1(iv), the continuity of g_D^p, g_D along with the continuity in Lemma 2.28(b). The excessiveness of $g_D^p(x,\cdot)$ follows easily from the definition. □

Due to Lemma 3.4, if the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies Assumption 2.14 and D is a non-empty open subset of $\mathcal{X}$ such that the associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ of the part process X^D is transient, we adopt the convention to redefine the $g_D(\cdot,\cdot)$ from Proposition 3.1 to be equal to $g_D^p(\cdot,\cdot)$ from Lemma 3.4. In particular, $g_D(x,\cdot)$ is X^D-excessive for all $x \in D$.

In the next lemma, we show that the Green function has Dirichlet boundary condition in the sense of Definition 2.18.

Lemma 3.5 (Dirichlet boundary condition of Green function)**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and D be a non-empty open subset of $\mathcal{X}$ satisfying the assumption of Proposition 3.1. For any fixed $y_0 \in D$, the function $x \mapsto g_D(x,y)$ is in $\mathcal{F}^0_{\text{loc}}(D, D\backslash\{y_0\})$, and is harmonic in $D\backslash\{y_0\}$.*

Proof. The following argument is a variant of [BM19, Proof of Lemma 4.10].

Then by [FOT, Theorems 1.5.4(i) and 4.2.6], there exists a $(0,\infty)$-valued integrable function $f_0 = f_{D,0}$ on D such that $\int_D f_0 G_D f_0 dm < \infty$ and $G_D f_0 \in (\mathcal{F}^0(D))_e$. Let us adopt the convention that f_0 is extended to the whole space by setting $f_0 := 0$ on D^c and similarly for $G_D f$ for any non-negative Borel function f on D.)

Let $y_0 \in D$, and let K be any compact subset of $\mathcal{X}$ such that $y_0 \notin K$. Choose ϕ so that $\phi \in \mathcal{F} \cap C_c(\mathcal{X})$, ϕ is $[0,1]$-valued, $\phi = 1$ on K, and $y_0 \notin \text{supp}[\phi]$. For each $r > 0$ with $B(y_0, 2r) \subset D$ and $r < \text{dist}(y_0, \text{supp}[\phi])$, consider the function

$$g_r := \phi \min\{1/r, G_D(f_r)\}, \quad \text{where } f_r := \left(\int_{B(y_0,r)} f_0 dm \right)^{-1} 1_{B(y_0,r)} f_0. \tag{3.9}$$

Then $G_D(f_r)$ is an element of $(\mathcal{F}^0(D))_e$ quasi-continuous on D by [FOT, Corollary 1.5.1 and Theorem 4.2.6], and hence is quasi-continuous on $\mathcal{X}$ by [CF, Theorem 3.4.9], [FOT, Theorem 4.4.3] and our convention that $g_r = 0$ on D^c. Since $(\mathcal{F}^0(D))_e \cap L^2(\mathcal{X}, m) = \mathcal{F}^0(D)$, it follows that $g_r \in \mathcal{F}^0(D)$. Also, $G_D(f_r)$ and g_r are continuous on $D \backslash \overline{B(y_0, r)}$ by the continuity of Green's function g_D on D and dominated convergence. Note that for any $r_0 > 0$ such that $B(y_0, 2r_0) \subset D$ and $r_0 < \text{dist}(y_0, \text{supp}[\phi])$, the function $(x,y) \mapsto g_D(x,y)$ stays bounded for $x \in D \backslash B(y_0, 2r_0)$ and $y \in B(y_0, r_0)$, by the latter of maximum principles (3.3) and the joint continuity of g_D. Therefore, there exists $\delta > 0$ such that $B(y_0, 2\delta) \subset D$ and $\delta < \text{dist}(y_0, \text{supp}[\phi])$, and for any $r < \delta$, we have

$$g_r := \phi \min\{1/r, G_D(f_r)\} = \phi G_D(f_r) \in L^\infty(\mathcal{X}) \cap (\mathcal{E}^0(D))_e.$$

Therefore for all $0 < r, s < \delta$, by [FOT, Corollary 1.5.1] we have $\phi^2(g_r - g_s) \in (\mathcal{E}^0(D))_e$ and hence by [FOT, (1.5.9)]

$$\mathcal{E}(g_r - g_s, \phi^2(g_r - g_s)) = \int_{\mathcal{X}} (f_r - f_s)\phi^2(g_r - g_s)\, dm = 0. \tag{3.10}$$

Now, as $r \to 0$, g_r converges pointwise on $\mathcal{X}$ to $\phi g_D(\cdot, y_0)$ (and uniformly on any compact subset of $D \backslash \{y_0\}$), by the (joint) continuity of g_D, and it thus remains to prove that this convergence takes place also in $(\mathcal{F}, \mathcal{E}_1)$. The convergence in $L^2(\mathcal{X}, m)$ is clear by dominated convergence because these functions are uniformly bounded and supported on $\text{supp}[\phi]$. These functions form an $\mathcal{E}$-Cauchy family as $r \to 0$ since we can apply dominated convergence to the right-hand side of the equality

$$\mathcal{E}(g_r - g_s, g_r - g_s) = \int_{\text{supp}[\phi]} (G_D(f_r) - G_D(f_s))^2 d\Gamma(\phi).$$

The above equality follows from the chain rule, (3.10) and the same calculation as (2.17). □

The following Dynkin-Hunt type formula is a basic ingredient in comparing the Green function on two domains.

Lemma 3.6 (Dyknin-Hunt formula). *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space satisfying the elliptic Harnack inequality. Let $D_1 \subset D_2$ be open subsets such that the associated Dirichlet form $(\mathcal{E}^{D_2}, \mathcal{F}^0(D_2))$ on $L^2(D_2; m\llcorner D_2)$ of the part process X^{D_2} is transient. Then there is a properly exception set $\mathcal{N}_{D_2}$ for X^{D_2} such that for all $x \in D_1 \backslash \mathcal{N}_{D_2}, y \in D_1$,*

$$g_{D_2}(x,y) = g_{D_1}(x,y) + \mathbb{E}_x[\mathbb{1}_{\{X_{\tau_{D_1}} \in D_2\}} g_{D_2}(X_{\tau_{D_1}}, y)] \tag{3.11}$$

In addition, if the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and the associated diffusion X satisfies Assumption 2.14, then (3.11) holds for all $(x,y) \in (D_1 \times D_1) \backslash (D_1)_{\mathrm{diag}}$.

Proof. By the occupation density formula (Proposition 3.1(iii)) and [BCM, Lemma 4.5], there exists a properly exceptional set $\mathcal{N}_{D_2}$ for X^{D_2} such that for all $f \in \mathcal{B}_+(D_2)$, and for all $x \in D_1 \backslash \mathcal{N}_{D_2}$ we have

$$\mathbb{E}_x \int_0^{\tau_{D_i}} f(X_s) ds = \int_{D_i} g_{D_i}(x,y) f(y)\, m(dy), \quad \text{for } i = 1, 2. \tag{3.12}$$

Therefore for all $f \in \mathcal{B}_+(D_2), x \in D_1 \backslash \mathcal{N}_{D_2}$, we have

$$\begin{aligned}
&\int_{D_2} g_{D_2}(x,z) f(z)\, m(dz) \\
&\overset{(3.12)}{=} \mathbb{E}_x \int_0^{\tau_{D_2}} f(X_s)\, ds = \mathbb{E}_x \int_0^{\tau_{D_1}} f(X_s)\, ds + \mathbb{E}_x \int_{\tau_{D_1}}^{\tau_{D_2}} f(X_s)\, ds \\
&\overset{(3.12)}{=} \int_{D_1} g_{D_1}(x,z) f(z)\, m(dz) + \mathbb{E}_x \left[\mathbb{1}_{\{X_{\tau_{D_1}} \in D_2\}} \mathbb{E}_{X_{\tau_{D_1}}} \left(\int_0^{\tau_{D_2}} f(X_s)\, ds \right) \right] \\
&\overset{(3.12)}{=} \int_{D_1} g_{D_1}(x,z) f(z)\, m(dz) + \int_{D_2} \mathbb{E}_x \left[\mathbb{1}_{\{X_{\tau_{D_1}} \in D_2\}} g_{D_2}(X_{\tau_{D_1}}, z) \right] f(z)\, m(dz),
\end{aligned} \tag{3.13}$$

where we use strong Markov property and Fubini's theorem in lines 3 and 4 above respectively.

For $y \in D_1$, set $f(\cdot) := (m(B(y,r)))^{-1} \mathbb{1}_{B(y,r)}(\cdot)$ and letting $r \downarrow 0$ in (3.13), we obtain (3.11). This is justified using the dominated convergence theorem, the joint continuity and maximum principles for the Green function (Proposition 3.1(ii),(v)).

If we assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the heat kernel estimate HKE(Ψ) and the corresponding diffusion process is defined at every starting point as given in Proposition 2.13, then we can use (3.5) instead of (3.12) to obtain the above conclusion. □

For a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfying the elliptic Harnack inequality and for a non-empty open subset $D \subset \mathcal{X}$ such that the associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ of the part process X^D is transient, we define (by a slight abuse of notation)

$$g_D(x,r) = \inf_{y \in S(x,r)} g_D(x,y) \quad \text{provided } \delta_D(x) < r, \quad S(x,r) := \{y \in \mathcal{X} : d(x,y) = r\}, \tag{3.14}$$

$$\mathrm{Cap}_D(A) = \inf\{\mathcal{E}(f,f) : f \in (\mathcal{F}^0(D))_e \text{ such that } \widetilde{f} \geqslant 1\ \mathcal{E}\text{-q.e. on } A\}, \quad A \Subset D, \tag{3.15}$$

where $(\mathcal{F}^0(D))_e$ is as given in (2.11).

It is known that there is a unique function called the *equilibrium potential* $e_{A,D} \in (\mathcal{F}^0(D))_e$ that attains the infimum above. We describe the corresponding equilibrium measures. The equality (3.18) in following lemma was claimed without a proof in [Fit, (2.7)]. Since it plays an important role in our work we provide a detailed proof.

Lemma 3.7. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $D \subset \mathcal{X}$ be an open set such that the Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ of the part process on D is transient. Let $A \Subset D$ be a relatively compact open subset of D.*

(a) There exists a unique $e_{A,D} \in (\mathcal{F}^0(D))_e$ and a Radon measure $\lambda^1_{A,D}$ such that

$$\mathrm{Cap}_D(A) = \mathcal{E}(e_{A,D}, e_{A,D}), \quad \widetilde{e}_{A,D} = 1 \ \mathcal{E}\text{-q.e. on } A, \quad \mathcal{E}(u, e_{A,D}) = \int \widetilde{u}\, d\lambda^1_{A,D} \tag{3.16}$$

for all $u \in (\mathcal{F}^0(D))_e$. Furthermore $\lambda^1_{A,D}$ is supported on ∂A with

$$\lambda^1_{A,D}(\mathcal{X}) = \lambda^1_{A,D}(\partial A) = \mathrm{Cap}_D(A). \tag{3.17}$$

(b) Furthermore if $\overline{D}$ is compact, there exists a measure $\lambda^0_{A,D}$ such that

$$\mathcal{E}(e_{A,D}, u) = \int_{\partial A} \widetilde{u}\, d\lambda^1_{A,D} - \int_{\partial B} \widetilde{u}\, d\lambda^0_{A,D}, \tag{3.18}$$

for any $u \in \mathcal{F} \cap L^\infty(\mathcal{X}, m)$, where $\widetilde{u}$ is a quasicontinuous version of u and $\lambda^1_{A,D}$ is the measure in part (a). Furthermore

$$\lambda^1_{A,D}(\partial A) = \lambda^0_{A,D}(\partial B) = \mathrm{Cap}(A, B). \tag{3.19}$$

Proof. (a) Note that $((\mathcal{F}^0(D))_e, \mathcal{E})$ is a Hilbert space by [FOT, Theorem 1.5.3]. Since $A \Subset D$, the regularity of $(\mathcal{E}, \mathcal{F})$ along with [CF, Theorem 2.3.4] implies that the set

$$\mathcal{L}_{A,D} := \{f \in (\mathcal{F}^0(D))_e : \widetilde{f} \geqslant 1 \ \mathcal{E}\text{-q.e. on } A\}$$

is non-empty, closed, convex subset of the Hilbert space $((\mathcal{F}^0(D))_e, \mathcal{E})$. Hence there exists a unique element $\widetilde{e}_{A,D} \in \mathcal{L}_{A,D}$ such that $\mathrm{Cap}_D(A) = \mathcal{E}(e_{A,D}, e_{A,D})$. Since $1 \wedge e_{A,D} \in \mathcal{L}_{A,D}$ and $\mathcal{E}(1 \wedge e_{A,D}, 1 \wedge e_{A,D}) \leqslant \mathcal{E}(e_{A,D}, e_{A,D})$, we conclude $\widetilde{e}_{A,D} = 1 \wedge \widetilde{e}_{A,D}$ q.e. and hence $\widetilde{e}_{A,D} = 1$ q.e. on A.

Let $v \in (\mathcal{F}^0(D))_e$ such that $v \geqslant 0$ m-a.e. Then for any $t > 0$, $e_{A,D} + tv \in \mathcal{L}_{A,D}$ and hence $\mathcal{E}(e_{A,D} + tv, e_{A,D} + tv) \geqslant \mathcal{E}(e_{A,D}, e_{A,D})$ or equivalently $\mathcal{E}(e_{A,D}, v) + (t/2)\mathcal{E}(v, v) \geqslant 0$. Letting $t \downarrow 0$, we conclude

$$\mathcal{E}(e_{A,D}, v) \geqslant 0, \quad \text{for all } v \in (\mathcal{F}^0(D))_e \text{ such that } v \geqslant 0 \ m\text{-a.e.}$$

The existence of a Radon measure $\lambda^1_{A,D}$ on D satisfying the last equality in (3.16) now follows from by applying [FOT, Theorem 2.2.5 and Lemma 2.2.10] to the Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$. We also consider it as a Radon measure on $\mathcal{X}$ by setting $\lambda^1_{A,D}(\cdot) := \lambda^1_{A,D}(\cdot \cap D)$. This concludes the proof of all claims in (3.16).

By the strong locality and $\widetilde{e}_{A,D} = 1$ $\mathcal{E}$-q.e. on A, we conclude that $e_{A,D}$ is harmonic in A. By the energy minimizing property of $e_{A,D}$, we have that $e_{A,D}$ is harmonic in $D\backslash\overline{A}$. Therefore any $u \in \mathcal{F} \cap (C_c(A) \cup C_c(D\backslash\overline{A}))$, we have $\mathcal{E}(u, e_{A,D}) = 0$ which implies that $\lambda^1_{A,D}(A \cup (D\backslash\overline{A})) = 0$. This implies $\lambda^1_{A,D}$ is supported on ∂A. The proof of (3.17) is contained in [BCM, Proof of Proposition 5.21].

(b) Let $\phi \in \mathcal{F} \cap C_c(\mathcal{X}), \operatorname{supp}[\phi] \cap A = \emptyset, \phi \leqslant 0$. Choose $0 \leqslant \psi \leqslant 1$ and $\psi\big|_V = 1$, where V is a neighborhood of $\operatorname{supp}[\phi]$. Since $e_{A,D}\psi$ is $\mathcal{E}$-harmonic on $(D^c\backslash A) \cap V$ and $\widetilde{e}_{A,D}\psi - (\widetilde{e}_{A,D}\psi + \phi)_+ = 0$ q.e. on $((D^c\backslash A) \cap V)^c$, we have $\mathcal{E}(e_{A,D}\psi, e_{A,D}\psi) = \mathcal{E}(e_{A,D}\psi, (\widetilde{e}_{A,D}\psi + \phi)_+)$ and therefore

$$\begin{aligned}
0 &\leqslant \mathcal{E}((e_{A,D}\psi + \phi)_+ - e_{A,D}\psi, (e_{A,D}\psi + \phi)_+ - e_{A,D}\psi) \\
&= \mathcal{E}((e_{A,D}\psi + \phi)_+, (e_{A,D}\psi + \phi)_+) - 2\mathcal{E}((e_{A,D}\psi + \phi)_+, e_{A,D}\psi) + \mathcal{E}(e_{A,D}\psi, e_{A,D}\psi) \\
&= \mathcal{E}((e_{A,D}\psi + \phi)_+, (e_{A,D}\psi + \phi)_+) - \mathcal{E}(e_{A,D}\psi, e_{A,D}\psi) \\
&\leqslant \mathcal{E}(e_{A,D}\psi + \phi, e_{A,D}\psi + \phi) - \mathcal{E}(e_{A,D}\psi, e_{A,D}\psi) \quad \text{(by Markov property).} \\
&= \mathcal{E}(\phi, \phi) + 2\mathcal{E}(e_{A,D}\psi, \phi) = \mathcal{E}(\phi, \phi) + 2\mathcal{E}(e_{A,D}, \phi). \quad \text{(by strong locality)}
\end{aligned}$$

By replacing ϕ with $t\phi$ and letting $t \downarrow 0$, we obtain

$$\mathcal{E}(e_{A,D}, \phi) \geqslant 0, \quad \text{for all } \phi \leqslant 0, \phi \in C_c(\mathcal{X}) \cap \mathcal{F} \text{ such that } \operatorname{supp}[\phi] \subset A^c. \tag{3.20}$$

It follows that there exists a Radon measure $\lambda^0_{A,D}$ on A^c such that for all $\phi \in \mathcal{F} \cap C_c(\mathcal{X})$ with $\operatorname{supp}[\phi] \subset A^c$, we have

$$\mathcal{E}(\phi, e_{A,D}) = -\int_{A^c} \phi \, d\lambda^0_{A,D}. \tag{3.21}$$

Furthermore by strong locality of $(\mathcal{E}, \mathcal{F})$ and $\mathcal{E}$-harmonicity of $e_{A,D}$ on $D^c\backslash A$ and the compactness of ∂D, we have

$$\lambda^0_{A,D}(A^c) = \lambda^0_{A,D}(\partial D) < \infty. \tag{3.22}$$

We consider $\lambda^0_{A,D}$ is a finite Borel measure on $\mathcal{X}$ by setting $\lambda^0_{A,D}(\cdot) := \lambda^0_{A,D}(\cdot \cap A^c)$. As before, we can consider $\lambda^1_{A,D}$ as a Borel measure on $\mathcal{X}$ such that

$$\lambda^1_{A,D}(\mathcal{X}) = \lambda^1_{A,D}(\partial A) < \infty. \tag{3.23}$$

Now let $\phi \in \mathcal{F} \cap C_c(\mathcal{X})$ and let $\psi \in \mathcal{F} \cap C_c(\mathcal{X})$ satisfy $\psi\big|_U = 1$ for some neighborhood U of A, $0 \leqslant \psi \leqslant 1$ on $\mathcal{X}$ and $\operatorname{supp}[\psi] \subset B$. Then

$$\begin{aligned}
\mathcal{E}(\phi, e_{A,D}) &= \mathcal{E}(\phi - \phi\psi, e_{A,D}) + \mathcal{E}(\phi\psi, e_{A,D}) \\
&= -\int_{\partial B} (\phi - \phi\psi) \, d\lambda^0_{A,D} + \int_{\partial A} \phi\psi \, d\lambda^1_{A,D} \\
&= -\int_{\partial B} \phi \, d\lambda^0_{A,D} + \int_{\partial A} \phi \, d\lambda^1_{A,D} \quad \text{(by (3.21),(3.22), (3.17),(3.23)).}
\end{aligned} \tag{3.24}$$

Also by [FOT, Theorem 4.4.3-(i),(ii) and Lemma 2.2.3], $\lambda^0_{A,D}, \lambda^1_{A,D}$ charge no set of zero capacity. Finally, for any $u \in \mathcal{F}_e \cap L^\infty(\mathcal{X}, m)$, by [FOT, Theorem 2.1.7 and Corollary 1.6.3], there exists $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \cap C_c(\mathcal{X})$ with $\sup_{\mathcal{X}} |u_n(\cdot)| \leqslant \|u\|_{L^\infty}$, $u_n \to \widetilde{u}$ q.e. on $\mathcal{X}$ and $\lim_{n\to\infty} \mathcal{E}(u - u_n, u - u_n) = 0$. This along with (3.24) applied to the sequence $\{u_n\}$, $\lambda^0_{A,D}(\partial B) < \infty, \lambda^1_{A,D}(\partial A) < \infty$ and the dominated convergence theorem implies the desired equality (3.18). □

We collect various useful estimates on the Green function from [BCM].

Lemma 3.8. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies the elliptic Harnack inequality. Let $D \subset \mathcal{X}$ be a non-empty open subset such that the Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ corresponding to the part process on D is transient.*

(a) If $D \neq \mathcal{X}$, there exist $C_1, A_0 \in (1, \infty)$ such that for all $x \in D, 0 < R \leqslant \delta_D(x)/A_0$ we have

$$\sup_{y\in S(x,r)} g_D(x, y) \leqslant C_1 \inf_{y\in S(x,r)} g_D(x, y), \quad g_D(x, r) \leqslant \mathrm{Cap}_D(B(x, r))^{-1} \leqslant C_1 g_D(x, r). \tag{3.25}$$

Furthermore, there exist $\theta, C_1 \in (0, 1)$ such that

$$g_D(x, R) \leqslant g_D(x, r) \leqslant C_2 \left(\frac{R}{r}\right)^\theta g_D(x, R), \quad \text{for all } x \in D, 0 < r < R \leqslant \delta_D(x)/A_1. \tag{3.26}$$

(b) There exist $A_0, C_0 \in (1, \infty)$ such that for all $y \in D, 0 < R < A_0^{-1}\delta_D(y)$, we have

$$C_0^{-1} \frac{g_D(x, y)}{g_D(y, R)} \leqslant \mathbb{P}_x[\sigma_{\overline{B(y,R)}} < \sigma_{D^c}] \leqslant C_0 \frac{g_D(x, y)}{g_D(y, R)} \quad \text{for } \mathcal{E}\text{-q.e. } x \in D\backslash\overline{B(y, R)}. \tag{3.27}$$

If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and the corresponding diffusion satisfy Assumption 2.14, then (3.27) holds for all $x \in D\backslash\overline{B(y, R)}$.

Proof. (a) The estimate (3.25) follows from [BCM, Lemma 5.10 and Proposition 5.7] and (3.26) follows from [BCM, Corollary 5.15] and maximum principle (Proposition 3.1(v)).

(b) By Lemma 2.23(a), we assume that $(\mathcal{X}, d)$ is K-relatively ball connected for some $K \in (1, \infty)$. Let $A_1 \in (1, \infty)$ be as given in (a) By [BCM, Lemma 5.10] and (a), there exist $A_1 \in (K, \infty), C_1 \in (1, \infty)$ such that

$$g_D(y, R) \leqslant \mathrm{Cap}_D(B(y, R))^{-1} \leqslant C_1 g_D(y, R), \quad g_D(y, R) \leqslant g_D(y, z) \leqslant C_1 g_D(y, R) \tag{3.28}$$

for all $y \in D, 0 < R < A_1^{-1}\delta_D(y)$. Let $y \in D, 0 < R < A_1^{-1}\delta_D(y)$, ν denote the equilibrium measure on $\partial B(y, R)$ corresponding to $\mathrm{Cap}_D(B(y, R))$.
Case 1, $d(x, y) > KR$: In this case, $g_D(x, \cdot)$ is harmonic on $B(y, d(x, y))$ and hence by (2.19) and (2.18), there exists $C_2 > 1$ such that

$$C_2^{-1} g_D(x, y) \leqslant g_D(x, z) \leqslant C_2 g_D(x, y) \quad \text{for all } z \in \partial B(y, R). \tag{3.29}$$

Therefore for q.e. $x \in D \setminus B(y, KR), 0 < R < A_1^{-1}\delta_D(y)$ and by [FOT, Theorem 4.3.3]

$$\begin{aligned}
\mathbb{P}_x[\sigma_{\overline{B(y,R)}} < \sigma_{D^c}] &= \int_{\partial B(y,R)} g_D(x,z)\,\nu(dy) \overset{(3.29)}{\leqslant} C_2 g_D(x,y)\,\mathrm{Cap}_D(B(y,R)) \\
&\overset{(3.28)}{\leqslant} C_2 \frac{g_D(x,y)}{g_D(y,R)}, \qquad (3.30) \\
\mathbb{P}_x[\sigma_{\overline{B(y,R)}} < \sigma_{D^c}] &= \int_{\partial B(y,R)} g_D(x,z)\,\nu(dy) \overset{(3.29)}{\geqslant} C_2^{-1} g_D(x,y)\,\mathrm{Cap}_D(B(y,R)) \\
&\overset{(3.28)}{\geqslant} C_2^{-1} C_1^{-1} \frac{g_D(x,y)}{g_D(y,R)}. \qquad (3.31)
\end{aligned}$$

Case 2, $R \leqslant d(x,y) \leqslant KR$: For q.e. $x \in D$ such that $R \leqslant d(x,y) \leqslant KR$ with $0 < R < A_1^{-1}\delta_D(y)$ and by [FOT, Theorem 4.3.3]

$$\begin{aligned}
\mathbb{P}_x[\sigma_{\overline{B(y,R)}} < \sigma_{D^c}] \geqslant \mathbb{P}_x[\sigma_{B(y,R/(2K))} < \sigma_{D^c}] &\overset{(3.31)}{\geqslant} C_2^{-1} C_1^{-1} \frac{g_D(x,y)}{g_D(y,R/(2K))} \\
&\overset{(3.26)}{\geqslant} C_2^{-1} C_1^{-1} c_1 (2K)^{-\theta} \frac{g_D(x,y)}{g_D(y,R)}, \qquad (3.32) \\
\mathbb{P}_x[\sigma_{\overline{B(y,R)}} < \sigma_{D^c}] \leqslant 1 \overset{(3.26)}{\leqslant} c_1^{-1} K^{\theta} \frac{g_D(x,y)}{g_D(y,R)}. \qquad (3.33)
\end{aligned}$$

By (3.30), (3.31), (3.32), and (3.33), we obtain (3.27).

If if the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and the associated diffusion X satisfies Assumption 2.14, then by Lemma 2.28(b) we obtain (3.27) for all $x \in D \setminus \overline{B(y,R)}$. □

3.2 Boundary Harnack principle

In this work, we need to understand the behavior of Green function near the boundary of a uniform domain. The following scale-invariant boundary Harnack principle is useful to describe the behavior of Green function near the boundary of a uniform domain. Boundary Harnack principle has been obtained in increasing generality over a long period of time [Kem, Anc78, Dah, Wu, JK, Aik01, Lie15, BM18].

Definition 3.9. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $U \subset \mathcal{X}$ be a proper domain. Then we say that U satisfies the *boundary Harnack principle* there exist $A_0, A_1, C_1 \in (1, \infty)$ such that for all $\xi \in \partial U$, for all $0 < r < \mathrm{diam}(U, d)/A_1$ and any two non-negative functions u, v that are harmonic on $U \cap B(\xi, A_0 r)$ with Dirichlet boundary condition along $\partial U \cap B(\xi, A_0 r)$, we have

$$\operatorname*{ess\,sup}_{x \in U \cap B(\xi,r)} \frac{u(x)}{v(x)} \leqslant C_1 \operatorname*{ess\,inf}_{x \in U \cap B(\xi,r)} \frac{u(x)}{v(x)}.$$

A standard consequence of the boundary Harnack principle is the following oscillation lemma and follows from [Aik01, Proof of Theorem 2]. It is an analogue of Moser's oscillation lemma for the elliptic Harnack inequality [Mos, §5] and has a similar proof.

Lemma 3.10. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $U \subset \mathcal{X}$ be a proper domain that satisfies the boundary Harnack principle. Then there exist $A_0, A_1, C_0 \in (1,\infty), \gamma > 0$ such that for all $\xi \in \partial U$, for all $0 < r < R < \operatorname{diam}(U,d)/A_1$ and any two non-negative functions u, v that are harmonic on $U \cap B(\xi, A_0 R)$ with Dirichlet boundary condition along $\partial U \cap B(\xi, A_0 R)$, we have*

$$\operatorname{osc}_{U \cap B(\xi,r)} \frac{u}{v} \leqslant C_0 \left(\frac{r}{R}\right)^{\gamma} \operatorname{osc}_{U \cap B(\xi,R)} \frac{u}{v}.$$

Another important consequence of the boundary Harnack principle is the Carleson estimate. The proof is a variant of [Aik08, Proof of Theorem 2] where we use estimates on Green function from [BM18, BCM] instead of known estimates of the Euclidean space. The basic idea is that Carleson estimate for one harmonic function with Dirichlet boundary condition (say, the Green function at a suitably chosen point) along with boundary Harnack principle implies Carleson estimate in general. The Carleson estimate for Green function can be obtained using the maximum principle and comparison estimates for the Green function obtained in [BM18, BCM]. It follows from a modification of the argument in [GyS, Proof of (4.28)].

Proposition 3.11 (Carleson estimate)**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the elliptic Harnack inequality. Let $U \subset \mathcal{X}$ be a uniform domain that satisfies the boundary Harnack principle. Then there exist $A_0, A_1, C_0 \in (1,\infty)$ such that for all $\xi \in \partial U$, for all $0 < r < R < \operatorname{diam}(U,d)/A_1$ and any non-negative function u that is harmonic and continuous on $U \cap B(\xi, A_0 R)$ with Dirichlet boundary condition along $\partial U \cap B(\xi, A_0 R)$, we have*

$$\sup_{x \in B(\xi,R)} u(x) \leqslant C u(\xi_{R/2}).$$

Proof. Let u be a harmonic function as given in the statement of the proposition. Let us choose A_0, A_1, C_1 as the constants in Definition 3.9.

First, we note that there exists $C_2, A_3 \in (1,\infty), A_4 > A_1$ such that

$$\sup_{U \cap B(\xi,R)} g_{U \cap B(\xi, A_3 r)}(\xi_{2A_0 R}, \cdot) \leqslant C_2 g_{U \cap B(\xi, A_3 r)}(\xi_{2A_0 R}, \xi_{R/2}), \tag{3.34}$$

for all $\xi \in \partial U, 0 < R < A_4^{-1} \operatorname{diam}(U,d)$. This follows from the chaining using elliptic Harnack inequality by a similar argument as given in the proof of Lemma 2.23(b), the maximum principle (Proposition 3.1) and comparison of Green functions in [BCM, Corollary 5.8].

Then by the boundary Harnack principle (Definition 3.9), we have

$$\sup_{B(\xi,R)} \frac{u(\cdot)}{g_{U \cap B(\xi, A_3 r)}(\xi_{2A_0 R}, \cdot)} \leqslant C_1 \frac{u(\xi_{R/2})}{g_{U \cap B(\xi, A_3 r)}(\xi_{2A_0 R}, \xi_{R/2})}, \tag{3.35}$$

for all $\xi \in \partial U, 0 < R < A_4^{-1} \operatorname{diam}(U,d)$. Therefore by (3.34) and (3.35), we conclude that for all $\xi \in \partial U, 0 < R < A_4^{-1} \operatorname{diam}(U,d)$ and for any non-negative function u that is harmonic and continuous on $U \cap B(\xi, A_0R)$ with Dirichlet boundary condition along $\partial U \cap B(\xi, A_0R)$, we have

$$\sup_{B(\xi,R)} u(\cdot) \leqslant C_1 \frac{u(\xi_{R/2})}{g_{U\cap B(\xi,A_3r)}(\xi_{2A_0R}, \xi_{R/2})} \sup_{B(\xi,R)} g_{U\cap B(\xi,A_3r)}(\xi_{2A_0R}, \cdot) \leqslant C_1C_2u(\xi_{R/2}).$$

□

Theorem 3.12 (Boundary Harnack Principle)**.** *[BM19, Theorem 1.1] Let $(\mathcal{X}, d)$ be a complete, separable, locally compact, length space, and let m be a non atomic Radon measure on $(\mathcal{X}, d)$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular strongly local Dirichlet form on $L^2(\mathcal{X}, m)$. Assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the elliptic Harnack inequality. Let $U \subsetneq \mathcal{X}$ be a length uniform domain. Then there exist $A_0, A_1, C_1 \in (1, \infty)$ such that for all $\xi \in \partial U$, for all $0 < r < \operatorname{diam}(U, d)/A_1$ and any two non-negative functions u, v that are harmonic on $U \cap B(\xi, A_0r)$ with Dirichlet boundary condition along $\partial U \cap B(\xi, A_0r)$, we have*

$$\operatorname*{ess\,sup}_{x\in U\cap B(\xi,r)} \frac{u(x)}{v(x)} \leqslant C_1 \operatorname*{ess\,inf}_{x\in U\cap B(\xi,r)} \frac{u(x)}{v(x)}.$$

If $\operatorname{diam}(U, d) < \infty$, the condition $0 < r < \operatorname{diam}(U, d)/A_1$ is not explicitly stated in [BM19] but it follows from the proof there.

It turns out that the assumption that $(\mathcal{X}, d)$ is a length space in Theorem 3.12 is unnecessary. In particular, elliptic Harnack inequality implies the boundary Harnack principle for uniform domains on any doubling metric space as shown in a work in preparation by Aobo Chen [Che] (instead of *length* uniform domains considered in Theorem 3.12). In other words, Theorem 3.12 can be generalized to uniform domains to metric spaces that need not contain any non-constant rectifiable curves.

3.3 Naïm kernel

We introduce the Naïm kernel and study some of its properties. For the remainder of the section we make the following running assumption.

Assumption 3.13. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the elliptic Harnack inequality such that $(\mathcal{X}, d)$ satisfies the metric doubling property. Let $U \subset \mathcal{X}$ be a uniform domain that satisfies the boundary Harnack principle and such that the Dirichlet form $(\mathcal{E}^U, \mathcal{F}^0(U))$ on $L^2(U; m\big|_U)$ corresponding to the part process on U is transient.

By the result of A. Chen mentioned above, the assumption that U satisfies the boundary Harnack principle is redundant but since [Che] is not yet available, we made this additional assumption throughout this work. In the case of *length* uniform domains in a length space, we can use Theorem 3.12 instead of the upcoming work [Che] to remove the assumption concerning the boundary Harnack principle.

For $x_0 \in U$, we define $\Theta^U_{x_0} : (U\backslash\{x_0\}) \times U\backslash\{x_0\})\backslash(U\backslash\{x_0\})_{\mathrm{diag}} \to [0, \infty)$ as

$$\Theta^U_{x_0}(x, y) := \frac{g_U(x, y)}{g_U(x_0, x)g_U(x_0, y)}. \tag{3.36}$$

The function $\Theta_{x_0}^U$ satisfies the following *local Hölder regularity* and bounds. The proofs are variants of Moser's oscillation inequality [Mos, §5].

Lemma 3.14. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain as given in Assumption 3.13. There exist $A, C_1, C_2 \in (1,\infty)$ and $\gamma > 0$ such that the following estimates hold:*

(a) For any $\eta \in \partial U, z \in U\setminus\{x_0\}, 0 < r < R < (2A)^{-1}\left(d(\eta, x_0) \wedge d(z, x_0) \wedge \delta_U(z)\right)$

$$\operatorname*{osc}_{(B(\eta,r)\cap(U\setminus\{x_0\}))\times(B(z,r)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U \leqslant A\left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{(B(\eta,R)\cap(U\setminus\{x_0\}))\times(B(z,R)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U.$$

(b) For $\eta, \xi \in \partial U$ with $\xi \neq \eta$ and for all $0 < r < R < (2A)^{-1}\left(d(\eta, x_0) \wedge d(\eta, \xi) \wedge d(\xi, x_0)\right)$

$$\operatorname*{osc}_{(B(\eta,r)\cap(U\setminus\{x_0\}))\times(B(\xi,r)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U \leqslant A\left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{(B(\eta,R)\cap(U\setminus\{x_0\}))\times(B(\xi,R)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U.$$

(c) For any $\eta \in \partial U, z \in U\setminus\{x_0\}, 0 < R < (2A)^{-1}\left(d(\eta, x_0) \wedge d(z, x_0) \wedge \delta_U(z)\right)$,

$$\sup_{(B(\eta,R)\cap(U\setminus\{x_0\}))\times(B(z,R)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U \leqslant C_1 \frac{g_U(z, \eta_{R/2})}{g_U(x_0, z) g_U(x_0, \eta_{R/2})}.$$

(d) For $\eta, \xi \in \partial U$ with $\xi \neq \eta$ and for all $0 < R < (2A)^{-1}\left(d(\eta, x_0) \wedge d(\eta, \xi) \wedge d(\xi, x_0)\right)$

$$\sup_{(B(\eta,R)\cap(U\setminus\{x_0\}))\times(B(\xi,R)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U \leqslant C_2 \frac{g_U(\eta_{R/2}, \xi_{R/2})}{g_U(x_0, \eta_{R/2}) g_U(x_0, \xi_{R/2})},$$

and

$$\inf_{(B(\eta,R)\cap(U\setminus\{x_0\}))\times(B(\xi,R)\cap(U\setminus\{x_0\}))} \Theta_{x_0}^U \geqslant C_2^{-1} \frac{g_U(\eta_{R/2}, \xi_{R/2})}{g_U(x_0, \eta_{R/2}) g_U(x_0, \xi_{R/2})}.$$

(e) For any $x_0, x \in U, \xi \in \partial U$ such that $0 < r < R < A^{-1} d(\xi, x_0) \leqslant A^{-1} d(\xi, x)$, we have

$$\sup_{y \in U \cap B(\xi,r)} \Theta_{x_0}^U(x,y) \leqslant C \Theta_{x_0}^U(x, \xi_{R/2}), \qquad \inf_{y \in U \cap B(\xi,r)} \Theta_{x_0}^U(x,y) \geqslant C \Theta_{x_0}^U(x, \xi_{R/2}), \tag{3.37}$$

and

$$\operatorname*{osc}_{y \in U \cap B(\xi,r)} \Theta_{x_0}^U(x,y) \leqslant C\left(\frac{r}{R}\right)^{\gamma} \Theta_{x_0}^U(x, \xi_{R/2}). \tag{3.38}$$

Proof. Let $A \in (1,\infty)$ be maximum of the constants δ^{-1} in EHI, A_0 and A_1 in Definition 3.9. Let us denote the corresponding constants C and C_1 by C_{EHI} and C_{BHP} respectively. We will use EHI and the boundary Harnack principle several times in the proof with the above constants $A, C_{\mathrm{EHI}}, C_{\mathrm{BHP}}$.

(a) For any $0 < r < (2A)^{-1} (d(\eta, x_0) \wedge d(z, x_0) \wedge \delta_U(z))$, define

$$M(r) := \sup_{(B(\eta,r)\cap(U\setminus\{x_0\}))\times(B(z,r)\cap(U\setminus\{x_0\}))} \Theta^U_{x_0},$$
$$m(r) := \inf_{(B(\eta,r)\cap(U\setminus\{x_0\}))\times(B(z,r)\cap(U\setminus\{x_0\}))} \Theta^U_{x_0}.$$

For any $(x_1, y_1), (x_2, y_2) \in (B(\eta, R/A) \cap (U\setminus\{x_0\})) \times (B(z, R/A) \cap (U\setminus\{x_0\}))$, we have

$$\begin{aligned}
&\frac{M(R)g_U(x_0, x_1)g_U(x_0, y_1) - g_U(x_1, y_1)}{g_U(x_0, x_1)g_U(x_0, y_1)} \\
&\quad \leqslant C_{\mathrm{BHP}} \frac{M(R)g_U(x_0, x_2)g_U(x_0, y_1) - g_U(x_2, y_1)}{g_U(x_0, x_2)g_U(x_0, y_1)} \\
&\quad \leqslant C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} \frac{M(R)g_U(x_0, x_2)g_U(x_0, y_2) - g_U(x_1, y_2)}{g_U(x_0, x_2)g_U(x_0, y_2)}.
\end{aligned} \tag{3.39}$$

In the first line above, we apply boundary Harnack principle to the functions $M(R)g_U(x_0, \cdot)g_U(x_0, y_1) - g_U(\cdot, y_1), g_U(x_0, \cdot)g_U(x_0, y_1) \in \mathcal{F}^0_{\mathrm{loc}}(U, B(\eta, Ar) \cap U)$ that are non-negative and harmonic on $B(\xi, Ar) \cap U$. In the last line, we use the elliptic Harnack inequality to $M(R)g_U(x_0, x_2)g_U(x_0, \cdot) - g_U(x_2, \cdot), g_U(x_0, x_2)g_U(x_0, \cdot) \in \mathcal{F}_{\mathrm{loc}}(B(z, R))$ that are non-negative and harmonic on $B(z, R)$

Taking supremum over (x_1, y_1) and infimum over (x_2, y_2) in (3.39), we obtain

$$M(R) - m(R/A) \leqslant C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} (M(R) - M(R/A)). \tag{3.40}$$

By considering $(x, y) \mapsto \Theta^U_{x_0} - m(r) = \frac{g_U(x,y) - m(R)g_U(x_0,x)g_U(x_0,y)}{g_U(x_0,x)g_U(x_0,y)}$ and using a similar argument as the proof of (3.40), we obtain

$$M(R/A) - m(R) \leqslant C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} (m(R/A) - m(R)). \tag{3.41}$$

Combining (3.40) and (3.41), we obtain

$$M(R/A) - m(R/A) \leqslant \frac{C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} - 1}{C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} + 1} (M(R) - m(R)).$$

Iterating the above estimate, we obtain (a) with $\gamma = (\log A)^{-1} \log \frac{C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} + 1}{C_{\mathrm{BHP}} C^2_{\mathrm{EHI}} - 1}$.

(b) For any $0 < r < (2A)^{-1} (d(\eta, x_0) \wedge d(\xi, x_0) \wedge d(\eta, \xi))$, define

$$M(r) := \sup_{(B(\eta,r)\cap(U\setminus\{x_0\}))\times(B(\xi,r)\cap(U\setminus\{x_0\}))} \Theta^U_{x_0},$$
$$m(r) := \inf_{(B(\eta,r)\cap(U\setminus\{x_0\}))\times(B(\xi,r)\cap(U\setminus\{x_0\}))} \Theta^U_{x_0}.$$

For any $(x_1,y_1),(x_2,y_2) \in (B(\eta,R/A) \cap (U\backslash\{x_0\})) \times (B(\xi,R/A) \cap (U\backslash\{x_0\}))$, we have

$$\begin{aligned}
&\frac{M(R)g_U(x_0,x_1)g_U(x_0,y_1) - g_U(x_1,y_1)}{g_U(x_0,x_1)g_U(x_0,y_1)} \\
&\qquad \leqslant C_{\mathrm{BHP}} \frac{M(R)g_U(x_0,x_2)g_U(x_0,y_1) - g_U(x_2,y_1)}{g_U(x_0,x_2)g_U(x_0,y_1)} \\
&\qquad \leqslant C_{\mathrm{BHP}}^2 \frac{M(R)g_U(x_0,x_2)g_U(x_0,y_2) - g_U(x_1,y_2)}{g_U(x_0,x_2)g_U(x_0,y_2)}. \qquad (3.42)
\end{aligned}$$

In the first line above, we apply boundary Harnack principle to the functions $M(R)g_U(x_0,\cdot)g_U(x_0,y_1) - g_U(\cdot,y_1), g_U(x_0,\cdot)g_U(x_0,y_1) \in \mathcal{F}^0_{\mathrm{loc}}(U, B(\eta,Ar)\cap U)$ that are non-negative and harmonic on $B(\xi,Ar)\cap U$. In the last line, we use boundary Harnack principle to $M(R)g_U(x_0,x_2)g_U(x_0,\cdot) - g_U(x_2,\cdot), g_U(x_0,x_2)g_U(x_0,\cdot) \in \mathcal{F}^0_{\mathrm{loc}}(U, U\cap B(\xi,R))$ that are non-negative and harmonic on $U \cap B(\xi,R)$

Taking supremum over (x_1,y_1) and infimum over (x_2,y_2) in (3.39), we obtain

$$M(R) - m(R/A) \leqslant C_{\mathrm{BHP}}^2 \left(M(R) - M(R/A)\right). \qquad (3.43)$$

By considering $(x,y) \mapsto \Theta^U_{x_0} - m(r) = \frac{g_U(x,y) - m(R)g_U(x_0,x)g_U(x_0,y)}{g_U(x_0,x)g_U(x_0,y)}$ and using a similar argument as the proof of (3.40), we obtain

$$M(R/A) - m(R) \leqslant C_{\mathrm{BHP}}^2 \left(m(R/A) - m(R)\right). \qquad (3.44)$$

Combining (3.40) and (3.41), we obtain

$$M(R/A) - m(R/A) \leqslant \frac{C_{\mathrm{BHP}}^2 - 1}{C_{\mathrm{BHP}}^2 + 1} \left(M(R) - m(R)\right).$$

Iterating the above estimate, we obtain (a) with $\gamma = (\log A)^{-1} \log \frac{C_{\mathrm{BHP}}^2 + 1}{C_{\mathrm{BHP}}^2 - 1}$.

(c) Let $(x,y) \in (B(\eta,R) \cap (U\backslash\{x_0\})) \times (B(z,R) \cap (U\backslash\{x_0\}))$, where η, z, R are as given in the statement of the lemma. Then by applying the boundary Harnack principle for the harmonic functions $g_U(\cdot,y)$ and $g_U(x_0,\cdot)$ on $B_U(\eta,AR)$ and by elliptic Harnack inequality for the harmonic functions $g_U(\eta_{R/2},\cdot)$ and $g_U(x_0,\cdot)$ on $B(z,AR)$, we obtain

$$\Theta^U_{x_0}(x,y) \leqslant C_{\mathrm{BHP}} \frac{g_U(\eta_{R/2},y)}{g_U(x_0,\eta_{R/2})g_U(x_0,y)} \leqslant C_{\mathrm{BHP}} C_{\mathrm{EHI}}^2 \frac{g_U(\eta_{R/2},z)}{g_U(x_0,\eta_{R/2})g_U(x_0,z)}.$$

(d) Let $(x,y) \in (B(\eta,R) \cap (U\backslash\{x_0\})) \times (B(\xi,R) \cap (U\backslash\{x_0\}))$, where η, ξ, R as given. Then by using the boundary Harnack principle for the harmonic functions $g_U(\cdot,y)$ and $g_U(x_0,\cdot)$ on $B_U(\eta,AR)$ and for the harmonic functions $g_U(\eta_{R/2},\cdot)$ and $g_U(x_0,\cdot)$ on $B_U(\xi,AR)$, we deduce

$$\Theta^U_{x_0}(x,y) \leqslant C_{\mathrm{BHP}} \frac{g_U(\eta_{R/2},y)}{g_U(x_0,\eta_{R/2})g_U(x_0,y)} \leqslant C_{\mathrm{BHP}}^2 \frac{g_U(\eta_{R/2},\xi_{R/2})}{g_U(x_0,\eta_{R/2})g_U(x_0,\xi_{R/2})},$$

and

$$\Theta^U_{x_0}(x,y) \geqslant C_{\mathrm{BHP}}^{-1} \frac{g_U(\eta_{R/2},y)}{g_U(x_0,\eta_{R/2})g_U(x_0,y)} \geqslant C_{\mathrm{BHP}}^{-2} \frac{g_U(\eta_{R/2},\xi_{R/2})}{g_U(x_0,\eta_{R/2})g_U(x_0,\xi_{R/2})}.$$

(e) By the boundary Harnack principle applied to the harmonic functions $g_U(x, \cdot)$ and $g_U(x_0, x) g_U(x_0, \cdot)$ on $B(\xi, AR) \cap U$ we obtain (3.37). By Lemma 3.10, we obtain

$$\operatorname*{osc}_{y \in U \cap B(\xi,r)} \Theta^U_{x_0}(x,y) \leqslant C_0 \left(\frac{r}{R}\right)^\gamma \operatorname*{osc}_{y \in U \cap B(\xi,r)} \Theta^U_{x_0}(x,y) \leqslant C_0 \left(\frac{r}{R}\right)^\gamma \sup_{y \in U \cap B(\xi,r)} \Theta^U_{x_0}(x,y).$$

The above estimate along with (3.37) implies (3.38).

□

Thanks to the Hölder regularity estimates obtained in Lemma 3.14, we can extended $\Theta^U_{x_0}$ to $(\overline{U} \backslash \{x_0\}) \times (\overline{U} \backslash \{x_0\}) \backslash (\overline{U} \backslash \{x_0\})_{\mathrm{diag}}$ as shown below.

Proposition 3.15. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain as given in Assumption 3.13. For any $x_0 \in U$, the function $\Theta^U_{x_0}(\cdot, \cdot)$ defined in (3.36) admits a continuous extension, which is again denoted by $\Theta^U_{x_0} : (\overline{U} \backslash \{x_0\}) \times (\overline{U} \backslash \{x_0\}) \backslash (\overline{U} \backslash \{x_0\})_{\mathrm{diag}} \to [0, \infty)$. There exist $C_1, C_2, A_1 \in (1, \infty), c_0 \in (0, 1/4), \gamma \in (0, \infty)$ such that the following estimates hold:*

$$C_1^{-1} \frac{g_U(\eta_r, \xi_r)}{g_U(x_0, \eta_r) g_U(x_0, \xi_r)} \leqslant \Theta^U_{x_0}(\eta, \xi) \leqslant C_1 \frac{g_U(\eta_r, \xi_r)}{g_U(x_0, \eta_r) g_U(x_0, \xi_r)}, \tag{3.45}$$

where $r = c_0(d(x_0, \eta) \wedge d(x_0, \xi) \wedge d(\eta, \xi))$;

$$\left|\Theta^U_{x_0}(\xi, \eta) - \Theta^U_{x_0}(x, y)\right| \leqslant C_2 \Theta^U_{x_0}(\xi, \eta) \left(\frac{d(\eta, x)^\gamma}{R^\gamma} + \frac{d(\xi, y)^\gamma}{R^\gamma}\right) \tag{3.46}$$

for all $\eta, \xi \in \partial U$ with $\eta \neq \xi$, $0 < R < (2A)^{-1}(d(\eta, x_0) \wedge d(\eta, \xi) \wedge d(\xi, x_0))$, $x \in \overline{U} \cap B(\eta, R), y \in \overline{U} \cap B(\xi, R)$. Furthermore $\Theta^U_{x_0}(\cdot, \cdot)$ is symmetric in $(\overline{U} \backslash \{x_0\}) \times (\overline{U} \backslash \{x_0\}) \backslash (\overline{U} \backslash \{x_0\})_{\mathrm{diag}}$.

Proof. The existence of a continuous extension to $(\overline{U} \backslash \{x_0\}) \times (\overline{U} \backslash \{x_0\}) \backslash (\overline{U} \backslash \{x_0\})_{\mathrm{diag}}$ of the function defined in (3.36) follows from Lemma 3.14. More precisely, the existence of continuous extension at all points in $\partial U \times (U \backslash \{x_0\})$ and $(U \backslash \{x_0\}) \cup \partial U$ follows from Lemma 3.14(a,c) along with the symmetry of Green function. On the other hand, the existence of continuous extension at all points in $(\partial U \times \partial U) \backslash (\partial U)_{\mathrm{diag}}$ and follows from Lemma 3.14(b,d).

The estimates (3.45) and (3.46) are direct consequences of Lemma 3.14(b,d). The symmetry of $\Theta^U_{x_0}$ follows from the symmetry of the Green function and the continuity of $\Theta^U_{x_0}$. □

Definition 3.16. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain as given in Assumption 3.13. The function $\Theta^U_{x_0} : (\overline{U} \backslash \{x_0\}) \times (\overline{U} \backslash \{x_0\}) \backslash (\overline{U} \backslash \{x_0\})_{\mathrm{diag}} \to [0, \infty)$ defined as the continuous extension of (3.36) is called the **Naïm kernel** of the domain U with base point $x_0 \in U$.

This function is essentially same as the one introduced by L. Naïm in [Naï] where she extends to function considered in (3.36) to the Martin boundary instead of the topological boundary as considered above. Another difference from [Naï] is the use of Martin topology and fine topology of H. Cartan instead of the topology arising from the metric.

3.4 Martin kernel

We recall the definition of the closely related *Martin kernel* introduced by R. S. Martin [Mar].

Definition 3.17. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain as given in Assumption 3.13. We define the **Martin kernel** $K^U_{x_0}(\cdot,\cdot) : U \times (\overline{U}\backslash\{x_0\})\backslash U_{\text{diag}} \to [0,\infty)$ as

$$K_{x_0}(x,\xi) := \begin{cases} \frac{g_U(x,\xi)}{g_U(x_0,\xi)} & \text{if } x,\xi \in U, x \neq \xi \\ \lim_{y\to\xi, y\in U} \frac{g_U(x,y)}{g_U(x_0,y)} & \text{if } \xi \in \partial U. \end{cases} \tag{3.47}$$

The above limit exists in the second case due to the *boundary Harnack principle* (by Lemma 3.10).

The following oscillation lemma is an analogue of Lemma 3.10.

Lemma 3.18. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain as given in Assumption 3.13. There exist $C, A \in (1,\infty)$ and $\gamma > 0$ such that the following estimates hold:*

(a) For any $x_0 \in U, z \in U, \xi \in \partial U, 0 < r < R < (2A)^{-1}(\delta_U(z) \wedge d(x_0,\xi))$, we have

$$\operatorname*{osc}_{(U\cap B(z,r))\times(\overline{U}\cap B(\xi,r))} K^U_{x_0}(\cdot,\cdot) \leqslant A\left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{(U\cap B(z,R))\times(\overline{U}\cap B(\xi,R))} K^U_{x_0}(\cdot,\cdot) \tag{3.48}$$

(b) For any $x_0 \in U, z \in U, \xi \in \partial U, 0 < r < R < (2A)^{-1}(\delta_U(z) \wedge d(x_0,\xi))$, we have

$$\sup_{(U\cap B(z,R))\times(\overline{U}\cap B(\xi,R))} K^U_{x_0}(\cdot,\cdot) \leqslant C K^U_{x_0}(z,\xi_{R/2}).$$

(c) For any $(\eta,\xi) \in (\partial U)\times(\partial U)\backslash(\partial U)_{\text{diag}}$, for all $0 < r < R < (2A)^{-1}(d(\xi,x_0) \wedge d(\eta,x_0) \wedge d(\xi,\eta))$, we have

$$\sup_{x\in U\cap B(\eta,R)} \operatorname*{osc}_{y\in\overline{U}\cap B(\xi,r)} K^U_{x_0}(x,y) \leqslant C\left(\frac{r}{R}\right)^{\gamma} K^U_{x_0}(\eta_{R/2},\xi_{R/2}) \tag{3.49}$$

Proof. We will omit the proofs (a) and (b) as it similar to that Lemma 3.10. Both estimates follow from applying the elliptic Harnack inequality and boundary Harnack principle to the first and second arguments respectively of the Martin kernel.

(c) By Lemma 3.10

$$\operatorname*{osc}_{y\in\overline{U}\cap B(\xi,r)} K^U_{x_0}(x,y) \lesssim \left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{y\in\overline{U}\cap B(\xi,R)} K^U_{x_0}(x,y) \lesssim \left(\frac{r}{R}\right)^{\gamma} K^U_{x_0}(x,\xi_{R/2})$$

for all $x \in U \cap B(\eta,R)$. By the Carleson's estimate (Proposition 3.11), we have

$$\sup_{x\in U\cap B(\eta,R)} K^U_{x_0}(x,\xi_{R/2}) \lesssim K^U_{x_0}(\eta_{R/2},\xi_{R/2}).$$

Combining the above two estimates, we obtain the desired result. □

Lemma 3.19. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain as given in Assumption 3.13. For all $\xi \in \overline{U}$, the function $K_{x_0}(\cdot, \xi) : U \to [0, \infty)$ belongs to $\mathcal{F}_{\mathrm{loc}}(U)$, harmonic in U. Furthermore $K_{x_0}(\cdot, \xi)$ has Dirichlet boundary condition on $\partial U \backslash \{\xi\}$ in the following sense: for any open subset V of U such that $\xi \notin \overline{V}$, $K_{x_0}(\cdot, \xi) \in \mathcal{F}^0_{\mathrm{loc}}(U, V)$.*

Proof. Let $y_n \in U$ be a sequence with $\lim_{n\to\infty} y_n = \xi$. Define $h_n : U\backslash\{y_n\} \to [0, \infty)$ as $h_n := K^U_{x_0}(\cdot, y_n)$ for all $n \geqslant 1$.

If $K \subset U$ is compact then $K \subset U\backslash\{y_n\}$ for all but finitely many n. By Lemma 3.18(a)-(b), the sequence h_n converges uniformly on compact subsets of U and is bounded on compact sets. Therefore by Proposition 3.1(iv) and Lemma 2.19, the function $K(\cdot, \xi) : U \to [0, \infty)$ belongs to $\mathcal{F}_{\mathrm{loc}}(U)$ and is harmonic in U.

Let V be an open subset of U such that $\xi \notin \overline{V}$ and let $A \subset V$ be relatively compact in $\overline{U}$ with $\mathrm{dist}(A, U\backslash V) > 0$. Then by Lemma 3.18(c), h_n converges uniformly to $K^U_{x_0}(\cdot, \xi)$ on A. Therefore by Lemma 2.19(b), $K^U_{x_0}(\cdot, \xi) \in \mathcal{F}^0_{\mathrm{loc}}(U, V)$. □

Next, we relate Martin and Naïm kernels. Due to Lemma 3.19 and the continuity of $\Theta^U_{x_0}$, the Naïm kernel can be expressed in terms of the Martin kernel as

$$\Theta^U_{x_0}(x, y) = \begin{cases} \frac{K^U_{x_0}(x,y)}{g_U(x_0,x)}, & x \in U, \\ \lim_{z\to x, z\in U} \frac{K^U_{x_0}(z,y)}{g_U(x_0,z)}, & x \in \partial U. \end{cases} \tag{3.50}$$

The above limit can be shown using to the Boundary Harnack principle using Lemmas 3.19 and Lemma 3.10. We chose the approach based on Lemma 3.14 because the symmetry of $\Theta^U_{x_0}$ and the joint continuity are immediate using our approach while these properties need to be shown if we use (3.50). The equality (3.50) is closer to the original approach to define Naïim kernel as the extension to boundary is done for one argument at a time in [Naï].

It is well known that any unbounded domain satisfying the boundary Harnack principle has a unique Martin kernel point at infinity. Following [GyS, Chapter 4], we call the *harmonic profile* of U [Anc78, Théorème 6.1, Lemme 6.2] as the Martin kernel point at infinity. We recall the short argument to prove its uniqueness.

Lemma 3.20. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a unbounded domain satisfying Assumption 3.13. Let $h_1, h_2 : U \to (0, \infty)$ be two continuous functions such that $h_1, h_2 \in \mathcal{F}^0_{\mathrm{loc}}(U, U)$ and h_1, h_2 are harmonic in U. Then there exists $c > 0$ such that $h_1(x) = c h_2(x)$ for all $x \in U$.*

Proof. Let $A \in (1, \infty)$ be the largest among constants A_0, A_1 in Definition 3.9 and Lemma 3.10. Let C be largest among the constants C_1, C_0 in Definition 3.9 and Lemma 3.10 respectively. Let γ be as given in Lemma 3.10.

Let $\xi \in \partial U, x_0 \in U$. For all $R > Ad(\xi, x_0)$, by Definition 3.9 we have

$$\sup_{B(x,R)\cap U} \frac{h_1(\cdot)}{h_2(\cdot)} \leqslant C \frac{h_1(x_0)}{h_2(x_0)}.$$

Letting $R \to \infty$, we obtain

$$\operatorname*{osc}_U h \leqslant \sup_U h \leqslant C \frac{h_1(x_0)}{h_2(x_0)}.$$

For any $0 < d(\xi, x_0) < r < R < \infty$, we have

$$\operatorname*{osc}_{B(x,r)\cap U} \frac{h_1(\cdot)}{h_2(\cdot)} \leqslant C\left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{B(x,R)\cap U} \frac{h_1(\cdot)}{h_2(\cdot)} \leqslant C\left(\frac{r}{R}\right)^{\gamma} \sup_U \frac{h_1(\cdot)}{h_2(\cdot)} \leqslant C^2 \left(\frac{r}{R}\right)^{\gamma} \frac{h_1(x_0)}{h_2(x_0)}.$$

Let $R \to \infty$, we obtain $\operatorname{osc}_{B(x,r)\cap U} \frac{h_1(\cdot)}{h_2(\cdot)} = 0$ for any $r \in (d(x_0, \xi), \infty)$. Letting $r \to \infty$, we obtain $\operatorname{osc}_U \frac{h_1(\cdot)}{h_2(\cdot)} = 0$. □

We recall a standard construction of the harmonic profile [GyS, Chapter 4].

Proposition 3.21 (Harmonic profile)**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be an unbounded domain satisfying Assumption 3.13. For any $x_0 \in U$ and a sequence $(y_n)_{n\geqslant 1}$ in U such that $\lim_{n\to\infty} d(x_0, y_n) = \infty$, Then the sequence $K^U_{x_0}(\cdot, y_n) : U\backslash\{y_n\} \to \infty$ converges to a continuous function $h^U_{x_0} : U \to (0, \infty)$ uniformly on bounded subsets of U such that $h^U_{x_0} \in \mathcal{F}^0_{\mathrm{loc}}(U, U)$, $h^{x_0}_U(x_0) = 1, h^U_{x_0}(\cdot)$ is bounded on bounded subsets of U and is harmonic on U. Furthermore, the limit $h^U_{x_0}(\cdot)$ depends only on U, x_0 and not on the sequence $(y_n)_{n\geqslant 1}$.*

Proof. Let $A \in (1, \infty)$ be the largest among constants A_0, A_1 in Definition 3.9 and Lemma 3.10. Let C be largest among the constants C_1, C_0 in Definition 3.9 and Lemma 3.10 respectively.

Let $\xi \in \partial U$ and let $Ad(x_0, \xi) < r < R$. Then for any $n, k \in \mathbb{N}$ such that $AR < d(\xi, y_n) \wedge d(\xi, y_k)$, by Lemma 3.10 and Definition 3.9 we estimate

$$\begin{aligned}
\sup_{U\cap B(\xi,r)} \left| \frac{K^U_{x_0}(\cdot, y_n)}{K^U_{x_0}(\cdot, y_k)} - 1 \right| &= \sup_{U\cap B(\xi,r)} \left| \frac{K^U_{x_0}(\cdot, y_n)}{K^U_{x_0}(\cdot, y_k)} - \frac{K^U_{x_0}(x_0, y_n)}{K^U_{x_0}(x_0, y_k)} \right| \leqslant \operatorname*{osc}_{U\cap B(\xi,r)} \frac{K^U_{x_0}(\cdot, y_n)}{K^U_{x_0}(\cdot, y_k)} \\
&\leqslant C\left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{U\cap B(\xi,r)} \frac{K^U_{x_0}(\cdot, y_n)}{K^U_{x_0}(\cdot, y_k)} \\
&\leqslant C\left(\frac{r}{R}\right)^{\gamma} \sup_{U\cap B(\xi,r)} \frac{K^U_{x_0}(\cdot, y_n)}{K^U_{x_0}(\cdot, y_k)} \\
&\leqslant C^2\left(\frac{r}{R}\right)^{\gamma} \frac{K^U_{x_0}(x_0, y_n)}{K^U_{x_0}(x_0, y_k)} = C^2\left(\frac{r}{R}\right)^{\gamma}
\end{aligned}$$

By letting $R = (2A)^{-1}(d(\xi, y_n) \wedge d(\xi, y_k))$, we obtain that for all n, m such that $d(\xi, y_n) \wedge d(\xi, y_k) > 2A^2 d(\xi, x_0)$, we have

$$\sup_{U\cap B(\xi,r)} \left| \frac{K^U_{x_0}(\cdot, y_n)}{K^U_{x_0}(\cdot, y_k)} - 1 \right| \leqslant C^2 (2A)^{\gamma} r^{\gamma} (d(\xi, y_n) \wedge d(\xi, y_k))^{-\gamma}. \tag{3.51}$$

By Carelson's estimate (Proposition 3.11) for any $\xi \in \partial U, r > 0$, there exist $C_1 > 0, N \in \mathbb{N}$ such that

$$\sup_{U\cap B(\xi,r)} K^U_{x_0}(\cdot, y_n) \lesssim K^U_{x_0}(\xi_{r/2}, y_n) \quad \text{for all } n \geqslant N. \tag{3.52}$$

By Harnack chaining ([BM19, p. 391]), there exist $N \in \mathbb{N}$, $C_2 = C_2(x_0, \xi, r)$ such that

$$K_{x_0}^U(\xi_{r/2}, y_n) \leqslant C_2 \quad \text{for all } n \geqslant N. \tag{3.53}$$

Combining (3.51), (3.52), and (3.53), we obtain

$$\lim_{n,k\to\infty} \sup_{U \cap B(\xi,r)} \left|K_{x_0}^U(\cdot, y_n) - K_{x_0}^U(\cdot, y_k)\right| \leqslant \lim_{n,k\to\infty} C_1 C_2 C^2 (2A)^\gamma r^\gamma (d(\xi, y_n) \wedge d(\xi, y_k))^{-\gamma} = 0.$$

Since $r > 0$ is arbitrary, by letting $r \to \infty$, we conclude that the sequence $K_{x_0}^U(\cdot, y_n), n \in \mathbb{N}$ converges uniformly in bounded subsets of U, say $h_{x_0}^U : U \to (0, \infty)$. By the continuity of $K_{x_0}^U(\cdot, y_n)$, we conclude that $h_{x_0}^U$ is continuous. The estimate (3.52) implies that $h_{x_0}^U$ is bounded on bounded subsets of U. By Lemma 2.19, we obtain that $h_{x_0}^U \in \mathcal{F}_{\mathrm{loc}}^0(U, U)$ and is harmonic in U.

The assertion that the limit $h_{x_0}^U(\cdot)$ depends only on U, x_0 follows from $h_{x_0}^U(x_0) = 1$ and Lemma 3.20. □

4 Estimates for harmonic and elliptic measures

To goal of this section is to estimate the harmonic measure of balls on the boundary of a uniform domain using ratio of Green functions. We restrict to the class of uniform domains that satisfy the following **capacity density condition**.

4.1 The Capacity density condition

This is a slight variant of similar conditions considered in [Anc86, AH].

Definition 4.1. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space satisfying the elliptic Harnack inequality. Let $K \in (1, \infty)$ be such that $(\mathcal{X}, d)$ is K-relatively ball connected. We say that a uniform domain U satisfies the capacity density condition (CDC) if there exist $A_0 \in (8K, \infty), A_1, C \in (1, \infty)$ such that for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U, d)/A_1$ we have

$$\operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R)) \leqslant C \operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R) \backslash U), \tag{CDC}$$

We note that the capacity density condition implies transience.

Remark 4.2. Let D be a domain that satisfies the capacity density condition (CDC). Then by [FOT, Theorem 4.4.3(ii)], D^c is non-polar. Hence by [BCM, Proposition 2.1 and Theorem 4.8], the associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^0(D))$ on $L^2(D; m|_D)$ of the part process X^D is transient.

Due to remark 2.17, it would be convenient to assume the stronger sub-Gaussian heat kernel estimate HKE(Ψ) instead of the elliptic Harnack inequality EHI. Therefore, we make the following assumption.

Assumption 4.3. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space such that the corresponding diffusion satisfies Assumption 2.14. In particular by Remark 2.17 and Lemma 2.23(a), $(\mathcal{X}, d)$ is K-relatively ball connected for some $K \in (1, \infty)$. Let U be a uniform domain satisfying the capacity density condition (CDC) and the boundary Harnack principle. We recall by Remark 4.2 that the Dirichlet form $(\mathcal{E}^U, \mathcal{F}^0(U))$ on $L^2(U; m|_U)$ corresponding to the part process on U is transient.

Ancona shows that the capacity density condition in an Euclidean domain is equivalent to an estimate on the harmonic measure called the *uniform* Δ*-regularity* [Anc86, Definition 2 and Lemma 3]. Such a result can be extended to an arbitrary domain on any MMD space satisfying the elliptic Harnack inequality using the estimates on hitting probability from [BM18, BCM]. More precisely, we have the following relationships between hitting probabilities and the capacity density condition. Part (b) of the lemma below is the justification behind our requirement $A_0 \in (8K, \infty)$ in Definition 4.1.

Lemma 4.4. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the elliptic Harnack inequality.*

(a) Suppose there exist $A_0, A_1 \in (1, \infty)$ and $\gamma \in (0, 1)$ such that

$$\omega_x^{U \cap B(\xi, A_0 R)}(U \cap S(\xi, A_0 R)) \leqslant 1 - \gamma \quad \text{for q.e. } x \in B(\xi, R) \text{ and } 0 < R < \operatorname{diam}(U, d)/A_1. \tag{4.1}$$

Then for all $0 < R < \operatorname{diam}(U, d)/A_1$, $\xi \in \partial U$, we have

$$\operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R)) \leqslant \gamma^{-2} \operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R) \setminus U). \tag{4.2}$$

(b) Let $K \in (1, \infty)$ be such that $(\mathcal{X}, d)$ is K-relatively ball connected. Suppose there exist $A_0 \in (8K, \infty), A_1, C \in (1, \infty)$ such that for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U, d)/A_1$, we have

$$\operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R)) \leqslant C \operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R) \cap U^c). \tag{4.3}$$

Then for any $\widehat{A_0} \in (1, \infty)$, there exist $\widehat{A_1}, \widehat{C} \in (1, \infty)$ such that for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U, d)/\widehat{A_1}$, we have

$$\operatorname{Cap}_{B(\xi, \widehat{A_0} R)}(B(\xi, R)) \leqslant \widehat{C} \operatorname{Cap}_{B(\xi, \widehat{A_0} R)}(B(\xi, R) \cap U^c). \tag{4.4}$$

(c) Let $K \in (1, \infty)$ be such that $(\mathcal{X}, d)$ is K-relatively ball connected. Suppose there exist $A_0 \in (8K, \infty), A_1, C \in (1, \infty)$ such that for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U, d)/A_1$, we have

$$\operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R)) \leqslant C \operatorname{Cap}_{B(\xi, A_0 R)}(B(\xi, R) \setminus U), \tag{4.5}$$

then there exist $\widehat{A_0}, \widehat{A_1} \in (1, \infty), \gamma \in (0, 1)$ such that

$$\omega_x^{U \cap B(\xi, \widehat{A_0} R)}(U \cap S(\xi, \widehat{A_0} R)) \leqslant 1 - \gamma \tag{4.6}$$

for q.e. $x \in B(\xi, R)$ and $0 < R < \operatorname{diam}(U, d)/\widehat{A_1}$. If in addition $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies Assumption 2.14, then (4.6) holds for all $x \in B(\xi, R)$.

Proof. (a) Let $e := e_{B(\xi,R)\setminus U, B(\xi,A_0R)}$ denote the equilibrium potential for $\operatorname{Cap}_{B(\xi,A_0R)}(B(\xi,R)\setminus U)$. Then by [FOT, Theorem 4.3.3], for q.e. $x \in B(\xi,R) \cap U$, we have

$$\begin{aligned}\widetilde{e}(x) &= \mathbb{P}_x\left(\sigma_{B(\xi,R)\setminus U} < \sigma_{B(\xi,A_0R)^c}\right)\\ &\geqslant \mathbb{P}_x\left(\sigma_{U\cap S(\xi,A_0R)} > \sigma_{U^c}\right) = 1 - \mathbb{P}_x\left(\sigma_{U\cap S(\xi,A_0R)} < \sigma_{U^c}\right) \overset{(4.1)}{\geqslant} \gamma.\end{aligned}$$

Therefore $\gamma^{-1}\widetilde{e} \geqslant 1$ q.e. on $B(\xi, A_0^{-1}r)$ and $\operatorname{Cap}_{B(\xi,A_0R)}(B(\xi,R)) \leqslant \mathcal{E}(\gamma^{-1}e, \gamma^{-1}e) = \gamma^{-2}\operatorname{Cap}_{B(\xi,A_0R)}(B(\xi,R)\setminus U)$.

(b) By [BCM, Lemma 5.22] and domain monotonicity of capacity, in order to show (4.4), we may and will assume that $\widehat{A_0} > A_0$. By [BCM, Lemma 5.18] there exist $C_2 > 1, \widehat{A_1} \geqslant A_1$ such that for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U,d)/\widehat{A_1}$

$$g_{B(\xi,A_0R)}(y,z) \leqslant g_{B(\xi,\widehat{A_0}R)}(y,z) \leqslant C_1 g_{B(\xi,A_0R)}(y,z) \quad \text{for all } y,z \in B(\xi,R). \tag{4.7}$$

Let e_1, ν be the equilibrium potential and measure for $\operatorname{Cap}_{B(\xi,\widehat{A_1}R)}(B(\xi,R)\setminus D)$ such that $\operatorname{Cap}_{B(\xi,\widehat{A_1}r)}(B(\xi,R)\setminus D) = \mathcal{E}(e_1,e_1)$ and $e_1(\cdot) = \int g_{B(\xi,\widehat{A_1}r)}(\cdot,z)\nu(dz)$, where ξ, R satisfy the conditions associated with (4.7). Define

$$e(\cdot) := \int g_{B(\xi,A_1r)}(\cdot,z)\nu(dz).$$

By (4.7), for q.e. $y \in B(\xi,R)\setminus D, \xi \in \partial D$, we have

$$e(y) = \int g_{B(\xi,A_1R)}(y,z)\nu(dz) \geqslant C_1^{-1}\int g_{B(\xi,\widehat{A_1}R)}(y,z)\nu(dz) \geqslant C_1^{-1}.$$

Therefore

$$\begin{aligned}\operatorname{Cap}_{B(\xi,A_1R)}(D\setminus B(\xi,R)) &\leqslant \mathcal{E}(C_1e, C_1e) = C_1^2\int e(z)\,\nu(dz) \leqslant C_1^2\int e_1(z)\,\nu(dz)\\ &= C_1^2\mathcal{E}(e_1,e_1) = C_1^2\operatorname{Cap}_{B(\xi,\widehat{A_1}R)}(B(\xi,R)\setminus D).\end{aligned}$$

The above estimate along with (4.3) and [BCM, Lemma 5.22] implies (4.4).

(c) By [BCM, Lemma 5.9], there exist $\widehat{A_0}, \widehat{A_1}, C_1 \in (1,\infty)$ such that for all $\xi \in U, 0 < R < \operatorname{diam}(U,d)/\widehat{A_1}$, and for all $x,y \in \overline{B(\xi,R)}$, we have

$$g_{B(\xi,\widehat{A_0}r)}(x,y) \geqslant C_1^{-1} g_{B(\xi,\widehat{A_0}r)}(\xi,r). \tag{4.8}$$

By (b) and increasing $\widehat{A_0}, \widehat{A_1}$ if necessary, we may assume that (4.4) holds. By further increasing $\widehat{A_0}, \widehat{A_1}$ if necessary and using [BCM, Lemma 5.10], we may assume that there exists $C_2 > 1$ such that for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U,d)/\widehat{A_1}$, we have

$$g_{B(\xi,\widehat{A_0}r)}(\xi,r) \leqslant \operatorname{Cap}_{B(\xi,\widehat{A_0}r)}\left(B(\xi,r)\right)^{-1} \leqslant C_2 g_{B(\xi,\widehat{A_0}r)}(\xi,r). \tag{4.9}$$

Let $\xi \in \partial U, 0 < R < \operatorname{diam}(U,d)/\widehat{A_1}$ and let $e := e_{B(\xi,R)\setminus U, B(\xi,\widehat{A_0}R)}, \nu$ denote the equilibrium potential and measure respectively for $\operatorname{Cap}_{B(\xi,\widehat{A_0}R)}(B(\xi,R)\setminus U)$. By [FOT, Theorem 4.3.3], for q.e. $x \in B(\xi,R) \cap U$, we have

$$\begin{aligned}
\widetilde{e}(x) &= \mathbb{P}_x\left(\sigma_{B(\xi,R)\setminus U} < \sigma_{B(\xi,\widehat{A_0}R)^c}\right) = \int_{\overline{B(\xi,R)\setminus U}} g_{B(\xi,\widehat{A_0}R)^c}(x,y)\,\nu(dy) \\
&\overset{(4.8)}{\geqslant} C^{-1} g_{B(\xi,\widehat{A_0}r)}(\xi,r)\nu\left(\overline{B(\xi,R)\setminus U}\right) \\
&= C^{-1} g_{B(\xi,\widehat{A_0}r)}(\xi,r)\operatorname{Cap}_{B(\xi,\widehat{A_0}R)}(B(\xi,R)\cap U^c) \\
&\overset{(4.4)}{\geqslant} C^{-1}\widehat{C}^{-1} g_{B(\xi,\widehat{A_0}r)}(\xi,r)\operatorname{Cap}_{B(\xi,\widehat{A_0}R)}(B(\xi,R)) \overset{(4.9)}{\geqslant} C^{-1}\widehat{C}^{-1}C_2^{-1}. \qquad (4.10)
\end{aligned}$$

Setting $\gamma = C^{-1}\widehat{C}^{-1}C_2^{-1} \in (0,1)$, we conclude

$$\omega_x^{U\cap B(\xi,\widehat{A_0}R)}(U \cap S(\xi,\widehat{A_0}R)) \leqslant \mathbb{P}_x\left(\sigma_{B(\xi,R)\setminus U} > \sigma_{B(\xi,\widehat{A_0}R)^c}\right) \overset{(4.10)}{\leqslant} 1-\gamma.$$

The final assertion under Assumption 2.14 follows from the continuity of harmonic measure due to Lemma 2.28(b).

□

The estimate (4.11) in the above Lemma can be used repeatedly to obtain certain polynomial type decay rates on the harmonic measure.

Lemma 4.5 (Uniform Δ-regularity). *Let $(\mathcal{X},d,m,\mathcal{E},\mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be a uniform domain that satisfy Assumption 4.3.*

(a) There exist $C_1 > 1, A_1 > 1, \delta > 0$ such that for all $0 < r < R < \operatorname{diam}(U,d)/A_1$ and for all $\xi \in \partial U$, we have

$$\omega_x^{U\cap B(\xi,R)}(U \cap S(\xi,R)) \leqslant C_1 \left(\frac{r}{R}\right)^{\delta}, \quad \text{for all } x \in U \cap B(\xi,r). \qquad (4.11)$$

(b) Assume in addition that U satisfies the boundary Harnack principle. Then there exist $C_2, A_0, A_1 \in (1,\infty), \delta > 0$ such that for all $0 < r < R < \operatorname{diam}(U,d)/A_1$, for all $\xi \in \partial U$, and for all continuous non-negative function $h : B(\xi,A_0r)\cap U \to (0,\infty)$ that is harmonic in $B(\xi,A_0r)$ with Dirichlet boundary condition on $\partial U \cap \left(\overline{U \cap B(\xi,A_0r)}\right)$, we have

$$\frac{h(\xi_r)}{h(\xi_R)} \leqslant C_2\left(\frac{r}{R}\right)^{\delta}. \qquad (4.12)$$

Proof. (a) By Lemma 4.4(a), there exist $A_0, A_1 \in (1,\infty)$ and $\gamma \in (0,1)$ such that

$$\omega_x^{U\cap B(\xi,R)}(U \cap S(\xi,R)) \leqslant 1-\gamma \qquad (4.13)$$

for all $\xi \in \partial U, x \in B(\xi, A_0^{-1}R)$ and $0 < R < \mathrm{diam}(U,d)/A_1$. By the strong Markov property, for all $i \in \mathbb{N}, \xi \in \partial U, x \in B(\xi, A_0^{-i}R)$ and $0 < R < \mathrm{diam}(U,d)/A_1$

$$\begin{aligned}
&\omega_x^{U\cap B(\xi,R)}(U \cap S(\xi,R)) \\
&\quad \leqslant \omega_x^{U\cap B(\xi,A_0^{-i}R)}(U \cap S(\xi, A_0^{-i}R)) \sup_{y \in U\cap S(\xi,A_0^{-i}R)} \omega_y^{U\cap B(\xi,R)}(U\cap S(\xi,R)) \\
&\quad \overset{(4.13)}{\leqslant} (1-\gamma) \sup_{y\in U\cap S(\xi,A_0^{-i+1}R)} \omega_y^{U\cap B(\xi,R)}(U\cap S(\xi,R)).
\end{aligned}$$

By repeatedly using the above estimate, we obtain

$$\omega_x^{U\cap B(\xi,R)}(U\cap S(\xi,R)) \leqslant (1-\gamma)^i$$

for all $i \in \mathbb{N}, \xi \in \partial U, x \in B(\xi, A_0^{-i}R)$ and $0 < R < \mathrm{diam}(U,d)/A_1$. This implies (4.11).

(b) By the boundary Harnack principle and Proposition 3.1, it suffices to consider the case when h is a Green function. More precisely, it suffices to show that there exist $C_3, A_0, A_1 \in (1,\infty), \delta > 0$ such that for all $0 < r < R < \mathrm{diam}(U,d)/A_1$, for all $\xi \in \partial U$, and for all $x_0 \in U$ such that $d(\xi, x_0) > A_0 r$, we have

$$\frac{g_U(\xi_r, x_0)}{g_U(\xi_R, x_0)} \leqslant C_3 \left(\frac{r}{R}\right)^{\delta}. \tag{4.14}$$

Let us choose $A_0, A_1 \in (1,\infty)$ such that the conclusion on (a) and the boundary Harnack principle and Carleson's inequality hold. For all $\xi \in \partial U, 0 < r < R, x_0 \in U$ as above, we have

$$\begin{aligned}
g_U(\xi_r, x_0) &= \mathbb{E}_{\xi_r}\left[g_U\left(X^U_{\tau_{U\cap B(\xi,R)}}, x_0\right)\right] \quad \text{(by Lemma 3.4(b))} \\
&\leqslant \left(\sup_{U\cap S(\xi,R)} g_U(\cdot, x_0)\right) \omega_{\xi_r}^{U\cap B(\xi,R)}(U\cap S(\xi,R)) \\
&\lesssim g_U(\xi_R, x_0)\omega_{\xi_r}^{U\cap B(\xi,R)}(U\cap S(\xi,R)) \quad \text{(by Carleson's estimate)} \\
&\lesssim g_U(\xi_R, x_0)\left(\frac{r}{R}\right)^{\delta} \quad \text{(by (4.11))}.
\end{aligned}$$

□

4.2 Two-sided bounds on harmonic measure

The following estimate of harmonic measure is the main result of this section. It is an extension of [AH, Lemmas 3.5 and 3.6] obtained for the Brownian motion and uniform domains satisfying the capacity density condition in Euclidean space which in turn generalize similar results obtained by Jerison and Kenig for NTA domains [JK, Lemma 4.8] and Dahlberg for Lipschitz domains [Dah, Lemma 1]. While it is possible to follow the 'box argument' in [AH], our proof is new and avoids the use of a complicated iteration argument (called the 'box argument') to obtain upper bound on harmonic measure [AH, Proof of Lemma 3.6].

Theorem 4.6. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be a uniform domain that satisfy Assumption 4.3. Then there exist $C, A \in (1, \infty), c_0 \in (0,1)$ such that*

$$C^{-1} g_U(x, \xi_r) \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r)) \leqslant \omega_x^U(\partial U \cap B(\xi,r)) \leqslant C g_U(x,\xi_r) \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r)) \tag{4.15}$$

for all $\xi \in \partial U, x \in U$ such that $0 < r < d(\xi, x_0)/A$.

While it is possible to prove Theorem 4.6 by adapting the techniques of Aikawa and Hirata using the *box argument* and the notion of *capacitary width*, we follow a more probabilistic approach. An easy consequence of Theorem 4.6, harmonicity of $g_U(x,\cdot)$ on $U \backslash \{x\}$, Harnack chaining (Lemma 2.24), (2.12) and the doubling property of m is the following doubling property of harmonic measure.

Corollary 4.7. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be a uniform domain that satisfy Assumption 4.3. There exist $C, A \in (1, \infty), c_0 \in (0,1)$ such that*

$$\omega_x^U(\partial U \cap B(\xi,r)) \leqslant C \omega_x^U(\partial U \cap B(\xi, r/2)) \tag{4.16}$$

for all $\xi \in \partial U, x \in U$ such that $0 < r < d(\xi, x_0)/A$.

Thanks to the capacity density condition, we can compare Green's function on a the domain U with that of a ball chosen at a suitable scale. The following is an analogue of a lemma of Aikawa and Hirata for uniform domains in Euclidean space [AH, Lemma 3.2]. Our proof follows an argument in [BM18, Proof of Lemma 3.12] to compare Green functions in different domains.

Lemma 4.8. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be a uniform domain that satisfy Assumption 4.3. There exist $A_1 \in (1, \infty)$ and $c_0 \in (0,1)$ such that for any $0 < c \leqslant c_0$, there exists C_1 such that the following estimate holds: $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1$, we have*

$$C_1^{-1} \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \leqslant g_U(\xi_r, cr) \leqslant C_1 \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1}. \tag{4.17}$$

Proof. By Lemma 4.5(a), there exist $A_1, A_0 \in (4, \infty)$ such that for all $0 < r < \operatorname{diam}(U,d)/A_1$ for all $\xi \in \partial U$, we have

$$\sup_{z \in U \cap B(\xi,2r)} \omega_z^{U \cap B(\xi, A_0 r)}(U \cap S(\xi, A_0 r)) \leqslant \frac{1}{2}. \tag{4.18}$$

By [BCM, Lemmas 5.10, 5.20(a) and 5.24], there exist $c_0 \in (0, c_U/2), \widetilde{A}_1 \in (4, \infty)$ such that for all $c \in (0, c_0]$ there exists C_2 satisfying the following estimate: for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/\widetilde{A}_1, y \in S(\xi_r, cr)$, we have

$$C_2^{-1} \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \leqslant g_{B(\xi_r, c_U r/2)}(\xi_r, y) \leqslant g_{B(\xi_r, A_0 r)}(\xi_r, y) \leqslant C_2 \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1}. \tag{4.19}$$

Using Lemma 3.8(a) and reducing c_0 further if necessary, there exists $C_3 \in (1,\infty)$ such that

$$\sup_{S(\xi_r,cr)} g_U(\xi_r,\cdot) \leqslant C_3 \inf_{S(\xi_r,cr)} g_U(\xi_r,\cdot) \tag{4.20}$$

for all $c \in (0,c_0]$, $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/\widetilde{A}_1$ and ξ_r satisfying the conclusion of Lemma 2.6.

Let $\eta \in S(\xi_r, c\delta_U(\xi_r))$ be such that

$$g_U(\xi_r,\eta) = \sup_{y \in S(\xi_r, c\delta_U(\xi_r))} g_U(\xi_r, y). \tag{4.21}$$

Then by the maximum principle (the latter inequality of Proposition 3.1(v)) and Dynkin-Hunt formula (Lemma 3.6), for all $0 < r < \operatorname{diam}(U,d)/A_1$, for all $\xi \in \partial U$, by choosing $\eta \in S(\xi_r, c\delta_U(\xi_r))$ satisfying (4.21), we obtain

$$\begin{aligned} g_U(\xi_r,\eta) &= g_{U\cap B(\xi,A_0 r)}(\xi_r,\eta) + \mathbb{E}_\eta\left[\mathbb{1}_{\{\tau_{U\cap B(\xi,A_0 r)}<\infty, X_{\tau_{U\cap B(\xi,A_0 r)}}\in U\}} g_U(X_{\tau_{U\cap B(\xi,A_0 r)}},\xi_r)\right] \\ &\leqslant g_{U\cap B(\xi,A_0 r)}(\xi_r,\eta) + g_U(\xi_r,\eta)\mathbb{P}_\eta\left[\tau_{U\cap B(\xi,A_0 r)} < \infty, X_{\tau_{U\cap B(\xi,A_0 r)}} \in U\right] \\ &\leqslant g_{U\cap B(\xi,A_0 r)}(\xi_r,\eta) + \frac{1}{2} g_U(\xi_r,\eta) \quad \text{(by (4.18))}. \end{aligned}$$

and hence

$$g_{B(\xi,c_U r/2)}(\xi_r,\eta) \leqslant g_{U\cap B(\xi,A_0 r)}(\xi_r,\eta) \leqslant g_U(\xi_r,\eta) \leqslant 2 g_{U\cap B(\xi,A_0 r)}(\xi_r,\eta) \leqslant 2 g_{B(\xi,A_0 r)}(\xi_r,\eta). \tag{4.22}$$

Combining (4.19), (4.22) and (4.20), we obtain the desired estimate. □

Proof of Theorem 4.6. We first show the lower bound on the harmonic measure which is considerably easier than the upper bound.

Lower bound on harmonic measure: By Lemma 4.5, there exists $c_1 \in (0,1/2)$ such that for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, y \in U \cap B(\xi, 2c_1 r)$, then

$$\omega_y^U(B(\xi,r)\cap \partial U) \geqslant 1 - \omega_y^{U\cap B(\xi,r)}(U \cap S(\xi,r)) \geqslant \frac{1}{2}. \tag{4.23}$$

By Lemmas 3.8(b) and 4.8 and increasing A_1 if necessary, there exist $c_2 \in (0,c_1), C_1, C_2 \in (1,\infty)$ such that

$$C_1^{-1}\frac{g_U(x_0,\xi_{c_1 r})}{g_U(\xi_{c_1 r}, c_2 r)} \leqslant \mathbb{P}_{x_0}\left(\sigma_{\overline{B(\xi_{c_1 r}, c_2 r)}} < \sigma_{U^c}\right) \leqslant C_1 \frac{g_U(x_0,\xi_{c_1 r})}{g_U(\xi_{c_1 r}, c_2 r)} \tag{4.24}$$

and

$$C_2^{-1}\operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \leqslant g_U(\xi_{c_1 r}, c_2 r) \leqslant C_2 \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \tag{4.25}$$

for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1$ and $x_0 \in U\backslash B(\xi,2r)$.

The lower bound on the harmonic measure is obtained by estimating the probability of the event that diffusion first hits the set $\overline{B(\xi_{c_1 r}, c_2 r)}$ before exiting U along $\partial U \cap B(\xi, r)$. Setting $B_0 := \overline{B(\xi_{c_1 r}, c_2 r)}$, we estimate the harmonic measure as

$$\begin{aligned}
\omega_x^U(\partial U \cap B(\xi,r)) &\geqslant \mathbb{P}_x\left(\sigma_{B_0} < \sigma_{U^c}, X_{\sigma_{U^c}} \in \partial U \cap B(\xi,r)\right) \\
&= \mathbb{P}_x(\sigma_{B_0} < \sigma_{U^c})\mathbb{E}_x\left[\omega^U_{X_{\sigma_{B_0}}}(\partial U \cap B(\xi,r))\right] \quad \text{(strong Markov property)} \\
&\geqslant \mathbb{P}_x(\sigma_{B_0} < \sigma_{U^c}) \inf_{y \in B_0} \omega_y^U(\partial U \cap B(\xi,r)) \overset{(4.23)}{\geqslant} \frac{1}{2}\mathbb{P}_x(\sigma_{B_0} < \sigma_{U^c}) \\
&\overset{(4.24)}{\geqslant} (2C_1)^{-1}\frac{g_U(x,\xi_{c_1 r})}{g_U(\xi_{c_1 r}, c_2 r)} \overset{(4.25)}{\geqslant} (2C_1C_2)^{-1} g_U(x,\xi_{c_1 r/2}) \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))
\end{aligned} \tag{4.26}$$

for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1$ and $x \in U \backslash B(\xi, 2r)$.

By Lemma 2.24 there exist $A_0, C_3 \in (1,\infty)$ such that

$$C_3 g_U(x,\xi_r) \geqslant g_U(x,\xi_{c_1 r/2}) \geqslant C_3^{-1} g_U(x,\xi_r) \tag{4.27}$$

for all $\xi \in \partial U, r > 0, x \in U \backslash B(\xi, A_0 r)$. Combining (4.37) and (4.27), we obtain the desired lower bound.

Upper bound on harmonic measure: We consider two cases depending on whether on not $(B(\xi,4r)\backslash B(\xi,2r)) \cap \partial U$ is empty.

Case 1: $B(\xi,4r)\backslash B(\xi,2r)) \cap \partial U = \emptyset$. In this case, we use the estimate

$$\omega_x^U(\partial U, B(\xi,r)) \leqslant \mathbb{P}_x\left(\sigma_{S(\xi,3r)\cap U} < \sigma_{U^c}\right). \tag{4.28}$$

By Lemma 3.8(b) and the same argument as the proof of Lemma 4.8 (using [BCM, Lemmas 5.10, 5.20(a) and 5.24]), there exist $c_1 \in (0,1), A_1, C_3, C_4 \in (1,\infty)$ such that

$$g_U(y, c_1 r) \geqslant g_{B(y,r)}(y, c_1 r) \geqslant C_3^{-1} \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \geqslant C_4^{-1} \operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \tag{4.29}$$

and

$$\mathbb{P}_x\left(\sigma_{\overline{B(y,c_1 r)}} < \sigma_{U^c}\right) \leqslant C_3 \frac{g_U(y,x_0)}{g_U(y,c_1 r)} \tag{4.30}$$

for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, y \in U \cap S(\xi,3r), x_0 \in U\backslash B(\xi,4r)$. By Lemma 2.23(b) and the proof of Lemma 2.24, there exist $A_0, C_5 \in (1,\infty)$ such that for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, y \in U \cap S(\xi,3r), x_0 \in U\backslash B(\xi, A_0 r)$, we have

$$g_U(y,x_0) \leqslant C_5 g_U(\xi_r, x_0). \tag{4.31}$$

Using the metric doubling property and by choosing a maximal $c_1 r$ separated subset $\{y_i : 1 \leqslant i \leqslant N\}$ of $U \cap S(\xi,3r)$, we have $U \cap S(\xi,3r) \subset \cup_{i=1}^N B(y_i, c_1 r)$, where $y_i \in U \cap S(\xi,3r)$ for all $i = 1,\dots,N$ and N has an upper bound that depends only on the doubling property and c_1. Therefore by (4.28), we obtain

$$\omega_{x_0}^U(B(\xi,r)\cap\partial U) \leqslant \mathbb{P}_{x_0}\left(\sigma_{\cup_{i=1}^N B(y_i,c_1r)} < \sigma_{U^c}\right) \leqslant \sum_{i=1}^N \mathbb{P}_{x_0}\left(\sigma_{\overline{B(y_i,c_1r)}} < \sigma_{U^c}\right)$$
$$\overset{(4.30)}{\leqslant} \sum_{i=1}^N C_3 \frac{g_U(y,x_0)}{g_U(y,c_1r)} \overset{(4.31)}{\leqslant} NC_3 \frac{g_U(\xi_r,x_0)}{g_U(y,c_1r)} \overset{(4.29)}{\leqslant} NC_3C_4 \frac{g_U(\xi_r,x_0)}{g_U(y,c_1r)} \tag{4.32}$$

for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, x_0 \in U\backslash B(\xi, A_0r)$. The desired upper bound in this case follows from (4.32) and (4.29).

Case 2: $B(\xi,4r)\backslash B(\xi,2r)) \cap \partial U \neq \varnothing$. Let $V := \overline{U}\backslash(\partial U\backslash B(\xi,3r/2))$ (note that V is an open subset of $\overline{U}$). By Theorem 2.12 and Proposition 2.13, we may assume that the reflected diffusion X^{ref} can be defined from every starting point $x \in \overline{U}$. We denote the corresponding probability measures and expectations by $\mathbb{P}_x^{\text{ref}}, \mathbb{E}_x^{\text{ref}}$ respectively.

Let $\eta \in B(\xi,4r)\backslash B(\xi,2r)) \cap \partial U$. Since $B(\eta,r/2)\cap\partial U \subset \overline{U}\backslash V$, by Lemma 4.5(a), irreducibility of X^{ref}, [CF, Theorem 3.5.6] and the part process $(X^{\text{ref}})^V$ on V is transient. Hence there exists a green function on the domain V corresponding to $\overline{U}, d, m|_{\overline{U}}, \mathcal{E}^{\text{ref}}, \mathcal{F}(U))$ denoted by $g_V^{\text{ref}}(\cdot,\cdot)$ satisfying the properties in Proposition 3.1 and Lemma 3.4(b).

By Lemma 3.8 and arguing similarly as (4.29) and (4.30), there exist $A_1, C_1, C_2 \in (1,\infty), c_1 \in (0, c_U/4)$ such that

$$\sup_{y\in S(\xi_r,c_1r)} g_V^{\text{ref}}(\xi_r,y) \leqslant C_1 g_V^{\text{ref}}(\xi_r,c_1r), \tag{4.33}$$

$$C_1^{-1}\frac{g_V^{\text{ref}}(x,\xi_r)}{g_V^{\text{ref}}(\xi_r,c_1r)} \leqslant \mathbb{P}_x^{\text{ref}}\left(\sigma_{\overline{B(\xi_r,c_1r)}} < \sigma_{\partial U\backslash B(\xi,3r/2)}\right) \leqslant C_1\frac{g_V^{\text{ref}}(x,\xi_r)}{g_V^{\text{ref}}(\xi_r,c_1r)}, \tag{4.34}$$

$$g_V^{\text{ref}}(\xi_r,c_1r) \geqslant g_U(\xi_r,c_1r) \geqslant C_2^{-1}\operatorname{Cap}_{B(\xi,2r)}(B(\xi,r))^{-1} \tag{4.35}$$

for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, x \in \overline{U}\backslash B(\xi_r,c_1r)$. By Harnack chaining in the domain V and (4.33), there exist $C_3 > 1$ such that for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, z \in B(\xi,r)\cap\partial U$, we have

$$g_V^{\text{ref}}(\xi_r,z) \geqslant C_3^{-1} g_V^{\text{ref}}(\xi_r,c_1r). \tag{4.36}$$

Set $B := \overline{B(\xi_r,c_1r)}$ and by the strong Markov property, we have

$$\begin{aligned}\mathbb{P}_x^{\text{ref}}\left(\sigma_B < \sigma_{\partial U\backslash B(\xi,3r/2)}\right) &\geqslant \mathbb{P}_x^{\text{ref}}\left[\widetilde{X}_{\sigma_{\partial U}} \in \partial U\cap B(\xi,r), \sigma_B\circ\theta_{\sigma_{\partial U}} < \sigma_{\partial U\backslash B(\xi,3r/2)}\circ\theta_{\sigma_{\partial U}}\right]\\ &\geqslant \omega_x^U(\partial U\cap B(\xi,r)) \inf_{z\in B(\xi,r)\cap\partial U}\mathbb{P}_z^{\text{ref}}(\sigma_B < \sigma_{\partial U\backslash B(\xi,3r/2)})\\ &\geqslant C_1^{-1}C_3^{-1}\omega_x^U(\partial U\cap B(\xi,r)) \quad (\text{by (4.36) and (4.34)})\end{aligned} \tag{4.37}$$

for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, x \in \overline{U}\backslash\overline{B(\xi_r,c_1r)}$.

Next, we will obtain the estimate $g_V^{\text{ref}}(x_0,\xi_r) \lesssim g_U(x_0,\xi_r)$ for all $x_0 \in U\backslash B(\xi,A_0r)$ for suitably chosen $A_0 \in (1,\infty)$. Recall the Dynkin-Hunt formula (Lemma 3.6) that

$$g_V^{\text{ref}}(y,z) = g_U(y,z) + \mathbb{E}_y^{\text{ref}}\left[\mathbb{1}_{\{\tau_U<\infty, X_{\tau_U}^{\text{ref}}\in V\}} g_V^{\text{ref}}(X_{\tau_U}^{\text{ref}},z)\right] \quad \text{for all } y \in U, z \in U\backslash\{y\}. \tag{4.38}$$

By Lemma 3.4(b) for any $x_0 \in U\backslash B(\xi, 4r), z \in V \cap B(\xi, d(\xi,\eta))$, we have

$$g_V^{\mathrm{ref}}(z, x_0) = \mathbb{E}_z^{\mathrm{ref}}\left(g_V^{\mathrm{ref}}((X^{\mathrm{ref}})^V_{\tau_{V\cap B(\xi,d(\xi,\eta))}}, x_0)\right) \leqslant \sup_{U\cap S(\xi,d(\xi,\eta))} g_V^{\mathrm{ref}}(\cdot, x_0). \tag{4.39}$$

Therefore, we obtain for all $x_0 \in U\backslash B(\xi, 4r), y \in V \cap B(\xi, 2r)$

$$\begin{aligned} g_V^{\mathrm{ref}}(y, x_0) &\overset{(4.38)}{\leqslant} g_U(y, x_0) + \mathbb{P}_y^{\mathrm{ref}}\left[\tau_U < \infty, X_{\tau_U}^{\mathrm{ref}} \in V\right] \sup_{z\in V\backslash U} g_V^{\mathrm{ref}}(z, x_0) \\ &\overset{(4.39)}{\leqslant} g_U(y, x_0) + \mathbb{P}_y^{\mathrm{ref}}\left[\tau_U < \infty, X_{\tau_U}^{\mathrm{ref}} \in V\right] \sup_{z\in U\cap \partial B(\xi,d(\xi,\eta))} g_V^{\mathrm{ref}}(z, x_0). \end{aligned} \tag{4.40}$$

Next, we show that there exists $\delta \in (0,1)$ such that for all $y \in S(\xi, d(\xi,\eta)) \cap U$

$$\mathbb{P}_y^{\mathrm{ref}}\left[\tau_U < \infty, X_{\tau_U}^{\mathrm{ref}} \in V\right] \leqslant 1 - \delta. \tag{4.41}$$

By Lemma 2.28(b), the function $h(y) := \mathbb{P}_y^{\mathrm{ref}}[\tau_U < \infty, X_{\tau_U}^{\mathrm{ref}} \in V]$ is harmonic and continuous on U. By Lemma 4.5, there exists $c_2 \in (0, 1/2)$ such that if $y \in U \cap B(\xi, d(\xi,\eta))$ is such that $\delta_U(y) < c_2 r$, then

$$h(y) \leqslant \frac{1}{2}. \tag{4.42}$$

If $y \in U \cap B(\xi, d(\xi,\eta))$ is such that $\delta_U(y) < c_1 r$ then by Harnack chaining for the harmonic function $1 - h$ on U using Lemma 2.24(b), there exists $\delta \in (0,1)$ such that $h(y) \leqslant 1 - \delta$ for all $y \in U \cap B(\xi, d(\xi,\eta))$. This concludes the proof of (4.41).

In particular, taking supremum over $y \in \partial B(\xi, d(\xi,\eta))$ in (4.40) and using (4.41), we obtain

$$\sup_{y\in\partial B(\xi,d(\xi,\eta))} g_V^{\mathrm{ref}}(y, x_0) \leqslant \sup_{y\in\partial B(\xi,d(\xi,\eta))} g_U(y, x_0) + (1-\delta) \sup_{y\in\partial B(\xi,d(\xi,\eta))} g_V^{\mathrm{ref}}(y, x_0)$$

which implies for all $x_0 \in U\backslash B(\xi, 4r)$

$$\sup_{y\in\partial B(\xi,d(\xi,\eta))} g_V^{\mathrm{ref}}(y, x_0) \leqslant \delta^{-1} \sup_{y\in\partial B(\xi,d(\xi,\eta))} g_U(y, x_0). \tag{4.43}$$

By Carleson's estimate (Proposition 3.11) and Harnack chaining using Lemma 2.23(b), there exist $A_0 \in (4,\infty), A_1, C_4 \in (1,\infty)$ such that for all $\xi \in \partial U, 0 < r < \mathrm{diam}(U,d)/A_1$ and $x_0 \in U\backslash B(\xi, A_0 r)$, we have

$$\sup_{y\in\partial B(\xi,d(\xi,\eta))} g_V^{\mathrm{ref}}(y, x_0) \geqslant C_4^{-1} g_V^{\mathrm{ref}}(\xi_r, x_0), \quad \text{and} \quad \sup_{y\in\partial B(\xi,d(\xi,\eta))} g_U(y, x_0) \leqslant C_4 g_U(\xi_r, x_0). \tag{4.44}$$

Therefore we obtain the desired upper bound using

$$\begin{aligned} \omega_{x_0}^U(\partial U \cap B(\xi, r)) &\overset{(4.37)}{\leqslant} C_1 C_3 \mathbb{P}_{x_0}^{\mathrm{ref}}\left(\sigma_B < \sigma_{\partial U\backslash B(\xi,3r/2)}\right) \overset{(4.34)}{\leqslant} C_1^2 C_3 \frac{g_V^{\mathrm{ref}}(x_0, \xi_r)}{g_V^{\mathrm{ref}}(\xi_r, c_1 r)} \\ &\overset{(4.35)}{\leqslant} C_1^2 C_2 C_3 g_V^{\mathrm{ref}}(x_0, \xi_r) \,\mathrm{Cap}_{B(\xi,2r)}(B(\xi, r)) \\ &\leqslant C_1^2 C_2 C_3 C_4^2 \delta^{-1} g_U(x_0, \xi_r) \,\mathrm{Cap}_{B(\xi,2r)}(B(\xi, r)) \quad \text{(by (4.43) and (4.44))}. \square \end{aligned}$$

Under an additional assumption which for instance is satisfied for the Brownian motion on $\mathbb{R}^n$ with $n \geqslant 2$, the capacity density condition on a domain U implies the uniform perfectness of its boundary ∂U. The uniform perfectness of boundary is relevant for the stable-like heat kernel estimates for the trace process in Theorem 5.19.

Definition 4.9. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the elliptic Harnack inequality such that $(\mathcal{X}, d)$ such that $(\mathcal{X}, d)$ is a doubling metric space. Then we say that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the **capacity non-increasing condition** if there exist $C, A \in (1, \infty)$ such that

$$\mathrm{Cap}_{B(x,2R)}(B(x,R)) \leqslant C\, \mathrm{Cap}_{B(x,2r)}(B(x,r)), \quad \text{for all } x \in \mathcal{X}, 0 < r < R < \mathrm{diam}(\mathcal{X}, d)/A. \tag{4.45}$$

We remark that the number 2 in (4.45) can be replaced with any constant larger than 1 due to [BCM, Lemma 5.22]. If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the stronger the heat kernel estimate HKE(Ψ) for some scale function Ψ, then by [GHL15, Theorem 1.2], (4.45) is equivalent to the following estimate: there exist $C, A \in (1, \infty)$ such that

$$\frac{\Psi(R)}{m(B(x,R))} \leqslant C \frac{\Psi(r)}{m(B(x,r))}, \quad \text{for all } x \in \mathcal{X}, 0 < r < R < \mathrm{diam}(\mathcal{X}, d)/A. \tag{4.46}$$

The condition (4.46) was called *fast volume growth* in [JM, Definition 1.5]. The following lemma follows from the estimates of harmonic measure in Theorem 4.6 along with Lemma 4.5(a) and Carleson's estimate (Proposition 3.11). We omit the proof as it follows from a straightforward modification of the argument in [AHMT1, Remark 2.56].

Lemma 4.10 (Cf. [AHMT1, Remark 2.56])**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be a uniform domain that satisfy Assumption 4.3. Furthermore assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the capacity non-increasing condition. Then ∂U is uniformly perfect.*

We provide some sufficient conditions for the capacity density condition below.

Remark 4.11. (a) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the elliptic Harnack inequality such that $(\mathcal{X}, d)$ is a doubling metric space. Let U satisfy the exterior corkscrew condition (see [JK, (3.2)] for the definition). Then the capacity estimates in [BCM, §5] imply the capacity density density condition for U. In particular, non-tangentially accessible domains (see [p. 93]JK) satisfy the capacity density condition.

(b) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the sub-Gaussian heat kernel estimate HKE(Ψ), where $\Psi(r) = r^{d_w}$ for all $r > 0$, where $d_w \geqslant 2$. Assume that m is a d_f-Ahlfors regular measure; that is, there exist $C \in (1, \infty)$ such that

$$C^{-1} r^{d_f} \leqslant m(B(x,r)) \leqslant C r^{d_f} \quad \text{for all } x \in \mathcal{X}, 0 < r < \mathrm{diam}(\mathcal{X}, d).$$

If the boundary ∂U admits a p-Ahlfors regular measure for some $p > d_f - d_w$, then U satisfies the capacity density condition. The desired lower bound on the capacity can be obtained by adapting the arguments in [HeiK, Proof of Theorem 5.9]. In particular, this shows that the uniform domains obtained by removing the bottom line or the outer square boundary of the Sierpiński carpet satisfy the capacity density condition of the Brownian motion on the Sierpiński carpet,

We recall a simple consequence of Lebesgue differentiation theorem. We note that the condition (4.47) is satisfied by harmonic measure on ∂U due to Corollary 4.7.

Lemma 4.12 (Lebesgue differentiation theorem)**.** *Let $(\mathcal{X}, d, m)$ be a metric measure space such that*

$$\limsup_{r \downarrow 0} \frac{m(B(x,2r))}{m(B(x,r))} < \infty \quad \text{for all } x \in \mathcal{X}. \tag{4.47}$$

Then for any locally integrable function $f : X \to \mathbb{R}$ almost every point is a Lebesgue point of f; that is,

$$\lim_{r \downarrow 0} \fint_{B(x,r)} |f(y) - f(x)|\, dm(y). \tag{4.48}$$

for m-almost every $x \in \mathcal{X}$. In particular, for any point $x \in \mathcal{X}$ satisfying (4.48) and if $\psi_r, r > 0$ be a family of measurable functions such that $\mathbb{1}_{B(x,r)} \leqslant \psi_r \leqslant \mathbb{1}_{B(x,2r)}$, then

$$\lim_{r \downarrow 0} \frac{\int \psi_r f\, dm}{\int \psi_r\, dm} = f(x). \tag{4.49}$$

Proof. The assertion given in (4.48) follows from [HKST, (3.4.10) and Theorem 3.4.3]. The conclusion (4.49) follows from (4.48) as

$$0 \leqslant \limsup_{r \downarrow 0} \int |\psi_r(y) f(y) - \psi_r(y) f(x)|\, m(dy) \leqslant \limsup_{r \downarrow 0} \int_{B(x,2r)} |f(y) - f(x)|\, m(dy) = 0.$$

□

The following proposition shows that the harmonic measure is the distributional Laplacian of the Green function. In the proof, we use the following notation to denote the 0-th order hitting distribution of a quasicontinuous function $u \in \mathcal{F}(U)_e$, where $\mathcal{F}(U)_e$ denotes the extended Dirichlet space corresponding to $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U)_e)$ on $L^2(\overline{U}, m|_{\overline{U}})$.

$$H^{\mathrm{ref}}_{\partial U} u(x) := \mathbb{E}^{\mathrm{ref}}_x \left[u(X^{\mathrm{ref}}_{\sigma_{\partial U}}) \mathbb{1}_{\{\sigma_{\partial U} < \infty\}} \right], \quad \text{for all } x \in \overline{U}, u \in \mathcal{F}(U)_e. \tag{4.50}$$

Proposition 4.13. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 2.14 and let $U \subset \mathcal{X}$ be an open subset such that the Dirichlet form $(\mathcal{E}^U, \mathcal{F}^0(U))$ on $L^2(U; m|_U)$ corresponding to the part process on U is transient. For all $x \in U$ and for all $u \in \mathcal{F}(U) \cap L^\infty(\overline{U})$ such that $x \notin \mathrm{supp}_{\overline{U}}[u]$, we have*

$$\mathcal{E}^{\mathrm{ref}}(g_U(x, \cdot), u) = -\int_{\partial U} \widetilde{u}\, d\omega^U_x.$$

Proof. By the transience of $(\mathcal{E}^U, \mathcal{F}^0(U))$ on $L^2(U; m|_U)$, there exists $f \in L^1(U, m)$ such that f is a strictly positive function, $\int_U f G_U f \, dm < \infty$ and $\mathrm{supp}_U[f] \cap \mathrm{supp}_{\overline{U}}[u] = \emptyset$. We note that $G_U f \in (\mathcal{F}^0(U))_e$ by [FOT, Theorem 4.2.6]. Then

$$\begin{aligned}\mathcal{E}^{\mathrm{ref}}(G_U f, u) &= \mathcal{E}^{\mathrm{ref}}(G_U f, u) - \mathcal{E}^{\mathrm{ref}}(G_U f, H^{\mathrm{ref}}_{\partial U}\widetilde{u}) \\ &\quad \text{(since } G_U f \in (\mathcal{F}^0(U))_e \text{ and } H^{\mathrm{ref}}_{\partial U}\widetilde{u} \text{ is harmonic in } U) \\ &= \mathcal{E}^{\mathrm{ref}}(G_U f, u - H^{\mathrm{ref}}_{\partial U}\widetilde{u}) \\ &= \int_U f(u - H^{\mathrm{ref}}_{\partial U}\widetilde{u})\, dm \quad \text{(by [FOT, Theorem 4.4.1(ii), (4.3.11)])} \\ &= -\int_U f H^{\mathrm{ref}}_{\partial U}\widetilde{u}\, dm \quad \text{(since } \mathrm{supp}_U[f] \cap \mathrm{supp}_{\overline{U}}[u] = \emptyset). \end{aligned} \tag{4.51}$$

We choose a sequence $f_n \geqslant 0$ such that $\int f_n \, dm = 1$, $\mathrm{supp}_m[f_n] \downarrow \{x\}$ as $n \to \infty$. The existence of such a sequence follows by considering f_r defined in (3.9). By quasi-continuity of $\widetilde{u}$ and using (4.51) with $f = f_n$ and letting $n \to \infty$, we obtain the desired conclusion. □

The Martin kernel can be viewed as the Radon-Nikodym derivative of harmonic measures at different starting points. A similar statement on non-tangentially accessible (NTA) domains in the Euclidean space was observed in [KT, Theorem 3.1] which is an easy consequence of the results in [JK]. Jerison and Kenig *define* the Martin kernel as such a Radon-Nikodym derivative [JK, Definition 1.3]. For NTA domains in the Euclidean space the equivalence of our definition with [JK, Definition 1.3] follows from the uniqueness theorem in [JK, Theorem 5.5]. Our next result is a generalization of [KT, Theorem 3.1].

Proposition 4.14. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be a uniform domain that satisfy Assumption 4.3. For all $x, x_0 \in U$, we have*

$$\frac{d\omega^U_x}{d\omega^U_{x_0}}(\cdot) = K^U_{x_0}(x, \cdot). \tag{4.52}$$

Proof. Let $A = B(\xi, r) \cap \partial U$, $B = B(\xi, 2r)^c \cap \overline{U}$ and $e_{A,B}$ denote the equilibrium potential for A with respect to the Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ for the reflected diffusion on $\overline{U}$ with Dirichlet boundary condition on B. By Proposition 4.13 and Lemma 3.7 there exist measures $\lambda^1_{A,B}, \lambda^0_{A,B}$ supported on $\overline{A}$ and $\overline{U} \cap \partial B(\xi, 2r)$ respectively such that

$$\begin{aligned} 0 < \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_x &= -\mathcal{E}^{\mathrm{ref}}(g_U(x,\cdot), e_{A,B}) \\ &= -\left(\int_{\overline{A}} g_U(x,y)\, d\lambda^1_{A,B}(y) - \int_{\overline{U} \cap \partial B(\xi, 2r)} g_U(x,y)\, d\lambda^0_{A,B}(y) \right) \\ &= \int_{\overline{U} \cap \partial B(\xi, 2r)} g_U(x,y)\, d\lambda^0_{A,B}(y). \end{aligned} \tag{4.53}$$

Taking ratio of (4.53) for x and for x_0 in place of x, we obtain

$$\left|\frac{\int_{\partial U} \widetilde{e}_{A,B}\, d\omega_x^U}{\int_{\partial U} \widetilde{e}_{A,B}\, d\omega_{x_0}^U} - K_{x_0}(x,\xi)\right| = \left|\frac{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x,y)\, d\lambda_{A,B}^0(y)}{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,y)\, d\lambda_{A,B}^0(y)} - K_{x_0}(x,\xi)\right|$$
$$\leqslant \left|\frac{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,y)\left(K_{x_0}(x,y) - K_{x_0}(x,\xi)\right)\, d\lambda_{A,B}^0(y)}{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,y)\, d\lambda_{A,B}^0(y)}\right|$$
$$\leqslant \frac{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,y)|K_{x_0}(x,y) - K_{x_0}(x,\xi)|\, d\lambda_{A,B}^0(y)}{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,y)\, d\lambda_{A,B}^0(y)} \quad (4.54)$$

By the boundary Hölder regularity of the Martin kernel (Lemma 3.10), there exist $C_1, A_1 \in (1,\infty), \gamma \in (0,\infty)$ such that for all $x_0, x \in U, \xi \in \partial U, 0 < r < A_1^{-1}(d(x_0,\xi) \wedge d(x,\xi)), y \in \overline{U} \cap B(\xi,r)$, we have

$$|K_{x_0}(x,y) - K_{x_0}(x,\xi)| \leqslant C_1 K_{x_0}(x,\xi)\left(\frac{r}{(d(x_0,\xi) \wedge d(x,\xi))}\right)^{\gamma}. \quad (4.55)$$

By using (4.54), (4.55) and letting $r \downarrow 0$, and use the continuity of g_U to take the limit to obtain the desired conclusion (4.52) $\omega_{x_0}^U$-a.e. using (4.49) in Lemma 4.12. The use of Lemma 4.12 is justified by the mutually absolutely continuity in Lemma 2.28(c) and the asymptotic doubling property in Corollary 4.7. □

4.3 The elliptic measure at infinity on unbounded domains

On unbounded uniform domain the harmonic measure need not be doubling. For instance if ∂U were unbounded and connected, due to [Hei, Exercise 13.1] every doubling measure on ∂U must necessarily be an infinite measure. In particular, there are no doubling *probability* measures on ∂U. Nevertheless, as we will see there is a canonical *doubling* measure on ∂U obtained as a limit of scaled harmonic measures ω_x^U as $x \to \infty$. Propositions 3.21 and 4.13 suggest to consider the limit of scaled harmonic measures $g_U(x_0,x)^{-1}\omega_x^U$ as $x \to \infty$. Following [BTZ, Lemma 3.5], we call this limit the **elliptic measure at infinity**. Alternately, the distributional Laplacian of the harmonic profile defines the elliptic measure at infinity on the boundary ∂U as shown below.

Proposition 4.15 (Elliptic measure at infinity)**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and let $U \subset \mathcal{X}$ be an unbounded uniform domain that satisfy Assumption 4.3. Let $(x_n)_{n\in\mathbb{N}}, x_0 \in U$ be a sequence such that $x_i \in U$ for all $i \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_0,x_n) = \infty$. Let $h_U^{x_0}(\cdot) = \lim_{n\to\infty} K_{x_0}(x_n,\cdot)$ denote the Martin kernel at infinity. Then the sequence of measure $\nu_n(\cdot) := g_U(x_n,x_0)^{-1}\omega_{x_n}^U(\cdot)$ converge weakly to $\nu_{x_0}^U$ and*

$$\mathcal{E}^{\mathrm{ref}}(h_{x_0}^U(\cdot), u) = -\int \widetilde{u}\, d\nu_{x_0}^U \quad \textit{for all } u \in \mathcal{F}(U). \quad (4.56)$$

In particular, the measure $\nu_{x_0}^U$ does not depend on the choice of the sequence $(x_n)_{n\geqslant 1}$.

The measure $\nu_{x_0}^U$ satisfies the following properties:

(a) *The measures* $\nu^U_{x_0}$ *and* $\omega^U_{x_0}$ *are mutually absolutely continuous. Furthermore, the Radon-Nikodym derivative* $\frac{d\nu^U_{x_0}}{d\omega^U_{x_0}} : \partial U \to (0,\infty)$ *can be chosen to be a strictly positive continuous function satisfying the following bound: there exist* $C, A \in (1,\infty)$ *such that for all* $\xi \in \partial U, 0 < R < A^{-1}d(\xi, x_0), \eta \in \partial U \cap B(\xi, R)$*, we have*

$$C^{-1}\frac{h^U_{x_0}(\xi_R)}{g_U(x_0,\xi_R)} \leqslant \frac{d\nu^U_{x_0}}{d\omega^U_{x_0}}(\eta) \leqslant C h^U_{x_0}(\xi_R) g_U(x_0,\xi_R). \tag{4.57}$$

(b) *The measure* $\nu^U_{x_0}$ *is smooth with quasi-support* ∂U *for the Dirichlet form* $(\mathcal{E}^{\text{ref}}, \mathcal{F}(U))$*.*

(c) *There exists* $C > 0$ *such that for all* $\xi \in \partial U, R > 0$*, we have*

$$C^{-1} h^U_{x_0}(\xi_R)\operatorname{Cap}_{B(\xi,2R)}(B(\xi,R)) \leqslant \nu^U_{x_0}(B(\xi,r)) \leqslant C h^U_{x_0}(\xi_R)\operatorname{Cap}_{B(\xi,2R)}(B(\xi,R)). \tag{4.58}$$

In particular, the measure $\nu^U_{x_0}$ *is doubling.*

Proof. For any $u \in C_c(\overline{U}) \cap \mathcal{F}(U)$, there exists $\phi \in C_c(\mathcal{X}) \cap \mathcal{F}, N \in \mathbb{N}$ such that $x_n \notin \operatorname{supp}[\phi]$ for all $n \geqslant N$ and $\operatorname{supp}_m[u] \subset \operatorname{supp}[\phi]$. By Proposition 4.13 and strong locality, we have

$$\int_{\partial U} \widetilde{u}\, d\nu_n = \mathcal{E}^{\text{ref}}\left(\frac{g_U(x_n,\cdot)}{g_U(x_n,x_0)}, u\right) = -\mathcal{E}^{\text{ref}}\left(\Phi(\cdot)\frac{g_U(x_n,\cdot)}{g_U(x_n,x_0)}, u\right), \tag{4.59}$$

where we adapt the convention of extending $g_U(x_n,\cdot)$ by 0 on U^c. If we similarly extend $h^U_{x_0}$ as 0 on U^c by Proposition 3.21 and Remark 2.20(b), we obtain that

$$\lim_{n\geqslant N, n\to\infty} \mathcal{E}^{\text{ref}}_1\left(\phi(\cdot)\frac{g_U(x_n,\cdot)}{g_U(x_n,x_0)} - \phi h^U_{x_0}, \phi(\cdot)\frac{g_U(x_n,\cdot)}{g_U(x_n,x_0)} - \phi h^U_{x_0}\right) = 0. \tag{4.60}$$

Combining (4.59), (4.60) and by the strong locality of $(\mathcal{E}^{\text{ref}}, \check{\mathcal{F}}(U))$, we obtain

$$\lim_{n\to\infty}\int_{\partial U} \widetilde{u}\, d\nu_n = -\mathcal{E}^{\text{ref}}\left(\phi(\cdot)h^U_{x_0}, u\right) = -\mathcal{E}^{\text{ref}}\left(h^U_{x_0}, u\right).$$

Therefore the measures $\nu_n = g_U(x_n,x_0)^{-1}\omega^U_{x_n}(\cdot)$ weakly converge to $\nu^U_{x_0}$. The claim that $\nu^U_{x_0}$ does not depend on the choice of the sequence $(x_n)_{n\geqslant 1}$ follows from (4.56) and the similar claim in Proposition 3.21.

(a) By Proposition 4.14 and (3.50),

$$\frac{d\nu_n}{d\omega^U_{x_0}}(\cdot) = \frac{1}{g_U(x_n,x_n)}\frac{d\omega^U_{x_n}}{d\omega^U_{x_0}}(\cdot) \overset{(4.52)}{=} \frac{K^U_{x_0}(x_n,\cdot)}{g_U(x_n,x_0)} \overset{(3.50)}{=} \Theta^U_{x_0}(x_n,\cdot). \tag{4.61}$$

By (3.37) and joint continuity of $\Theta^U_{x_0}$, the sequence $\Theta^U_{x_0}(x_n,\cdot)$ is uniformly bounded on every compact subset of ∂U. Similarly by (3.38) and joint continuity of $\Theta^U_{x_0}$, the sequence $\Theta^U_{x_0}(x_n,\cdot)$ is equicontinuous on every compact subset of ∂U. By Arzela-Ascoli theorem and passing to a subsequence, we may assume that $\Theta^U_{x_0}(x_n,\cdot)$ converges uniformly on compact subsets of ∂U to a continuous function, say $\Theta^U_{x_0}(\infty,\cdot) : \partial U \to [0,\infty)$. Hence by (4.56) and (4.61)

$$\nu^U_{x_0} \ll \omega^U_{x_0}, \quad \text{and} \quad \frac{d\nu^U_{x_0}}{d\omega^U_{x_0}}(\cdot) = \Theta^U_{x_0}(\infty,\cdot). \tag{4.62}$$

By (4.61), (3.37), joint continuity of $\Theta^U_{x_0}$, for all $\xi \in \partial U, R > 0, \eta \in \partial U \cap B(\xi, R)$, we have

$$\frac{d\nu_n}{d\omega^U_{x_0}}(\eta) \overset{(4.61)}{=} \Theta^U_{x_0}(x_n,\eta) \overset{(3.37)}{\asymp} \Theta^U_{x_0}(x_n,\xi_R) = \frac{g_U(x_n,\xi_R)}{g_U(x_n,x_0)}\frac{1}{g_U(x_0,\xi_R)}$$

for all n sufficiently large. Letting $n \to \infty$ and using Proposition 3.21, we obtain the estimate (4.57). Since $\Theta^U_{x_0}(\infty,\cdot)$ is strictly positive in ∂U, we conclude that $\nu^U_{x_0}$ and $\omega^U_{x_0}$ are mutually absolutely continuous.

(b) By the mutual absolute continuity of $\nu^U_{x_0}$ and $\omega^U_{x_0}$, the quasi-supports are equal. Hence the desired conclusion follows from Lemma 2.28(d).

(c) By Proposition 3.21 and (4.56), we have

$$h^U_{x_0}(\cdot) = h^U_{x_0}(y)h^U_y(\cdot), \quad \nu^U_y(\cdot) = \frac{1}{h^U_{x_0}(y)}\nu^U_{x_0}(\cdot), \quad \text{for all } y \in U. \tag{4.63}$$

For $\xi \in \partial U, R > 0$, we choose $y \in U \backslash B(\xi, AR)$ and estimate

$$\nu^U_y(B(\xi,R)) \overset{(4.57)}{\asymp} \omega^U_y(B(\xi,R))\frac{h^U_y(\xi_R)}{g_U(y,\xi_R)} \overset{(4.15)}{\asymp} h^U_{x_0}(\xi_R)\,\mathrm{Cap}_{B(\xi,2R)}(B(\xi,R)). \tag{4.64}$$

The estimate (4.58) follows from (4.63) and (4.64).

The doubling property of $\nu^U_{x_0}$ follows from (4.58) along with Proposition 3.21, Lemma 2.24, and [BCM, add. text]. □

5 The trace process on the boundary

In this section, we *always* assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is an MMD space and U is a uniform domain satisfying Assumption 4.3.

5.1 The boundary measure and the corresponding PCAF

To define the boundary trace process, we choose a reference measure on the boundary ∂U as given in the following definition.

Definition 5.1. If U is bounded, we choose $x_0 = \hat{\xi}_r$ using Lemma 2.6, where $\hat{\xi} \in \partial U$ is chosen arbitrarily and $r = \mathrm{diam}(U,d)/5$. If U is unbounded, let $x_0 \in U$ be an arbitrary point. We define the measure μ supported on the boundary as

$$\mu(\cdot) := \begin{cases} \omega^U_{x_0}(\cdot), & \text{if } U \text{ is bounded}, \\ \nu^U_{x_0}(\cdot), & \text{if } U \text{ is unbounded}, \end{cases} \tag{5.1}$$

where $\omega^U_{x_0}(\cdot), \nu^U_{x_0}(\cdot)$ are the harmonic measure (Definition 2.27) and elliptic measure at infinity (Proposition 4.15) respectively.

In order to describe properties of μ, we define $\widetilde{\Phi} : \partial U \times (0, \operatorname{diam}(U,d)/4) \to (0,\infty)$,

$$\widetilde{\Phi}(\xi, r) = \begin{cases} g_U(x_0, \xi_r), & \text{if } \operatorname{diam}(U,d) < \infty, \\ h^U_{x_0}(\xi_r), & \text{if } \operatorname{diam}(U,d) = \infty, \end{cases} \tag{5.2}$$

where ξ_r is chosen using Lemma 2.6.

Note that by [GHL15, Theorem 1.2], and [BCM, Lemma 5.24], there exist $C, A \in (1,\infty)$ such that

$$C^{-1}\frac{m(B(\xi,R))}{\Psi(R)} \leqslant \operatorname{Cap}_{B(\xi,2R)}(B(\xi,R)) \leqslant C\frac{m(B(\xi,R))}{\Psi(R)} \tag{5.3}$$

for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U,d)/A$. Let us recall that the function $\widetilde{\Phi}(\cdot,\cdot)$ is useful to estimate the measure μ. Indeed, by Theorem 4.6, Proposition 4.15(c), and (5.3), there exist $C, A \in (1,\infty)$ such that

$$C^{-1}\frac{m(B(\xi,R))}{\Psi(R)} \leqslant \frac{\mu(B(\xi,R))}{\widetilde{\Phi}(\xi,R)} \leqslant C\frac{m(B(\xi,R))}{\Psi(R)}, \quad \text{for all } \xi \in \partial U, 0 < R < \operatorname{diam}(U,d)/A. \tag{5.4}$$

We record some basic estimates on $\widetilde{\Phi}(\cdot,\cdot)$ and show that $\widetilde{\Phi}(\cdot,\cdot)$ is comparable to a function $\Phi(\cdot,\cdot)$ that has better continuity properties.

Lemma 5.2. *There exist a regular scale function $\Phi : \partial U \times (0,\infty) \to (0,\infty)$ in the sense of Definition 2.30 and $C_1, A_1 \in (1,\infty)$ such that*

$$C_1^{-1}\widetilde{\Phi}(\xi,r) \leqslant \Phi(\xi,r) \leqslant C_1\widetilde{\Phi}(\xi,r), \quad \textit{for all } \xi \in \partial U, 0 < r < \operatorname{diam}(U)/A_1. \tag{5.5}$$

Proof. First, we show that there exist $C, \beta_1, \beta_2 > 0, A \in (4,\infty)$ such that, for all $\eta, \xi \in \partial U$, $0 < r \leqslant R$ with $R \vee d(\xi,\eta) < \operatorname{diam}(U,d)/A$

$$C^{-1}\Big(\frac{R}{d(\xi,\eta)\vee R}\Big)^{\beta_2}\Big(\frac{d(\xi,\eta)\vee R}{r}\Big)^{\beta_1} \leqslant \frac{\widetilde{\Phi}(\xi,R)}{\widetilde{\Phi}(\eta,r)} \leqslant C\Big(\frac{R}{d(\xi,\eta)\vee R}\Big)^{\beta_1}\Big(\frac{d(\xi,\eta)\vee R}{r}\Big)^{\beta_2}. \tag{5.6}$$

By Lemmas 2.24 and 4.5 and by the harmonicity and Dirichlet boundary conditions of $g_U(x_0,\cdot)$ and $h^U_{x_0}(\cdot)$ in Propositions 3.1(iv), 3.21, Lemma 2.18, there exist $C_1, C_2, A \in (1,\infty), \beta_1, \beta_2 \in (0,\infty)$ such that

$$C_1^{-1}\left(\frac{R}{r}\right)^{\beta_1} \leqslant \frac{\widetilde{\Phi}(\xi,R)}{\widetilde{\Phi}(\xi,r)} \leqslant C_1\left(\frac{R}{r}\right)^{\beta_2}, \quad \text{for all } \xi \in \partial U, 0 < r < R < \operatorname{diam}(U,d)/A, \tag{5.7}$$

and

$$C_2^{-1} \leqslant \frac{\widetilde{\Phi}(\xi, R \vee d(\xi,\eta))}{\widetilde{\Phi}(\eta, R\vee d(\xi,\eta))} \leqslant C_2, \tag{5.8}$$

for all $\eta, \xi \in \partial U, 0 < r \leqslant R$ with $R \vee d(\xi, \eta) < \mathrm{diam}(U, d)/A$.

The conclusion (5.6) follow from (5.7), (5.8) using the expression

$$\frac{\widetilde{\Phi}(\xi, R)}{\widetilde{\Phi}(\xi, r)} = \frac{\widetilde{\Phi}(\xi, R)}{\widetilde{\Phi}(\xi, R \vee d(\xi, \eta))} \cdot \frac{\widetilde{\Phi}(\xi, R \vee d(\xi, \eta))}{\widetilde{\Phi}(\eta, R \vee d(\xi, \eta))} \cdot \frac{\widetilde{\Phi}(\eta, R \vee d(\xi, \eta))}{\widetilde{\Phi}(\xi, r)}.$$

By (2.12), there exists $A_2 \in (1, \infty)$ such that for all $\xi \in \partial U, R < \mathrm{diam}(U)/A$, we have

$$\widetilde{\Phi}(\xi, A_2^{-1} R) \leqslant \frac{1}{2} \widetilde{\Phi}(\xi, R) \tag{5.9}$$

Using (5.9), we define the function as follows: If U is unbounded, we define

$$\Phi(\xi, A_2^k) = \widetilde{\Phi}(\xi, A_2^k) \quad \text{for all } \xi \in \partial U, k \in \mathbb{Z},$$

and extend $\Phi(\xi, \cdot)$ by piecewise linear interpolation to $(0, \infty)$ for each $\xi \in \partial U$. Using (5.6) and (5.5), the estimate (2.36) in Definition 2.30. The fact that $\Phi(\xi, \cdot)$ is an increasing homeomorphism follows from (5.9). This concludes the proof if U is unbounded.

If U is bounded, we define

$$\Phi(\xi, A_2^k (2A)^{-1} \mathrm{diam}(U)) = \widetilde{\Phi}(\xi, A_2^k (2A)^{-1} \mathrm{diam}(U)) \quad \text{for all } \xi \in \partial U, k \in \mathbb{Z}, k \geqslant 0,$$

and extend $\Phi(\xi, \cdot)$ by piecewise linear interpolation to $(0, \infty)$. The conclusion follows from the same reasoning as the bounded case. □

It will be convenient to use $\Phi(\cdot, \cdot)$ in Lemma 5.2 instead of $\widetilde{\Phi}(\cdot, \cdot)$ due to its better continuity property. So we set $\Phi(\cdot, \cdot)$ to denote the function in Lemma 5.2 for the remainder of the work.

The following lemma is an upper bound on the integral of heat kernel with respect to μ. This upper bound is later used to show that μ is a smooth measure in the strict sense (Lemma 5.4) and to identify the support of the corresponding PCAF with the topological boundary (Proposition 5.7).

Lemma 5.3. *There exists $C \in (1, \infty)$ such that for all $\xi \in \partial U, t \in (0, \infty)$, we have*

$$\int_{\partial U} p_t^{\mathrm{ref}}(x, y) \, \mu(dy) \leqslant C \frac{\mu\left(B(\xi_x, \Psi^{-1}(t))\right)}{m(B(\xi_x, \Psi^{-1}(t))}, \tag{5.10}$$

where $\xi_x \in \partial U$ is any point such that $\mathrm{dist}(x, U^c) = d(x, \xi_x)$.

Proof. By HKE(Ψ), [GT12, Lemma 3.19] and (2.12), there exists $C_1 \in (1, \infty), c_2 \in (0, 1), 0 < \alpha_1 < \alpha_2 < \infty$ such that for all $x, y \in \overline{U}$, we have

$$p_t^{\mathrm{ref}}(x, y) = p_t^{\mathrm{ref}}(y, x) \leqslant \frac{C_1}{m(B(x, \Psi^{-1}(t)))} \exp\left(-c_2 \min\left(\left(\frac{d(x, y)}{\Psi^{-1}(t)} \right)^{\alpha_1}, \left(\frac{d(x, y)}{\Psi^{-1}(t)} \right)^{\alpha_2} \right) \right). \tag{5.11}$$

If $\xi_x \in \partial U$ satisfies $\mathrm{dist}(x, U^c) = d(x, \xi_x)$, then

$$d(\xi_x, y) \leqslant d(x,y) + d(x,\xi_x) \leqslant 2d(x,y) \quad \text{for all } y \in \partial U. \tag{5.12}$$

By (5.11), (5.12) and (2.1), there exists $C_2 \in (1,\infty), c_3 \in (0,1)$ such that

$$\begin{aligned} p_t^{\mathrm{ref}}(x,y) &\leqslant \frac{C_1}{m(B(y,\Psi^{-1}(t)))} \exp\left(-c_3 \min\left(\left(\frac{d(\xi_x,y)}{\Psi^{-1}(t)}\right)^{\alpha_1}, \left(\frac{d(\xi_x,y)}{\Psi^{-1}(t)}\right)^{\alpha_2}\right)\right) \\ &\leqslant \frac{C_2}{m(B(\xi_x,\Psi^{-1}(t)))} \exp\left(-\frac{c_3}{2} \min\left(\left(\frac{d(\xi_x,y)}{\Psi^{-1}(t)}\right)^{\alpha_1}, \left(\frac{d(\xi_x,y)}{\Psi^{-1}(t)}\right)^{\alpha_2}\right)\right) \end{aligned} \tag{5.13}$$

for all $x \in \overline{U}, y \in \partial U$ where $\xi_x \in \partial U$ satisfies $\mathrm{dist}(x,U^c) = d(x,\xi_x)$.

For all $x \in \overline{U}, \xi_x \in \partial U$ such that $\mathrm{dist}(x,U^c) = d(x,\xi_x)$ and for all $t > 0$ using (5.13) and (2.1), we estimate

$$\begin{aligned} &\int_{\partial U} p_t^{\mathrm{ref}}(x,y)\,\mu(dy) \\ &= \int_{B(\xi_x,\Psi^{-1}(t))} p_t^{\mathrm{ref}}(x,\cdot)\,d\mu(y) + \sum_{k=1}^{\infty} \int_{B(\xi_x,2^k\Psi^{-1}(t))\setminus B(\xi_x,2^{k-1}\Psi^{-1}(t))} p_t^{\mathrm{ref}}(x,\cdot)\,d\mu \\ &\overset{(5.13)}{\lesssim} \frac{\mu(B(x,\Psi^{-1}(t)))}{m(B(\xi_x,\Psi^{-1}(t))} + \sum_{k=1}^{\infty} \frac{\mu(B(\xi_x,2^k\Psi^{-1}(t)))}{m(B(\xi_x,\Psi^{-1}(t))} \exp(-c2^{\alpha_1 k}) \\ &\lesssim \frac{\mu(B(\xi_x,\Psi^{-1}(t)))}{m(B(\xi_x,\Psi^{-1}(t))} + \sum_{k=1}^{\infty} \frac{\mu(B(\xi_x,2^k\Psi^{-1}(t)))}{m(B(\xi_x,\Psi^{-1}(t))} \exp(-c2^{\alpha_1 k}) \\ &\lesssim \frac{\mu(B(\xi_x,\Psi^{-1}(t)))}{m(B(\xi_x,\Psi^{-1}(t))} \left[1 + \sum_{k=1}^{\infty} 2^{k\beta} \exp(-c2^{\alpha_3 k})\right] \quad \text{(by (2.1))} \\ &\lesssim \frac{\mu(B(\xi_x,\Psi^{-1}(t)))}{m(B(\xi_x,\Psi^{-1}(t))}. \end{aligned} \tag{5.14}$$

□

Next, we show that μ is a smooth measure in the strict sense for the Dirichlet form corresponding to the reflected diffusion on $\overline{U}$.

Lemma 5.4. *The measure μ is a smooth measure in the strict sense for the reflected Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ on $L^2(\overline{U}, m|_{\overline{U}})$ with quasi-support ∂U.*

Proof. We only consider the case when U is unbounded (the bounded case is similar and easier).

Fix $\xi \in \partial U$. For any $n \in \mathbb{N}$, we consider the measure $\mu_{\xi,n}(\cdot) := \mu(\cdot \cap B(\xi,n))$. By the same argument in (5.11) there exists $C_1 \in (1,\infty), c_2 \in (0,1), 0 < \alpha_1 < \alpha_2 < \infty$ such that for all $x,y,z \in \overline{U}$ with $d(x,y) \leqslant d(x,z)$, we have

$$p_t^{\mathrm{ref}}(x,z) \leqslant \frac{C_1}{m(B(y,\Psi^{-1}(t)))} \exp\left(-c_1 \min\left(\left(\frac{d(x,y)}{\Psi^{-1}(t)}\right)^{\alpha_1}, \left(\frac{d(x,y)}{\Psi^{-1}(t)}\right)^{\alpha_2}\right)\right). \tag{5.15}$$

Note that for any $x \notin B(\xi, 2n), z \in B(\xi, n)$, we have $d(x,z) \geqslant d(\xi, z)$. Hence by (5.15) and the same argument as (5.14), we obtain

$$\int_{\partial U} p_s^{\mathrm{ref}}(x,\cdot)\, d\mu_{\xi,n} \lesssim \frac{\mu(B(\xi,\Psi^{-1}(s)))}{m(B(\xi,\Psi^{-1}(s))} \quad \text{for all } x \notin B(\xi, 2n). \tag{5.16}$$

If $x \in B(\xi, 2n)$, then by Lemma 5.3

$$\int_{\partial U} p_t^{\mathrm{ref}}(x,y)\mu_{\xi,n}(dy) \leqslant \int_{\partial U} p_t^{\mathrm{ref}}(x,y)\mu(dy) \lesssim \frac{\mu(B(\xi_x,\Psi^{-1}(s)))}{m(B(\xi_x,\Psi^{-1}(s))}, \tag{5.17}$$

where $\xi_x \in B(\xi, 3n) \cap \partial U$ satisfies $\mathrm{dist}(x, U^c) = d(x, \xi_x)$. For all $\eta \in \partial U$ using the doubling property of m and μ, we have

$$\begin{aligned}
\int_0^1 e^{-s}\frac{\mu(B(\eta,\Psi^{-1}(s)))}{m(B(\eta,\Psi^{-1}(s))}\,ds &= \sum_{k=0}^{\infty}\int_{2^{-k}}^{2^{-k+1}} e^{-s}\frac{\mu(B(\eta,\Psi^{-1}(s)))}{m(B(\eta,\Psi^{-1}(s))}\,ds \\
&\asymp \sum_{k=0}^{\infty}\frac{\mu(B(\eta,\Psi^{-1}(2^{-k})))}{m(B(\eta,\Psi^{-1}(2^{-k}))}2^{-k} \\
&\overset{(5.4)}{\asymp} \sum_{k=0}^{\infty}\Phi(\eta,\Psi^{-1}(2^{-k})) \\
&\asymp \Phi(\eta,\Psi^{-1}(1)) \quad \text{(by Lemmas 5.2 and [GT12, Lemma 3.19])},
\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}
\int_1^{\infty} e^{-s}\frac{\mu(B(\eta,\Psi^{-1}(s)))}{m(B(\eta,\Psi^{-1}(s))}\,ds &= \sum_{k=1}^{\infty}\int_{2^{k-1}}^{2^{k}} e^{-s}\frac{\mu(B(\eta,\Psi^{-1}(s)))}{m(B(\eta,\Psi^{-1}(s))}\,ds \\
&\lesssim \sum_{k=1}^{\infty}\frac{\mu(B(\eta,\Psi^{-1}(2^{k})))}{m(B(\eta,\Psi^{-1}(2^{k}))}2^{k}e^{-2^{k-1}} \\
&\asymp \sum_{k=0}^{\infty}\Phi(\eta,\Psi^{-1}(2^{k}))e^{-2^{k-1}} \quad \text{(by (5.4))} \\
&\lesssim \sum_{k=0}^{\infty}\Phi(\eta,\Psi^{-1}(1))2^{k\beta}e^{-2^{k-1}} \lesssim \Phi(\eta,\Psi^{-1}(1)).
\end{aligned} \tag{5.19}$$

In the last line above, we use Lemmas 5.2 and [GT12, Lemma 3.19]. Combining (5.16), (5.17), (5.18), (5.19) and using Lemma 5.2, we obtain

$$\int_{\partial U}\int_0^{\infty} e^{-t}p_t^{\mathrm{ref}}(x,y)\,dt\,\mu_{\xi,n}(dy) \lesssim \sup_{\eta\in\partial U\cap B(\xi,3n)}\Phi(\eta,\Psi^{-1}(1)) \lesssim \Phi(\xi,n) \tag{5.20}$$

for all $x \in \overline{U}$. Since $\mu_{\xi,n}$ is a finite measure such that the corresponding 1-potential $x \mapsto \int_{\partial U}\int_0^{\infty} e^{-t}p_t^{\mathrm{ref}}(x,y)\,dt\,\mu_{\xi,n}(dy)$ is bounded, we conclude that $\mu_{\xi,n}$ is of finite energy integral for all $n \in \mathbb{N}$ [FOT, Exercise 4.2.2]. Therefore μ is a smooth measure in the strict sense. The assertion that ∂U is a quasi-support of μ follows from Lemma 2.28(d) and Proposition 4.15(b). □

We record another upper bound on an integral of heat kernel with respect to μ similar to Lemma 5.3.

Lemma 5.5. *There exist $C \in (1,\infty), A \in (4,\infty)$ such that for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A$*

$$\int_{\partial U \cap B(\xi,r)} \int_0^\infty p_t^{\mathrm{ref},\overline{U}\cap B(\xi,r)}(x,y)\,dt\,\mu(dy) \leqslant C\Phi(\xi,r), \tag{5.21}$$

where $p_t^{\mathrm{ref},\overline{U}\cap B(\xi,r)}(\cdot,\cdot) : (\overline{U}\cap B(\xi,r)) \times (\overline{U}\cap B(\xi,r))$ is the continuous heat kernel as given in Lemma 3.4(d).

Proof. By Fubini's theorem and Lemma 5.3, there exists $A_1 \in (1,\infty)$ such that for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, x \in \overline{U} \cap B(\xi,r)$ we have

$$\begin{aligned}
&\int_{\partial U \cap B(\xi,r)} \int_0^{\Psi(r)} p_t^{\mathrm{ref},\overline{U}\cap B(\xi,r)}(x,y)\,dt\,\mu(dy) \\
&\quad\leqslant \int_{\partial U} \int_0^{\Psi(r)} p_t^{\mathrm{ref}}(x,y)\,dt\,\mu(dy) \quad (\text{since } p^{\mathrm{ref},\overline{U}\cap B(\xi,r)}(\cdot,\cdot) \leqslant p^{\mathrm{ref}}(\cdot,\cdot)) \\
&\quad\lesssim \int_0^{\Psi(r)} \frac{\mu\left(B(\xi_x,\Psi^{-1}(t))\right)}{m(B(\xi_x,\Psi^{-1}(t)))}\,dt = \sum_{k=0}^{\infty} \int_{\Psi(2^{-k}r)}^{\Psi(2^{-(k-1)}r)} \frac{\mu\left(B(\xi_x,\Psi^{-1}(t))\right)}{m(B(\xi_x,\Psi^{-1}(t)))}\,dt \quad (\text{by } (5.10)) \\
&\quad\lesssim \sum_{k=0}^{\infty} \frac{\mu\left(B(\xi_x,2^{-k}r)\right)}{m(B(\xi_x,2^{-k}r))}\Psi(2^{-k}r) \overset{(5.4)}{\asymp} \sum_{k=0}^{\infty} \Phi(\xi_x,2^{-k}r) \overset{(5.6)}{\asymp} \Phi(\xi_x,r) \overset{(5.6)}{\asymp} \Phi(\xi,r),
\end{aligned} \tag{5.22}$$

where $\xi_x \in \partial U$ is chosen as given in Lemma 5.3.

By [HS, Proof of Theorem 2.5], there exist $C_1, A_1 \in (1,\infty)$ such that for all $x \in \overline{U}, 0 < r < \operatorname{diam}(U,d)/A_1$, the first Dirichlet eigenvalue

$$\lambda_0(B(x,r)\cap\overline{U}) := \inf\left\{ \frac{\mathcal{E}^{\mathrm{ref}}(f,f)}{\int_{B(x,r)\cap\overline{U}} f^2\,dm} : f \in \mathcal{F}(U), f\big|_{(B(x,r)\cap\overline{U})^c} = 0 \ m\text{-a.e.} \right\}$$

satisfies

$$\frac{C_1^{-1}}{\Psi(r)} \leqslant \lambda_0(B(x,r)\cap\overline{U}) \leqslant \frac{C_1}{\Psi(r)}. \tag{5.23}$$

Hence by [HS, Proof of Lemma 3.9(3)] and (5.23), there exist $C_2, A_1 \in (1,\infty), c_1 \in (0,\infty)$ such that all $x \in \overline{U}, 0 < r < \operatorname{diam}(U,d)/A_1, y,z \in \overline{U}\cap B(x,r), t \geqslant \Psi(r)$, we have

$$p_t^{\mathrm{ref},\overline{U}\cap B(\xi,r)}(y,z) \leqslant \frac{C_2}{m(B(x,r)\cap\overline{U})} \exp\left(-\frac{c_1 t}{\Psi(r)}\right). \tag{5.24}$$

Therefore for all $\xi \in \partial U, 0 < r < \operatorname{diam}(U,d)/A_1, x \in \overline{U} \cap B(\xi, r)$ we have

$$
\begin{aligned}
&\int_{\partial U \cap B(\xi,r)} \int_{\Psi(r)}^{\infty} p_t^{\mathrm{ref},\overline{U}\cap B(\xi,r)}(x,y)\, dt\, \mu(dy) \\
&\quad \leqslant \int_{\partial U \cap B(\xi,r)} \int_{\Psi(r)}^{\infty} \frac{C_2}{m(B(\xi,r))} \exp\left(-\frac{c_1 t}{\Psi(r)}\right) dt\, \mu(dy) \quad (\text{by } (5.24)) \\
&\quad \lesssim \int_{\partial U \cap B(\xi,r)} \frac{\Psi(r)}{m(B(\xi,r))}\, \mu(dy) \asymp \frac{\mu(B(\xi,r))\Psi(r)}{m(B(\xi,r))} \overset{(5.4)}{\asymp} \Phi(\xi, r).
\end{aligned} \tag{5.25}
$$

By (5.22) and (5.25), we obtain the desired upper bound (5.21). □

Since μ is a smooth measure in the strict sense, it defines a PCAF in the strict sense due to the Revuz correspondence.

Definition 5.6. Let A denote the positive continuous additive functional (PCAF) in the strict sense for the reflected Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ on $L^2(\overline{U}, m|_{\overline{U}})$ whose Revuz measure is μ. Note that by Lemma 5.4 and [FOT, Theorem 5.1.7], there exists a PCAF in the strict sense $(A_t)_{t \geqslant 0}$ whose Revuz measure in μ.

The state space of the trace process corresponding to the PCAF (A_t) is the support of the PCAF. To this end, we show that the support of A is ∂U in the following proposition.

Proposition 5.7. *The support of the positive continuous additive functional in the strict sense A corresponding to μ is ∂U; that is,*

$$
\partial U = \{x \in \overline{U} : \mathbb{P}_x^{\mathrm{ref}}[A_t > 0 \text{ for any } t > 0] = 1\}. \tag{5.26}
$$

Proof. Set

$$
R := \inf\{t > 0 | A_t > 0\}, \quad S(\mu) := \{x \in \overline{U} : \mathbb{P}_x^{\mathrm{ref}}[R = 0] = 1\}.
$$

First we show that

$$
\mathbb{P}_x^{\mathrm{ref}}(R \geqslant \sigma_{\partial U}) = 1 \quad \text{for all } x \in U. \tag{5.27}
$$

Let $p_t^U(\cdot,\cdot)$ denote the continuous heat kernel for the associated part Dirichlet form of $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ on U given by Lemma 3.4(a) and Theorem 2.12. Then for all $x \in U$, we obtain

$$
\begin{aligned}
\mathbb{E}_x^{\mathrm{ref}}\left[A_{\sigma_{\partial U}}\right] &= \lim_{t \downarrow 0} \mathbb{E}_x^{\mathrm{ref}}\left[\int_t^{\tau_U} dA_s\right] \quad (\text{by monotone convergence theorem}) \\
&= \lim_{t \downarrow 0} \mathbb{E}_{p_t^U(x,\cdot)\cdot m}^{\mathrm{ref}}\left[\int_0^{\tau_U} dA_s\right] = 0 \quad (\text{by [CF, (4.1.25), Proposition 4.1.10]}).
\end{aligned}
$$

Therefore $\mathbb{P}_x^{\mathrm{ref}}(A_{\sigma_{\partial U}} = 0) = 1$ and hence we obtain (5.27). By the right continuity of sample paths, $\mathbb{P}_x^{\mathrm{ref}}(\sigma_{\partial U} > 0) = 1$ for all $x \in U$ and hence by (5.27), we conclude

$$
\partial U \supseteq \{x \in \overline{U} : \mathbb{P}_x^{\mathrm{ref}}[A_t > 0 \text{ for any } t > 0] = 1\}. \tag{5.28}
$$

Note that by [CF, (A.3.12) in Proposition A.3.6], we have

$$\mathbb{P}_x^{\mathrm{ref}}[R = \sigma_{S(\mu)}] = 1 \quad \text{for any } x \in \overline{U}. \tag{5.29}$$

Therefore in order to obtain (5.26), by (5.28) and (5.29) it suffices to prove that

$$\mathbb{P}_x^{\mathrm{ref}}[\sigma_{S(\mu)} = 0] = 1 \quad \text{for any } x \in \partial U. \tag{5.30}$$

We adapt [BCM, Proof of Proposition 6.16] to obtain (5.30). We collect a few preliminary estimates on Green function. By Lemma 5.5, there exist $C_1, A_1 \in (1,\infty)$ such that,

$$\int_{B(\xi,r)\cap\overline{U}} g^{\mathrm{ref}}_{B(\xi,r)\cap\overline{U}}(y,z)\,\mu(dz) \leqslant C_1\Phi(\xi,r) \quad \text{for all } y \in B(\xi,r). \tag{5.31}$$

By increasing A_1 if necessary and by [GHL15, Theorem 1.2] and Theorem 2.12, there exist $C_2, A_0 \in (1,\infty)$ such that for all $x \in \overline{U}, 0 < r < \operatorname{diam}(U,d)/A_1$, we have

$$C_2^{-1}\frac{\Psi(r)}{m(B(x,r))} \leqslant g^{\mathrm{ref}}_{B(x,r)\cap\overline{U}}(x, A_0^{-1}r) \leqslant C_2\frac{\Psi(r)}{m(B(x,r))}. \tag{5.32}$$

Next we show

$$\mathbb{P}_\xi^{\mathrm{ref}}[\tau_\xi = 0] = 1 \quad \text{for all } \xi \in \partial U. \tag{5.33}$$

Indeed, for any $x \in \overline{U}$ and any $t > 0$, we have

$$\mathbb{P}_x^{\mathrm{ref}}[X_t^{\mathrm{ref}} = x] = \int_{\overline{U}} 1_{\{x\}} \cdot p_t^{\mathrm{ref}}(x,\cdot)dm = 0$$

by the existence of heat kernel of the reflected diffusion X^{ref} and the fact that $m(\{x\}) = 0$ thanks to the reverse volume doubling property, and hence $\mathbb{P}_x^{\mathrm{ref}}[\tau_x \leqslant t] = 1$ for any $x \in \overline{U}$. Now letting $t \downarrow 0$ yields (5.33). Fix any $\xi \in \partial U$ and let $t > 0$ and $\epsilon > 0$ be arbitrary. By (5.33), we have

$$\mathbb{P}_\xi^{\mathrm{ref}}(T < t) > 1 - \varepsilon, \quad \text{for all } \xi \in \overline{U}, \text{ where } T = \tau_{B(\xi,r)\cap\overline{U}}, \tag{5.34}$$

for some $r = r(\xi,t,\epsilon) > 0$. By decreasing $r = r(\xi,t,\varepsilon)$ if necessary, we may assume that $0 < r < \operatorname{diam}(U,d)/A_1$, where $A_1 \in (1,\infty)$ is as above. Fixing $r = r(\xi,t,\epsilon)$ as above, we define

$$K_1 = B(\xi, A_0^{-1}r) \cap S(\mu).$$

We show that there exists a constant $c_0 \in (0,1)$ that depends only on the constants involved in the assumption such that

$$\mathbb{P}_\xi^{\mathrm{ref}}(\sigma_{K_1} < T) \geqslant c_0. \tag{5.35}$$

Let e denote the equilibrium measure for K_1 such that $e(\overline{K_1}) = \operatorname{Cap}_B(K_1)$, where $B = B(x,r) \cap \overline{U}$ and $\operatorname{Cap}_B(K_1)$ denotes the capacity as defined in (3.15) corresponding to the reflected diffusion $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ on $L^2(\overline{U}, m|_{\overline{U}})$. To prove (5.35), we observe that

$$\mathbb{P}_z^{\mathrm{ref}}(\sigma_{K_1} < \tau_B) = \int_{\overline{K_1}} g_B^{\mathrm{ref}}(z,y)\,e(dy) \quad \text{for all } z \in B. \tag{5.36}$$

To obtain (5.36), we use [FOT, Theorem 4.3.3 and the 0-order version of Exercise 4.2.2] to conclude that both sides of (5.36) are quasi-continuous versions of the 0-order equilibrium potential for K_1 with respect to the part Dirichlet form on B. Furthermore, both sides of (5.36) are $(X^{\mathrm{ref}})^B$-excessive from [CF, Lemma A.2.4(ii)] and Lemma 3.4(b) respectively. By the absolute continuity property from Lemma 3.4(a) and [CF, Theorem A.2.17(iii)], we obtain (5.36). By (5.36) and the maximum principle (3.3),

$$\mathbb{P}_\xi^{\mathrm{ref}}(\sigma_{K_1} < T) = \int_{\overline{K_1}} g_B^{\mathrm{ref}}(\xi, y)\, e(dy) \geqslant g_B^{\mathrm{ref}}(\xi, A_0^{-1}r)\, \mathrm{Cap}_B(K_1). \tag{5.37}$$

Since $S(\mu)$ is a quasi-support of the Revuz measure μ of $(A_t)_{t\geqslant 0}$ by [FOT, Theorem 5.1.5], we have $\mu(S(\mu)^c) = 0$ and hence

$$\mu(K_1) = \mu\left(B(\xi, A_0^{-1}r)\right). \tag{5.38}$$

We recall the following inequality for capacity ([FOT, p.441, Solution to Exercise 2.2.2]): for any Radon measure ν on B with $\int_B g_B^{\mathrm{ref}}(\cdot, z)\, \nu(dz) \leqslant 1$ $\mathcal{E}^{\mathrm{ref}}$-q.e. on B and $\nu(B \backslash K_1) = 0$,

$$\nu(K_1) \leqslant \mathrm{Cap}_B(K_1).$$

By considering the measure $\nu(\cdot) = \mu(K_1 \cap \cdot)/(C_1\Phi(\xi, r))$, (5.31) and the above inequality, we obtain

$$\mathrm{Cap}_B(K_1)^{-1} \leqslant \nu(K_1)^{-1} = C_1\Phi(\xi, r)/\mu(K_1) \overset{(5.38)}{=} C_1 \frac{\Phi(\xi, r)}{\mu(B(\xi, A_0^{-1}r))}. \tag{5.39}$$

To establish (5.35), we estimate $\mathbb{P}_\xi^{\mathrm{ref}}(\sigma_{K_1} < T)$ as

$$\begin{aligned}
\mathbb{P}_\xi^{\mathrm{ref}}(\sigma_{K_1} < T) &\overset{(5.37)}{\geqslant} g_B^{\mathrm{ref}}(\xi, A_0^{-1}r)\, \mathrm{Cap}_B(K_1) \overset{(5.39)}{\geqslant} C_1^{-1} \frac{g_B^{\mathrm{ref}}(\xi, A_0^{-1}r)\mu(B(\xi, A_0^{-1}r))}{\Phi(\xi, r)} \\
&\overset{(5.32)}{\geqslant} (C_1C_2)^{-1} \frac{\Psi(r)\mu(B(\xi, A_0^{-1}r))}{m(B(\xi, r))\Phi(\xi, r)} \overset{(5.4)}{\gtrsim} \frac{\mu(B(\xi, A_0^{-1}r))}{\mu(B(\xi, r))} \\
&\gtrsim 1 \quad \text{(by Corollary 4.7 and Proposition 4.15(c))}.
\end{aligned}$$

By choosing $\epsilon = c_0/2$ and using $\{\sigma_{K_1} < T\} \subset \left\{\sigma_{S(\mu)} \leqslant t\right\} \cup \{T \geqslant t\}$ $\varepsilon = c_0/2$, we obtain

$$\begin{aligned}
\mathbb{P}_\xi^{\mathrm{ref}}(\sigma_{S(\mu)} \leqslant t) &\geqslant \mathbb{P}_\xi^{\mathrm{ref}}(\sigma_{K_1} < T) - \mathbb{P}^x(T \geqslant t) \\
&> c_0 - \varepsilon = \tfrac{1}{2}c_0 \quad \text{(by (5.34) and (5.35))}.
\end{aligned}$$

Since $t > 0$ is arbitrary, the Blumenthal 0-1 law [CF, Lemma A.2.5] gives (5.30). □

Remark 5.8. By the estimate in (5.39) along with [CF, Theorem 3.3.8(iii)] or [FOT, Theorem 4.4.3(ii)], for the MMD space $(\overline{U}, d, m|_{\overline{U}}, \mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ we have

$$\mathrm{Cap}_1(B(\xi, r) \cap \partial U) > 0 \quad \text{for all } \xi \in \partial U, r > 0.$$

5.2 The Doob-Naïm formula

We describe trace process on the boundary and the associated trace Dirichlet form corresponding to the PCAF defined in Definition 5.6.

Definition 5.9 (The boundary trace process)**.** Let $\{X_t^{\text{ref}}, (\mathbb{P}_x^{\text{ref}})_{x\in\overline{U}}\}$ denote the reflected diffusion process on $\overline{U}$ that satisfies the Feller and strong Feller property defined at every starting point corresponding to the MMD space $(\overline{U}, d, m\big|_{\overline{U}}, \mathcal{E}^{\text{ref}}, \mathcal{F}(U))$ (see Theorem 2.12 and Proposition 2.13). Let μ and A be the smooth measure in strict sense and the corresponding positive continuous additive functional in strict sense as defined in Definitions 5.1 and 5.6 respectively. We recall that the associated trace process is defined as

$$\check{X}_t^{\text{ref}} := X_{\tau_t}^{\text{ref}}, \quad \tau_t := \inf\{s > 0 : A_s > t\}. \tag{5.40}$$

By Proposition 5.7 and [FOT, Theorem A.2.12] or [CF, Theorem A.3.9], X_t^{ref} defines a strong Markov process with right continuous sample paths on ∂U such that the corresponding law $(\check{\mathbb{P}}^{\text{ref}})_{x\in\partial U}$ satisfies $\check{\mathbb{P}}_x^{\text{ref}}(\check{X}_0^{\text{ref}} = x) = 1$ for all $x \in \partial U$. By [FOT, Theorem 6.2.1(i)] and Proposition 5.7, the corresponding transition semigroup $(\check{T}_t^{\text{ref}})_{t\geqslant 0}$ is a strongly continuous semigroup on $L^2(\partial U, \mu)$.

To describe the Dirichlet form $(\check{\mathcal{E}}^{\text{ref}}, \check{\mathcal{F}}(U))$ on $L^2(\partial U, \mu)$ associated to the semigroup $(\check{T}_t^{\text{ref}})_{t\geqslant 0}$ we adopt the convention that every function in the extended Dirichlet space $\mathcal{F}(U)_e$ of $(\mathcal{E}^{\text{ref}}, \mathcal{F}(U))$ is denoted by its $\mathcal{E}^{\text{ref}}$-quasicontinuous version. By [FOT, Theorem 6.2.1(ii)] or [CF, Theorem 5.2.2] and Proposition 5.7, the associated trace Dirichlet form is given by

$$\begin{aligned}\check{\mathcal{F}}(U) &:= \{\phi \in L^2(\partial U, \mu) : \phi = u \quad \mu\text{-a.e. on } \partial U \text{ and } u \in \mathcal{F}(U)_e,\}\\ \check{\mathcal{E}}^{\text{ref}}(\phi, \phi) &:= \mathcal{E}^{\text{ref}}(H_{\partial U}^{\text{ref}} u, H_{\partial U}^{\text{ref}} u), \quad \text{where } \phi = u \ \mu\text{-a.e. on } \partial U \text{ and } u \in \mathcal{F}(U)_e,\end{aligned} \tag{5.41}$$

where $H_{\partial U}^{\text{ref}} u$ is the 0-order hitting distribution corresponding to X^{ref} given in (4.50). By [FOT, Lemma 6.2.1] the form in (5.41) is well-defined.

The Dirichlet form $(\check{\mathcal{E}}^{\text{ref}}, \check{\mathcal{F}}(U))$ on $L^2(\partial U, \mu)$ associated to the trace process is regular by [FOT, Theorem 6.2.1(iii)]. By the Beurling-Deny decomposition ([FOT, Theorem 3.2.1] or [CF, Theorem 4.3.3]) every regular Dirichlet form can be uniquely decomposed into a strongly local (or diffusion) part, a jump part and a killing part. To describe this decomposition, let us denote by the extended Dirichlet space associated with $(\check{\mathcal{E}}^{\text{ref}}, \check{\mathcal{F}}(U))$ as $\check{\mathcal{F}}(U)_e$. Then by [CF, Theorem 4.3.3], there exist symmetric strongly local bi-linear form $\check{\mathcal{E}}^{\text{ref},(c)} : \check{\mathcal{F}}(U)_e \times \check{\mathcal{F}}(U)_e$, a symmetric Radon measure J on $(\partial U \times \partial U)\backslash(\partial U)_{\text{diag}}$, and a Radon measure κ on ∂U such that

$$\begin{aligned}\check{\mathcal{E}}^{\text{ref}}(u, v) &= \check{\mathcal{E}}^{\text{ref},(c)}(u, v) + \frac{1}{2}\int_{(\partial U\times\partial U)\backslash(\partial U)_{\text{diag}}} (\widetilde{u}(x) - \widetilde{u}(y))(\widetilde{v}(x) - \widetilde{v}(y))\, J(dx\, dy)\\ &\quad + \int_{\partial U} \widetilde{u}(x)\widetilde{v}(x)\, \kappa(dx),\end{aligned} \tag{5.42}$$

for all $u, v \in \check{\mathcal{F}}(U)_e$, where $\widetilde{u}, \widetilde{v}$ denote $\check{\mathcal{E}}^{\mathrm{ref}}$-quasicontinuous versions of u, v respectively. The measure κ and J are called the killing measure and jumping measure respectively.

The following lemma is the main ingredient to show that the killing measure κ is zero as is an easy consequence of the Δ-regularity estimate shown in Lemma 4.5(a).

Lemma 5.10. *Under the capacity density condition, we have*

$$\mathbb{P}_x^{\mathrm{ref}}[\sigma_{\partial U} = \infty] = 0, \quad \textit{for all } x \in \overline{U}.$$

Proof. First, since the reflected diffusion has the property that

$$\mathbb{P}_x^{\mathrm{ref}}[X_t^{\mathrm{ref}} \in U] = 1 \quad \text{for any } x \in \overline{U} \text{ and any } t > 0$$

by (AC) for X^{ref} and $m(\partial U) = 0$, it suffices to show the claim for $x \in U$. Then by the Markov property at any time $t > 0$, X^{ref} hits ∂U after time t $\mathbb{P}_x^{\mathrm{ref}}$-a.s. for any $x \in \partial U$. In particular, we can work with the original diffusion X on the ambient space $\mathcal{X}$ rather than the reflected diffusion X^{ref} on $\overline{U}$.

We claim that any relatively compact open subset $D \subset \mathcal{X}$ with D^c non-$\mathcal{E}$-polar,

$$\mathbb{P}_x[\tau_D < \infty] = 1 \quad \text{for any } x \in D. \tag{5.43}$$

This follows by [BCM, Proposition 3.2], and (AC) for the part process X^D on D. In particular, if U is bounded, then the desired claim follows by $U \neq \mathcal{X}$, (CDC) and Remark 4.2.

Thus we may and will assume that U is unbounded. Let $x \in U$ and choose $\xi \in \partial U, R > 0$ such that $R > d(x, \xi)$. By Lemma 4.5(a) there exist $C_1, \delta > 0$ such that for all $K \in (1, \infty)$

$$\begin{aligned}
\mathbb{P}_x\left(\tau_U < \infty\right) &\overset{(5.43)}{\geqslant} \mathbb{P}_x\left(\tau_U \leqslant \tau_{B(\xi,KR)}\right) = 1 - \mathbb{P}_x\left(\tau_{B(\xi,KR)} < \tau_U\right) \\
&\geqslant 1 - \omega_x^{U \cap B(\xi,KR)}\left(U \cap S(\xi, KR)\right) \\
&\geqslant 1 - C_1 K^{-\delta} \quad \text{(by Lemma 4.5(a))}.
\end{aligned}$$

Letting $K \to \infty$, we obtain the desired conclusion. □

Our next result shows that the only non-vanishing term in the Beurling-Deny decomposition (5.42) is the jump part. Our main tool is [CF, Corollary 5.6.1] that identifies the Beurling-Deny decomposition in terms of the energy measure and the supplementary Feller measures and Feller measures.

Proposition 5.11. *The trace Dirichlet form $(\check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ on $L^2(\partial U, \mu)$ is of pure jump type; that is κ and $\check{\mathcal{E}}^{\mathrm{ref},(c)}$ in (5.42) are identically zero.*

Proof. By [CF, Corollary 5.6.1] the killing measure is the supplementary Feller measure as defined in [CF, (5.5.7)] which in turn vanishes due to Lemma 5.10.

By [CF, Corollary 5.6.1] and [Mur23+, Theorem 2.9], the strongly local part of $(\check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ vanishes. More precisely, we view U as a uniform domain in $\overline{U}$ and apply [Mur23+, Theorem 2.9] on the MMD space $(\overline{U}, d, m|_{\overline{U}}, \mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ to conclude that the energy measure of any function $f \in \mathcal{F}(U)$ on the boundary ∂U is zero. This concludes the proof that the trace form is of pure jump type. □

The goal of this section is the Doob-Naïm formula stated in Theorem 5.12. We discuss relevant previous works and approaches of proving the Doob-Naïm formula. As mentioned in the introduction, this was first shown by Doob in the setting of Green spaces [Doo]. Green spaces are locally Euclidean and hence the result does not apply to diffusion on fractals [BC]. Doob's work relies on existence of fine limits to define the Naïm kernel and existence of 'fine normal derivatives' [Doo, §8] shown by Naïm [Naï]. It is unclear to the authors whether these results of Naïm can be extended to our setting and we leave it as an interesting direction for future work. M. Silverstein showed Doob-Naïm formula for Markov chains on countable spaces using an excursion measure [Sil, Theorem 1.3]. While it is possible to construct similar excursions in our setting [CF, §5.7], we choose a direct approach starting from the definition of the trace Dirichlet form in (5.41) and performing a fairly simple computation. The joint continuity of the Naïm kernel established using the boundary Harnack principle in Proposition 3.15 and the description of Martin kernel as the Radon-Nikodym derivative of harmonic measure in Proposition 4.14 are important ingredients in our proof.

For random walks on certain trees, the trace Dirichlet form on the boundary is amenable to explicit computations. This was first done by Kigami [Kig10, Theorem 5.6] and was later shown to coincide with the Doob-Naïm formula in [BGPW, Theorem 6.4]. Kigami also obtained stable-like heat kernel estimates [Kig10, Theorem 7.6] for the trace process on boundary.

By extending the results of [Doo, Fuk, Sil], we show that $\Theta^U_{x_0}(\cdot,\cdot)$ is the jump kernel of the trace process with respect to $\omega^U_{x_0} \times \omega^U_{x_0}$.

Theorem 5.12 (Doob-Naïm formula). *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain satisfying Assumption 4.3. Then the jump measure J of the Beurling-Deny decomposition of the trace Dirichlet form $(\check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ on $L^2(\partial U, \mu)$ as given in (5.42) is*

$$dJ(\xi, \eta) = \Theta^U_{x_0}(\xi, \eta)\, d\omega^U_{x_0}(\xi)\, d\omega^U_{x_0}(\eta).$$

Equivalently,

$$\check{\mathcal{E}}^{\mathrm{ref}}(u, v) = \frac{1}{2} \int_{(\partial U \times \partial U) \setminus (\partial U)_{\mathrm{diag}}} (\widetilde{u}(x) - \widetilde{u}(y))(\widetilde{v}(x) - \widetilde{v}(y))\, \Theta^U_{x_0}(\xi, \eta)\, d\omega^U_{x_0}(\xi)\, d\omega^U_{x_0}(\eta)$$

for all $u, v \in \check{\mathcal{F}}(U)$, where $\widetilde{u}, \widetilde{v}$ denote $\check{\mathcal{E}}^{\mathrm{ref}}$-quasicontinuous versions of u, v respectively.

Proof. Let $\xi, \eta \in \partial U$ be distinct and $r < d(\xi, \eta)/4$. Let $A = B(\xi, r) \cap \partial U$, $B = B(\xi, 2r)^c \cap \overline{U}$ and $e_{A,B} \in \mathcal{F}(U)$ denote the equilibrium potential for $\mathrm{Cap}_B(A)$ for the Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ as given in Lemma 3.7 such that

$$\mathrm{Cap}_B(A) = \mathcal{E}^{\mathrm{ref}}(e_{A,B}, e_{A,B}), \quad \widetilde{e}_{A,B} = 1\ \mathcal{E}^{\mathrm{ref}}\text{-q.e. on } A, \quad \widetilde{e}_{A,B} = 0\ \mathcal{E}^{\mathrm{ref}}\text{-q.e. on } \overline{U} \setminus B,$$

where $\widetilde{e}_{A,B}$ is a $\mathcal{E}^{\mathrm{ref}}$-quasicontinuous version of $e_{A,B}$. Let $\lambda^1_{A,B}, \lambda^0_{A,B}$ denote the associated measures as given in Lemma 3.7 supported in $\overline{A}$ and $\overline{U} \cap \partial B(\xi, 2r)$ respectively. By (4.53), we have

$$0 < \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_{x_0} = \int_{\overline{U}\cap\partial B} g_U(x_0, y)\, d\lambda^0_{A,B}(y). \tag{5.44}$$

Let $u \in C_c(\overline{U}) \cap \mathcal{F}(U)$ be such that $\mathbb{1}_{B(\eta,r)} \leqslant u \leqslant \mathbb{1}_{B(\eta,2r)}$. Since $H^{\mathrm{ref}}_{\partial U} u$ is harmonic in U and $H^{\mathrm{ref}}_{\partial U}(\widetilde{e}_{A,B}) - \widetilde{e}_{A,B} = 0$ $\mathcal{E}^{\mathrm{ref}}$-q.e. on ∂U, we have

$$\begin{aligned}
\mathcal{E}^{\mathrm{ref}}(H^{\mathrm{ref}}_{\partial U} u, H^{\mathrm{ref}}_{\partial U}(\widetilde{e}_{A,B})) &= \mathcal{E}^{\mathrm{ref}}(H^{\mathrm{ref}}_{\partial U}(u), \widetilde{e}_{A,B}) \quad \text{(by [FOT, (4.3.11), (4.3.12)])}\\
&= -\int_{\overline{U}\cap\partial B(\xi,2r)} H^{\mathrm{ref}}_{\partial U} u\, d\lambda^0_{A,B} \quad \text{(by Lemma (3.18))}\\
&= -\int_{\overline{U}\cap\partial B(\xi,2r)} \left(\int_{\partial U} u(z)\, d\omega^U_y(z)\right) d\lambda^0_{A,B}(y)\\
&\overset{(4.52)}{=} -\int_{\overline{U}\cap\partial B(\xi,2r)} \left(\int_{\partial U} u(z) K_{x_0}(y,z)\, d\omega^U_{x_0}(z)\right) d\lambda^0_{A,B}(y).
\end{aligned} \tag{5.45}$$

Note that by [FOT, Lemma 6.2.4] and [CF, Theorem 5.2.8],

$$\widetilde{e}_{A,B}\big|_{\partial U} \in \check{\mathcal{F}}(U) \text{ and } \widetilde{e}_{A,B}\big|_{\partial U} \text{ is } \check{\mathcal{E}}^{\mathrm{ref}}\text{-quasicontinuous.} \tag{5.46}$$

Therefore by, the Beurling-Deny decomposition (5.42), (5.46), and Proposition 5.11, we obtain

$$\begin{aligned}
&\mathcal{E}^{\mathrm{ref}}(H^{\mathrm{ref}}_{\partial U} u, H^{\mathrm{ref}}_{\partial U}(\widetilde{e}_{A,B}))\\
&\quad = \check{\mathcal{E}}^{\mathrm{ref}}(u\big|_{\partial U}, \widetilde{e}_{A,B}\big|_{\partial U}), \quad \text{(by (5.41))}\\
&\quad = \frac{1}{2}\int_{(\partial U\times\partial U)\setminus(\partial U)_{\mathrm{diag}}} (u(x)-u(y))(\widetilde{e}_{A,B}(x) - \widetilde{e}_{A,B}(y))\, J(dx,dy) \quad \text{(by (5.42), (5.46))}\\
&\quad = -\int_{(\partial U\times\partial U)\setminus(\partial U)_{\mathrm{diag}}} u(x)\widetilde{e}_{A,B}(y)\, J(dx,dy),
\end{aligned} \tag{5.47}$$

where in the last line above we use that $u, \widetilde{e}_{A,B}$ have disjoint supports (note that $r < d(\xi,\eta)/4$) and J is symmetric (see [CF, Proposition 4.3.2]). Therefore, we obtain

$$\begin{aligned}
&\frac{\int_{(\partial U\times\partial U)\setminus(\partial U)_{\mathrm{diag}}} u(x)\widetilde{e}_{A,B}(y)\, J(dx,dy)}{\int_{\partial U} u\, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_{x_0}}\\
&\quad = \frac{-\mathcal{E}^{\mathrm{ref}}(H^{\mathrm{ref}}_{\partial U} u, H^{\mathrm{ref}}_{\partial U}(\widetilde{e}_{A,B}))}{\int_{\partial U} u\, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_{x_0}} \quad \text{(by (5.47))}\\
&\quad = \frac{\int_{\overline{U}\cap\partial B(\xi,2r)} \left(\int_{\partial U} u(z) K_{x_0}(y,z)\, d\omega^U_{x_0}(z)\right) d\lambda^0_{A,B}(y)}{\int_{\partial U} u\, d\omega^U_{x_0} \int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,y)\, d\lambda^0_{A,B}(y)} \quad \text{(by (5.44) and (5.45))}\\
&\quad \overset{(3.50)}{=} \int_{\overline{U}\cap\partial B(\xi,2r)} \int_{\partial U} \Theta^U_{x_0}(y,z) \frac{u(z)}{\int_{\partial U} u\, d\omega^U_{x_0}}\, d\omega^U_{x_0}(z) \frac{g_U(x_0,y)}{\int_{\overline{U}\cap\partial B(\xi,2r)} g_U(x_0,\cdot)\, d\lambda^0_{A,B}}\, d\lambda^0_{A,B}(y).
\end{aligned} \tag{5.48}$$

Let ρ be the metric on $\partial U \times \partial U$ defined by $\rho((x_1,y_1),(x_2,y_2)) = \max(d(x_1,x_2), d(y_1,y_2))$. For $(x_1,x_2) \in \partial U \times \partial U$, let $B_\rho((x_1,x_2),r)$ denote the open ball of radius r in the metric ρ centered at (x_1,x_2). By [FOT, Lemma 4.5.4(i)] and using $\widetilde{e}_{A,B} = 1$ $\mathcal{E}^{\mathrm{ref}}$-q.e. on A, we have

$$u(x)\widetilde{e}_{A,B}(y) = 1 \quad \text{for } J\text{-almost every } (x,y) \in (B(\eta,r) \times B(\xi,r)) \cap (\partial U \times \partial U).$$

Hence

$$\int_{\partial U}\int_{\partial U} u(x)\widetilde{e}_{A,B}(y)\, J(dx,dy) \geqslant J(B_\rho((\eta,\xi),r)). \tag{5.49}$$

By Corollary 4.7, there exist $C_1 \in (1,\infty), A_1 \in (6,\infty)$ such that for all $(\xi,\eta) \in \partial U \times \partial U, 0 < r < A_1^{-1}(d(x_0,\xi) \wedge d(x_0,\eta))$, we have

$$\left(\omega^U_{x_0} \times \omega^U_{x_0}\right)\left(B_\rho((\eta,\xi),2r)\right) \leqslant C_1 \left(\omega^U_{x_0} \times \omega^U_{x_0}\right)\left((B_\rho((\eta,\xi),r)\right). \tag{5.50}$$

Since $\omega^U_{x_0}$ is smooth, $\widetilde{e}_{A,B} \leqslant \mathbb{1}_{B(\xi,2r)}$ $\mathcal{E}^{\mathrm{ref}}$-q.e. implies $\widetilde{e}_{A,B} \leqslant \mathbb{1}_{B(\xi,2r)}$ $\omega^U_{x_0}$-a.e. and hence

$$\int_{\partial U} u\, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_{x_0} \leqslant \int_{\partial U} \mathbb{1}_{B(\eta,2r)}\, d\omega^U_{x_0} \int_{\partial U} \mathbb{1}_{B(\xi,2r)}\, d\omega^U_{x_0} = \left(\omega^U_{x_0} \times \omega^U_{x_0}\right)\left(B_\rho((\eta,\xi),2r)\right). \tag{5.51}$$

Combining (5.51), (5.49) and (5.50), we obtain

$$\frac{J(B_\rho((\eta,\xi),r))}{\left(\omega^U_{x_0} \times \omega^U_{x_0}\right)\left(B_\rho((\eta,\xi),r)\right)} \leqslant C_1 \frac{\int_{(\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}} u(x)\widetilde{e}_{A,B}(y)\, J(dx,dy)}{\int_{\partial U} u\, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_{x_0}} \tag{5.52}$$

for all $(\xi,\eta) \in (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}, 0 < r < A_1^{-1}(d(x_0,\xi) \wedge d(x_0,\eta) \wedge d(\xi,\eta))$.

By using (3.46) in Proposition 3.15 and increasing A_1 if necessary, there exist $C_2 \in (1,\infty), \gamma \in (0,\infty)$ such that

$$\begin{aligned} &\left| \frac{\int_{(\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}} u(x)\widetilde{e}_{A,B}(y)\, J(dx,dy)}{\int_{\partial U} u\, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B}\, d\omega^U_{x_0}} - \Theta^U_{x_0}(\eta,\xi) \right| \\ &\qquad \leqslant C_2 \Theta^U_{x_0}(\eta,\xi) \left(\frac{r}{d(x_0,\xi) \wedge d(x_0,\eta) \wedge d(\xi,\eta)} \right)^\gamma \end{aligned} \tag{5.53}$$

for all $(\eta,\xi) \in (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}, 0 < r < A_1^{-1}(d(x_0,\xi) \wedge d(x_0,\eta) \wedge d(\xi,\eta))$. By (5.52) and (5.53), there exist $c_0 \in (0, A_1^{-1})$ such that for all $(\eta,\xi) \in (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}, 0 < r \leqslant c_0(d(x_0,\xi) \wedge d(x_0,\eta) \wedge d(\xi,\eta))$, we have

$$\frac{J(B_\rho((\eta,\xi),r))}{\left(\omega^U_{x_0} \times \omega^U_{x_0}\right)\left(B_\rho((\eta,\xi),r)\right)} \leqslant 2C_1 \Theta^U_{x_0}(\eta,\xi). \tag{5.54}$$

Using (5.54), we will show the absolute continuity of J with respect to $\omega^U_{x_0} \times \omega^U_{x_0}$; that is

$$J \ll \omega^U_{x_0} \times \omega^U_{x_0}. \tag{5.55}$$

By the inner regularity of J it suffices to to that if $K \subset (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}$ is compact and $(\omega^U_{x_0} \times \omega^U_{x_0})(K) = 0$, then

$$J(K) = 0. \tag{5.56}$$

If $K \subset (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}$ is compact and $(\omega^U_{x_0} \times \omega^U_{x_0})(K) = 0$, then by the outer regularity of $\omega^U_{x_0} \times \omega^U_{x_0}$, for any $\epsilon > 0$, there exists an open set $K_\epsilon \subset (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}$ such that $(\omega^U_{x_0} \times \omega^U_{x_0})(K_\epsilon) < \epsilon$. By the 5B-covering lemma [Hei, Theorem 1.2], there exists balls $B_\rho((y_i, z_i), r_i) \subset K_\epsilon, i \in I$ such that $(y_i, z_i) \in K, 0 < r_i \leqslant c_0(d(x_0, y_i) \wedge d(x_0, z_i) \wedge d(y_i, z_i))$ for all $i \in I$, $\bigcup_{i\in I} B_\rho((y_i, z_i), r_i) \supset K$ and $B_\rho((y_i, z_i), r_i)/5), i \in I$ are pairwise disjoint. Hence, we have

$$\begin{aligned}
J(K) &\leqslant \sum_{i\in I} J(B_\rho((y_i, z_i), r_i)) \overset{(5.54)}{\leqslant} \sum_{i\in I} 2C_1 \Theta^U_{x_0}(y_i, z_i)(\omega^U_{x_0} \times \omega^U_{x_0}) \left(B_\rho((y_i, z_i), r_i)\right) \\
&\leqslant 2C_1^4 \sup_K \Theta^U_{x_0}(\cdot, \cdot) \sum_{i\in I} (\omega^U_{x_0} \times \omega^U_{x_0}) \left(B_\rho((y_i, z_i), r_i/5)\right) \quad \text{(by (5.50))} \\
&\leqslant 2C_1^4 \sup_K \Theta^U_{x_0}(\cdot, \cdot)(\omega^U_{x_0} \times \omega^U_{x_0})(K_\epsilon) \\
&\qquad \text{(since } \textstyle\bigcup_{i\in I} B_\rho((y_i, z_i), r_i) \subset K_\epsilon \text{ and } B_\rho((y_i, z_i), r_i)/5), i \in I \text{ are pairwise disjoint)} \\
&\leqslant 2C_1^4 \sup_K \Theta^U_{x_0}(\cdot, \cdot)\epsilon
\end{aligned}$$

By letting $\epsilon \downarrow 0$, we obtain (5.56) since $\sup_K \Theta^U_{x_0}(\cdot, \cdot) < \infty$ due to continuity of $\Theta^U_{x_0}$ (Proposition 3.15) and compactness of K. This concludes the proof of (5.55).

By letting $r \downarrow 0$ in the Hölder continuity estimate (5.53) and using the asymptotic doubling property of harmonic measures in (5.51), absolute continuity in (5.55) along with Lebesgue differentiation theorem ((4.49) in Lemma 4.12), we obtain the desired conclusion. □

Remark 5.13. The absolute continuity (5.55) can alternately be obtained using the identification of the Feller measure with jumping measure in [CF, Theorem 5.6.3] along with [FHY, p. 3143, equation before Example 2.1]. However, we choose the more elementary approach using (5.48) because the identification of Feller measure with the jumping measure in [CF, Theorem 5.6.3] is quite involved.

The following corollary of Doob-Naïm formula relates the jump density to the boundary reference measure μ and the function $\Phi(\cdot, \cdot)$.

Corollary 5.14. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain satisfying Assumption 4.3. The jumping measure is given by $J(d\xi, d\eta) = J_\mu(\xi, \eta)\, \mu(d\xi)\, \mu(d\eta)$, where*

$$J_\mu(\xi, \eta) = \begin{cases} \Theta^U_{x_0}(\xi, \eta), & \text{if } U \text{ is bounded,} \\ \Theta^U_{x_0}(\xi, \eta) \left(\frac{d\nu^U_{x_0}}{d\omega^U_{x_0}}(\xi) \frac{d\nu^U_{x_0}}{d\omega^U_{x_0}}(\eta) \right)^{-1} & \text{if } U \text{ is unbounded.} \end{cases} \tag{5.57}$$

and there exists $C, A \in (1, \infty)$ such that for all $(\xi, \eta) \in (\partial U \times \partial U)\setminus(\partial U)_{\mathrm{diag}}$ such that $d(\xi, \eta) < \mathrm{diam}(U, d)/A$, we have

$$C^{-1} \frac{1}{\mu\left(B(\xi, d(\xi, \eta))\right) \Phi(\xi, d(\xi, \eta))} \leqslant J_\mu(\xi, \eta) \leqslant C \frac{1}{\mu\left(B(\xi, d(\xi, \eta))\right) \Phi(\xi, d(\xi, \eta))}. \tag{5.58}$$

Proof. The jump kernel formula (5.57) is a direct consequence of the Doob-Naim formula (Theorem 5.12) along with the mutual absolute continuity in Proposition 4.15(a).

By Theorem 4.6 and Proposition 4.15(c), there exist $C_1, A_1 \in (1,\infty)$ such that

$$C_1^{-1}\frac{\Phi(\xi,R)}{\Psi(R)}m(B(\xi,R)) \leqslant \mu(B(\xi,r)) \leqslant C_1\frac{\Phi(\xi,R)}{\Psi(R)}m(B(\xi,R)) \tag{5.59}$$

for all $\xi \in \partial U, 0 < R < \operatorname{diam}(U)/A_1$.

If U is unbounded, the estimate (5.58) follows from (4.57), Lemma 5.2 and (5.59) provided $d(\xi,\eta) < d(x_0,\xi)/A$ for some large enough $A \in (1,\infty)$. If $d(\xi,\eta) < d(x_0,\xi)/A$ is not satisfied, then by changing the base point from x_0 to y as given in the argument using (4.63) in the proof of Theorem 4.15(c) and using (4.57), we obtain (5.58) in the case when U is unbounded.

If U is bounded, then (3.45) in Proposition 3.15 along with Lemma 5.2, there exist $c_0 \in (0,1), C_2 \in (1,\infty)$ such that

$$C_2^{-1}\frac{g_U(\eta_{c_0 d(\xi,\eta)},\xi_{c_0 d(\xi,\eta)})}{g_U(x_0,\eta_{c_0 d(\xi,\eta)})g_U(x_0,\xi_{c_0 d(\xi,\eta)})} \leqslant J_\mu(\eta,\xi) \leqslant C_2\frac{g_U(\eta_{c_0 d(\xi,\eta)},\xi_{c_0 d(\xi,\eta)})}{g_U(x_0,\eta_{c_0 d(\xi,\eta)})g_U(x_0,\xi_{c_0 d(\xi,\eta)})}, \tag{5.60}$$

for all $(\xi,\eta) \in (\partial U \times \partial U)\setminus(\partial U)_{\text{diag}}$. By covering ∂U with balls of radii $c_1\operatorname{diam}(U)$ for $c_1 \in (0,1)$ sufficiently and using Lemma 5.2 and increasing C_1 if necessary, we can improve (5.59) as

$$C_1^{-1}\frac{\Phi(\xi,R)}{\Psi(R)}m(B(\xi,R)) \leqslant \mu(B(\xi,r)) \leqslant C_1\frac{\Phi(\xi,R)}{\Psi(R)}m(B(\xi,R)) \tag{5.61}$$

for all $\xi \in \partial U, 0 < R \leqslant \operatorname{diam}(U)$. Combining (5.61), (5.60) and Lemma 5.2, we obtain (5.58) in the bounded case as well. □

Remark 5.15. The estimates (5.61) and (5.59) along with the doubling property of m, Lemma 5.2 and (2.12) shows that μ is a doubling measure on ∂U.

5.3 Heat kernel bounds for the trace process

The following occupation density formula for the boundary trace process shows that the Green function for the trace process is same as that of the reflected diffusion.

Lemma 5.16. *Let $F \subset \partial U$ be a closed subset such that the part process $(X^{\text{ref}})^{\overline{U}\setminus F}$ of the reflected diffusion on $\overline{U}\setminus F$ is transient. Then we have the following occupation density formula:*

$$\check{\mathbb{E}}^{\text{ref}}_\xi \int_0^{\tau_{\partial U\setminus F}} f(\check{X}^{\text{ref}}_s)\,ds = \int_{\partial U\setminus F} g^{\text{ref}}_{\overline{U}\setminus F}(\xi,y)f(y)\,\mu(dy) \quad \textit{for all } \xi \in \partial U\setminus F, f \in \mathcal{B}_+(\partial U\setminus F). \tag{5.62}$$

Proof. Let $D = \overline{U}\setminus F$ and $p_t^{\text{ref},D}(\cdot,\cdot)$ denote the continuous heat kernel corresponding to the part process $(X^{\text{ref}})^{\overline{U}\setminus F}$ with respect to $m|_{\overline{U}}$ which exists due to Lemma 3.4 and

Theorem 2.12. Let $P_t^{\mathrm{ref},D}$ denote the corresponding semigroup.

$$\begin{aligned}
\check{\mathbb{E}}_\xi^{\mathrm{ref}} \int_0^{\tau_{\partial U\setminus F}} f(\check{X}_s^{\mathrm{ref}})\,ds &= \mathbb{E}_\xi^{\mathrm{ref}} \int_0^{\tau_{\partial U\setminus F}} f(X_{\tau_s}^{\mathrm{ref}})\,ds \\
&= \mathbb{E}_\xi^{\mathrm{ref}} \int_0^\infty f(X_{s\wedge\tau_D}^{\mathrm{ref}})\,dA_s \quad \text{(by [CF, Lemma A.3.7(i)])} \\
&= \lim_{\delta\downarrow 0} \mathbb{E}_\xi^{\mathrm{ref}} \int_\delta^\infty f(X_{s\wedge\tau_D}^{\mathrm{ref}})\,dA_s \\
&= \lim_{\delta\downarrow 0} \mathbb{E}_{p_\delta^{\mathrm{ref},D}(\xi,\cdot)\,dm}^{\mathrm{ref}} \int_0^\infty f(X_{s\wedge\tau_D}^{\mathrm{ref}})\,dA_s \\
&= \lim_{\delta\downarrow 0} \int_0^\infty \int_{\partial U\setminus F} \left(P_s p_\delta^{\mathrm{ref},D}(\xi,\cdot)\right)(y) f(y)\,\mu(dy)\,ds \quad \text{(by [CF, (4.1.25)])} \\
&= \lim_{\delta\downarrow 0} \int_\delta^\infty \int_{\partial U\setminus F} p_s^{\mathrm{ref},D}(\xi,y) f(y)\,\mu(dy)\,ds \\
&= \int_0^\infty \int_{\partial U\setminus F} p_s^{\mathrm{ref},D}(\xi,y) f(y)\,\mu(dy)\,ds \\
&= \int_{\partial U\setminus F} \left(\int_0^\infty p_s^{\mathrm{ref},D}(\xi,y)\,ds\right) f(y)\,\mu(dy) = \int_{\partial U\setminus F} g_{\overline{U}\setminus F}^{\mathrm{ref}}(\xi,y) f(y)\,\mu(dy).
\end{aligned}$$

□

Remark 5.17. A weaker version of (5.62) with *every* ξ replaced with *quasi-every* ξ can be obtained following [FOT, Proof of Lemma 6.2.2] (see in particular [FOT, (6.2.10) and (6.2.11)]).

The following exit time lower bound is a key ingredient in heat kernel estimates for the trace process. The proof uses sub-Gaussian heat kernel estimates for the reflected diffusion obtained in [Mur23+] (see Theorem 2.12).

Proposition 5.18. *There exist $C_1, A_1 \in (1,\infty)$ such that all $\xi \in \partial U, 0 < r < \mathrm{diam}(\partial U, d)/2$, we have*

$$\check{\mathbb{E}}_\xi^{\mathrm{ref}}[\tau_{B(\xi,r)}] \geqslant C_1^{-1}\Phi(\xi,r). \tag{5.63}$$

Proof. By Remark 5.8 for any $\xi \in \partial U, 0 < r < \mathrm{diam}(\partial U)/2$, then the part process $(X^{\mathrm{ref}})^{\overline{U}\setminus(\partial U\cap B(\xi,r)^c)}$ of the reflected diffusion is transient.

By Theorem 2.12 and [GHL15, Theorem 1.2], there exist $A_0, A_1, C_2 \in (1,\infty)$ such that for all $x \in \overline{U}, 0 < r < \mathrm{diam}(\overline{U})/2$, we have

$$g_{B(x,r)\cap\overline{U}}^{\mathrm{ref}}(y,z) \geqslant C_2^{-1}\frac{\Psi(r)}{m(B(x,r))} \quad \text{for all } y,z \in B(x, A_0^{-1}r). \tag{5.64}$$

By domain monotonicity of Green function, we have

$$g_{\overline{U}\setminus(\partial U\cap B(\xi,r)^c)}^{\mathrm{ref}}(\cdot,\cdot) \geqslant g_{B(\xi,r)\cap\overline{U}}^{\mathrm{ref}}(\cdot,\cdot) \tag{5.65}$$

for all $\xi \in \partial U, r < \operatorname{diam}(\partial U)/2$.

Therefore by applying (5.62) with $f \equiv 1$, for all $\xi \in \partial U, 0 < r < \operatorname{diam}(\partial U)/2$, we have

$$\begin{aligned}\check{\mathbb{E}}^{\mathrm{ref}}_{\xi}[\tau_{B(\xi,r)}] &= \int_{\partial U \setminus (B(\xi,r)^c)} g^{\mathrm{ref}}_{\overline{U}\setminus(\partial U \cap B(\xi,r)^c)}(\xi,y)\,\mu(dy) \overset{(5.65)}{\geqslant} \int_{\partial U \setminus (B(\xi,r)^c)} g^{\mathrm{ref}}_{B(\xi,r)\cap \overline{U}}(\xi,y)\,\mu(dy) \\ &\geqslant C_2^{-1}\frac{\Psi(r)}{m(B(\xi,r))}\mu(\partial U \cap B(\xi,r) \quad (\text{by (5.64)}).\end{aligned} \tag{5.66}$$

The exit time lower bound (5.63) follows from (5.66) and (5.61). □

Given the exit time bound (Proposition 5.18) and jump kernel bound (Corollary 5.14) for the trace process on the boundary, we obtain stable-like heat kernel bound for the trace process using Theorem 2.32.

Theorem 5.19. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain satisfying Assumption 4.3 such that $(\partial U, d)$ is a uniformly perfect metric space. Let $(\check{X}^{\mathrm{ref}}_t)$ denote the μ-symmetric boundary trace process of the reflected diffusion X^{ref} as given in Definition 5.9. Then $(\check{X}^{\mathrm{ref}}_t)$ admits a continuous heat kernel and satisfies the stable-like heat kernel bound SHK(Φ), where Φ is as given in Lemma 5.2 and (5.2).*

Proof. Let $(\check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ on $L^2(\partial U, \mu)$ denote the corresponding Dirichlet form as given in Definition 5.9. We recall from Proposition 5.11 that $(\check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ is of pure-jump type. By Theorem 2.32, the doubling property of μ in Remark 5.15, the exit time lower in Proposition 5.18 and the jump kernel bound in Corollary 5.14, we obtain that the stronlgy continuous contraction semigroup $(Q_t)_{t>0}$ corresponding to the trace Dirichlet form $(\check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ on $L^2(\partial U, \mu)$ admits a continuous heat kernel satisfying the stable-like estimate SHK(Φ).

Next, we identify this continuous heat kernel with the heat kernel of the transition semigroup $(P_t)_{t>0}$ corresponding to the trace process $\check{X}^{\mathrm{ref}}_t$ using an argument similar to Lemma 3.4(a). By the same argument as the proof of (5.62) using [CF, Lemma A.3.7(i) and (4.1.26)] we obtain that the resolvent is absolutely continuous with respect to μ. Hence by [FOT, Theorem 4.2.4.] the transition semigroup (P_t) satisfies the absolute continuity condition (AC) with respect to μ. Due to [FOT, Theorem 6.2.1], we can use [FOT, Proof of Theorem 4.2.8] to obtain

$$P_t(x, dy) = Q_t(x, dy) \quad \text{for any } t \in (0,\infty) \text{ and for q.e. } x \in \partial U.$$

Let f be a bounded continuous function on ∂U. Then for any $s, t > 0$ and any $x \in \partial U$, by $P_t f = Q_t f$ q.e. and (AC) of P_s we obtain

$$P_t(P_s f)(x) = (P_{t+s} f)(x) = P_s(P_t f)(x) = P_s(Q_t f)(x),$$

and letting $s \downarrow 0$ yields

$$(P_t f)(x) = (Q_t f)(x)$$

by dominated convergence theorem, since $(P_s f)(y) \to f(y)$ as $s \downarrow 0$ for any $y \in \partial U$ by the continuity of f, right continuity of sample paths, and $P_s(Q_t f)(x) \to (Q_t f)(x)$ as $s \downarrow 0$ by the continuity of $Q_t f$. The continuity of $Q_t^D f$ can be easily verified using HK(Φ). Thus $P_t(x, dy) = Q_t(x, dy)$ for any $t > 0$ and any $x \in \partial U$. ☐

Remark 5.20. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be a uniform domain satisfying the assumptions of Theorem 5.19. Let $\mathrm{Cap}, \mathrm{Cap}^{\mathrm{ref}}, \mathrm{Cap}^{\mathrm{tr}}$ denote the capacities for the spaces $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, $(\overline{U}, d, m|_{\overline{U}}, \mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$, and $(\partial U, d, \mu, \check{\mathcal{E}}^{\mathrm{ref}}, \check{\mathcal{F}}(U))$ as defined in (3.15) respectively. Using the Poincaré inequality in [CKW, Definition 7.5] for lower bound on capacity across annuli and [CKW, Proposition 2.3(5)] for a matching upper bound we obtain the following estimate: there exist $C, A \in (1, \infty)$ such that for all $\xi \in \partial U, 0 < r < \mathrm{diam}(\partial U, d)/A$, we obtain

$$C^{-1}\frac{\mu(B(\xi, r))}{\Phi(\xi, r)} \leqslant \mathrm{Cap}^{\mathrm{tr}}_{B(\xi,2r)\cap\partial U}(B(\xi, r) \cap \partial U) \leqslant C\frac{\mu(B(\xi, r))}{\Phi(\xi, r)} \tag{5.67}$$

On the other hand, by [GHL15, Theorem 1.2], Theorem 2.12 and [BCM, Lemma 5.24], there exist $C, A \in (1, \infty)$ such that

$$C^{-1}\frac{m(B(\xi, r))}{\Psi(r)} \leqslant \mathrm{Cap}^{\mathrm{ref}}_{B(x,2r)\cap\overline{U}}(B(x, r) \cap \overline{U}) \leqslant C\frac{m(B(x, r))}{\Psi(r)} \tag{5.68}$$

for all $x \in \overline{U}, 0 < r < \mathrm{diam}(U, d)/A$, and

$$C^{-1}\frac{m(B(x, r))}{\Psi(r)} \leqslant \mathrm{Cap}_{B(x,2r)}(B(x, r)) \leqslant C\frac{m(B(x, r))}{\Psi(r)} \tag{5.69}$$

for all $x \in \mathcal{X}, 0 < r < \mathrm{diam}(\mathcal{X}, d)/A$. Combining (5.67), (5.68) (5.69), and (5.4), there exist $A \in (1, \infty)$ such that

$$\mathrm{Cap}_{B(\xi,2r)}(B(\xi, r)) \asymp \mathrm{Cap}^{\mathrm{ref}}_{B(\xi,2r)\cap\overline{U}}(B(\xi, r) \cap \overline{U}) \asymp \mathrm{Cap}^{\mathrm{tr}}_{B(\xi,2r)\cap\partial U}(B(\xi, r) \cap \partial U)$$

for all $\xi \in \partial U, 0 < r < \mathrm{diam}(\partial U, d)/A$.

By Lemma 4.10 and Remark 4.11(a), Theorem 5.19 applies to the reflected Brownian motion on any non-tangentially accessible domain on $\mathbb{R}^n, n \geqslant 2$. Theorem 5.19 also applies to the reflected Brownian motion on the Sierpiński carpet domain formed by removing either the bottom line or the outer square boundary (by [Lie22, Proposition 4.4] and[CQ, Proposition 2.4] and Remark 4.11(b)).

Another related direction of research is the Calderón's inverse problem. In our setting, we can phrase it as follows: Does the Dirichlet form of the boundary trace process determine the Dirichlet form of the underlying reflected diffusion? We refer to [SU] for further context, background, and a solution to this problem for a class of Dirichlet forms in $\mathbb{R}^n$.

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