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Parabolic Harnack Inequality Implies the Existence of Jump Kernel

Guanhua Liu[1,2] · Mathav Murugan[2]

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Abstract
We prove that the parabolic Harnack inequality implies the existence of jump kernel for symmetric pure jump process. This allows us to remove a technical assumption on the jumping measure in the recent characterization of the parabolic Harnack inequality for pure jump processes by Chen, Kumagai and Wang. The key ingredients of our proof are the Lévy system formula and estimates on the heat kernel.

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1 Introduction

The parabolic Harnack inequality is a fundamental regularity estimate for non-negative solutions to the heat equation and its variants. Important applications of the parabolic Harnack inequality are apriori Hölder regularity of solutions, the existence of heat kernel, and bounds on the heat kernel. We refer to the survey [19] for an introduction to Harnack inequalities and variants.

A major result on the parabolic Harnack inequality is its characterization by simpler geometric and analytic properties. In the context of diffusions on Riemannian manifolds this characterization was established by Grigor'yan [15] and Saloff-Coste [21] for the classical space-time scaling (time scales like square of space). This characterization was extended and modified to many settings including diffusions on metric measure spaces [22], nearest neighbor walks on graphs [12], and for anomalous space time scaling [1, 2, 16] by several

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✉ Mathav Murugan
mathav@math.ubc.ca

Guanhua Liu
liu-gh17@mails.tsinghua.edu.cn

[1] Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China

[2] Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

authors. A similar characterization of the parabolic Harnack inequality for jump processes remained open until a recent breakthrough by Chen, Kumagai and Wang [9].

The purpose of this note is to show that the parabolic Harnack inequality implies the existence of the jump kernel for pure jump processes. In other words, we show that the jumping measure is absolutely continuous with respect to the product measure $\mu \otimes \mu$, where μ is the symmetric (reference) measure for the jump process. As a consequence, we remove a technical hypothesis on the jumping measure assumed in [9] for characterizing the parabolic Harnack inequality (see Remark 1.3).

1.1 Framework and Result

Let (M, d) be a complete, locally compact, separable metric space, and let μ be a positive Radon measure on M with full support. Such a triple (M, d, μ) is called a *metric measure space*. We assume that (M, d) is *unbounded*; that is, $\sup_{x,y\in M} d(x, y) = \infty$. We set $B(x, r) := \{y \in X \mid d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$ for $x \in M, r \in (0, \infty)$. We assume that the measure μ satisfies the following *volume doubling* property Eq. VD: there exists $C_D > 1$ such that

$$V(x, 2r) \le C_D V(x, r), \quad \text{for any } x \in M, r \in (0, \infty). \tag{VD}$$

We consider a *symmetric Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$. In other words, $\mathcal{F}$ is a dense linear subspace of $L^2(M, \mu)$, $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is symmetric, non-negative definite, bilinear form that is *closed* ($\mathcal{F}$ is a Hilbert space under the inner product $\mathcal{E}_1(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2(M,\mu)}$) and *Markovian* (for any $f \in \mathcal{F}$, we have $\widehat{f} := (0 \vee f) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\widehat{f}, \widehat{f}) \le \mathcal{E}(f, f)$). We assume that $(\mathcal{E}, \mathcal{F})$ is *regular*; that is, $\mathcal{F} \cap \mathcal{C}_c(X)$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(\mathcal{C}_c(X), \|\cdot\|_{\sup})$. We assume that $(\mathcal{E}, \mathcal{F})$ is a *pure jump* type Dirichlet form; that is, there exists a symmetric positive Radon measure on $M \times M \setminus \text{diag}$ such that

$$\mathcal{E}(f, f) = \int_{M\times M\setminus \text{diag}} (f(x) - f(y))^2 J(dx, dy), \quad \text{for all } f \in \mathcal{F},$$

where $\text{diag} = \{(x, x) \mid x \in M\}$ denotes the diagonal. The Radon measure J is called the *jumping measure*; cf. [14, Theorem 3.2.1]. We say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ *admits a jump kernel* if J is absolutely continuous with respect to the product measure $\mu \otimes \mu$ on $M \times M \setminus \text{diag}$. If the Dirichlet form admits a jump kernel, then the Radon-Nikodym derivative of J with respect to $\mu \otimes \mu$ is called the *jump kernel*. In other words, (if it exists) the jump kernel $j(\cdot, \cdot)$ is a measurable function such that $J(dx, dy) = j(x, y)\mu(dx)\mu(dy)$. The central question of this work whether or not a pure jump type, regular Dirichlet form admits a jump kernel.

Every regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$ has an associated μ-symmetric *Hunt process* $X = \{X_t, t \ge 0, \mathbb{P}^x, x \in M \setminus \mathcal{N}\}$, where $\mathcal{N}$ is a *properly exceptional* set for $(\mathcal{E}, \mathcal{F})$; that is, $\mu(\mathcal{N}) = 0$ and $\mathbb{P}^x(X_t \in \mathcal{N} \text{ for some } t > 0) = 0$. This Hunt process is unique up to the choice of a properly exceptional set [14, Theorems 4.2.8 and 7.2.1]. Let $Z_t = (V_t, X_t)$ be the associated space-time process ($\mathbb{R} \times M$-valued process) defined by $V_t = V_0 - t$. The law of the space time process $s \mapsto Z_s$ starting from (t, x) will be denoted by $\mathbb{P}^{(t,x)}$. The expectation with respect to $\mathbb{P}^{(t,x)}$ is denoted by $\mathbb{E}^{(t,x)}$. We say that $A \subset [0, \infty) \times M$ is *nearly Borel measurable* if for any Borel probability measure μ_0 on $[0, \infty) \times M$, there are Borel measurable subsets A_1, A_2 such that $A_1 \subset A \subset A_2$ and satisfies $\mathbb{P}^{\mu_0}(Z_t \in A_2 \setminus A_1 \text{ for some } t \ge 0) = 0$. The collection of nearly Borel measurable subsets of $[0, \infty) \times M$ forms a σ-field, which is called nearly Borel measurable σ-field. We recall the (probabilistic) definition of the parabolic Harnack inequality.

Definition 1.1 We say that a nearly Borel measurable function $u : [0,\infty) \times M \to \mathbb{R}$ is *caloric* on $D = (a,b) \times B(x_0, r)$ for the Markov process X if there is a property exceptional set $\mathcal{N}_u$ of the Markov process X such that for any relatively compact open subset U of D, we have

$$u(t,x) = \mathbb{E}^{(t,x)} u(Z_{\tau_U}), \quad \text{for all } (t,x) \in U \cap ([0,\infty) \times (M \setminus \mathcal{N}_u)).$$

Let $\phi : [0,\infty) \to [0,\infty)$ be a homeomorphism (and hence strictly increasing with $\phi(0) = 0$). We say that the parabolic Harnack inequality Eq. PHI(ϕ) holds for the process X, if there exist constants $c_0 \in (0,1)$, $0 < C_1 < C_2 < C_3 < C_4$ and $C_5 > 1$ such that for any $x_0 \in M, t_0 \geq 0, r > 0$, provided that $u : \mathbb{R}_+ \times M \to \mathbb{R}_+$ is caloric on $(t_0, t_0 + C_4\phi(r)) \times B(x_0, r)$, we always have

$$\operatorname*{ess\,sup}_{(t_0+C_1\phi(r), t_0+C_2\phi(r)) \times B(x_0, c_0 r)} u \leq C_5 \operatorname*{ess\,inf}_{(t_0+C_3\phi(r), t_0+C_4\phi(r)) \times B(x_0, c_0 r)} u. \tag{PHI(ϕ)}$$

The main result of our work is that the parabolic Harnack inequality implies the existence of jump kernel.

Theorem 1.2 *Let (M, d, μ) be an unbounded, complete, separable, locally compact metric measure space, where μ is a Radon measure with full support on (M, d) that satisfies the volume doubling property* Eq. VD*. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(M, \mu)$ of pure jump type and let X be the corresponding μ-symmetric Hunt process. Let $\phi : [0,\infty) \to [0,\infty)$ be a homeomorphism such that there exist constants $C_\phi \geq 1$, $\beta_2 \geq \beta_1 > 0$ such that*

$$C_\phi^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq C_\phi \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for all } 0 < r \leq R. \tag{1.1}$$

If the process X satisfies the parabolic Harnack inequality Eq. PHI(ϕ)*, then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$ admits a jump kernel.*

Remark 1.3 (a) Let $J(dx, dy)$ denote the jumping measure for $(\mathcal{E}, \mathcal{F})$. Assume that there is a kernel $\widetilde{J}(x, dy)$ (in other words, $x \mapsto \widetilde{J}(x, A)$ is a Borel measurable function for any Borel set A, and that $A \mapsto \widetilde{J}(x, A)$ is a Borel measure on M for any $x \in M$) such that

$$J(dx, dy) = \widetilde{J}(x, dy)\mu(dy). \tag{1.2}$$

Theorem 1.2 was shown under the additional assumption that a kernel $\widetilde{J}(x, dy)$ exists and satisfies Eq. 1.2 in [9, Proposition 3.3] (see also [4, Proposition 4.7] for a similar result and proof). This assumption can be viewed as a weaker form of the existence of jump kernel and was assumed throughout [9]. As a consequence of Theorem 1.2, we could remove the assumption Eq. 1.2 in the characterization of parabolic Harnack inequality in [9].

(b) As explained in [9, Remark 1.22], the condition that the metric space is unbounded can be relaxed. Our proof of Theorem 1.2 also extends to the case where there are non-zero diffusion and jump parts as considered in [11]. We discuss further extensions in Remark 2.6.

In the proof of Theorem 1.2, we consider the same caloric function used in [9, Proposition 3.3] and [4, Proposition 4.7]. However, the argument in [9] requires a Lévy system formula (see [9, Lemma 2.11]) that relies on the assumption Eq. 1.2. To overcome the difficulty, we use a more abstract Lévy system formula that does not rely on Eq. 1.2. The main new ingredient in our proof is the use of a near diagonal lower bound on the heat kernel to obtain useful quantitative estimates on the jumping measure. In particular, we use *both* upper and lower bounds on the heat kernel while the argument in [9, Proposition 3.3] uses only upper bound on the heat kernel. We remark that the use of near diagonal lower bound can be avoided by following the argument in [9, Proposition 3.3] with the abstract Lévy system formula (see the alternate proof of Lemma 2.4).

Notation *Throughout this paper, we use the following notations and conventions.*

(a) *For a measurable function $f \geq 0$ and a measure μ, by $f \cdot \mu$, we denote the measure $A \mapsto \int_A f \, d\mu$.*
(b) *For a measure μ and a function f, the integral $\int f \, d\mu$ is denoted by $\langle \mu, f \rangle$.*
(c) *The notation $A \lesssim B$ for quantities A and B indicates the existence of an implicit constant $C \geq 1$ depending on some inessential parameters such that $A \leq CB$. We write $A \asymp B$, if $A \lesssim B$ and $B \lesssim A$.*

2 Proof

The proof of Theorem 1.2 relies on two key ingredients. The ingredients are bounds on the heat kernel and a Lévy system formula, which we recall in Section 2.1 and Section 2.2 respectively. After these preliminaries, we present the proof of Theorem 1.2 in Section 2.3.

2.1 Heat Kernel

We recall the notion of *heat kernel*. Let (M, d, μ) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(M, \mu)$. Let $(X_t, t \geq 0, \mathbb{P}_x, x \in M \setminus \mathcal{N})$ be the corresponding μ-symmetric Hunt process, where $\mathcal{N}$ is a properly exceptional set for $(\mathcal{E}, \mathcal{F})$. Let $\{P_t\}$ note the corresponding *Markov semigroup* [14, Theorem 1.4.1]. The *heat kernel* associated with the Markov semigroup $\{P_t\}$ (if it exists) is a family of measurable functions $p(t, \cdot, \cdot) : M \times M \mapsto [0, \infty)$ for every $t > 0$, such that

$$P_t f(x) = \int p(t, x, y) f(y) \, \mu(dy), \quad \text{for all } f \in L^2(M, \mu), t > 0 \text{ and } x \in M, \tag{2.1}$$

$$p(t, x, y) = p(t, y, x), \quad \text{for all } x, y \in M \text{ and } t > 0, \tag{2.2}$$

$$p(t + s, x, y) = \int p(s, x, y) p(t, y, z) \, \mu(dy), \quad \text{for all } t, s > 0 \text{ and } x, y \in M. \tag{2.3}$$

For an open set B, let X^B denote the μ-symmetric Hunt process on B obtained from X killed upon exiting B [8, Theorems 3.3.8 and 3.3.9]; that is,

$$X^B(t) = \begin{cases} X(t) & \text{for } t < \tau_B, \\ \Delta & \text{for } t \geq \tau_B \end{cases},$$

where Δ denotes the cemetery state and $\tau_B = \inf\{s > 0 \mid X_t \notin B\}$ denote the exit time of B. Let $\left\{P_t^B\right\}$ denote the Markov semigroup on $L^2(B, \mu)$. The *heat kernel* associated with the Markov semigroup $\left\{P_t^B\right\}$ (if it exists) is denoted by $p^B(t, \cdot, \cdot)$. We recall the existence and bounds on the heat kernels for X and X^B from [9].

Proposition 2.1 [9, Propositions 3.1 and 3.2] *Let (M, d, μ) be an unbounded, complete, separable, locally compact metric measure space, where μ is a Radon measure with full support on (M, d) that satisfies the volume doubling property* Eq. VD. *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(M, \mu)$ of pure jump type and let X be the corresponding μ-symmetric Hunt process. Let $\phi : [0, \infty) \to [0, \infty)$ be a homeomorphism such that there exist constants $C_\phi \geq 1$, $\beta_2 \geq \beta_1 > 0$ satisfying* Eq. 1.1. *Assume further that X satisfies the parabolic Harnack inequality* Eq. PHI(ϕ). *Then*

(a) The process X has a continuous heat kernel $p : (0, \infty) \times M \times M \to [0, \infty)$ that satisfies the following upper bound. There exist a constant $C_U > 0$ and a properly exceptional set $\mathcal{N}$ for X such that,

$$p_t(x, y) \leq \frac{C_U}{V(x, \phi^{-1}(t))}, \quad \text{for all } x, y \in M \setminus \mathcal{N} \text{ and for all } t > 0, \quad (\mathrm{UHKD}(\phi))$$

(b) For every ball $B = B(x_0, r)$, let X^B denote the process obtained from X killed upon exiting B. Then X^B has a heat kernel $p^B : (0, \infty) \times B \times B \to [0, \infty)$ and satisfies the following lower bound: there exist $c_L > 0$, $\delta_N \in (0, 1)$ and a properly exceptional set $\mathcal{N}$ for X such that for any $x_0 \in M$, $r > 0$, $0 < t \leq \phi(\delta_N r)$ and $B = B(x_0, r)$,

$$p_t^{B(x_0,r)}(x, y) \geq \frac{c_L}{V(x_0, \phi^{-1}(t))}, \quad \text{for all } x, y \in B(x_0, \delta_N \phi^{-1}(t)) \setminus \mathcal{N}. \quad (\mathrm{NDL}(\phi))$$

Remark 2.2 (a) We remark that the proofs of [9, Propositions 3.1 and 3.2] do not rely on the assumption Eq. 1.2 or the reverse volume doubling property.

(b) Using a standard parabolic Hölder regularity estimate (see [5, Corollary 4.5 and Lemma 4.6]), we may assume that $(t, x, y) \mapsto p_t(x, y)$ is continuous on $(0, \infty) \times M \times M$. Similarly, for any ball $B(x_0, r)$, we may assume that $(t, x, y) \mapsto p_t^{B(x_0,r)}(x, y)$ is continuous in $(0, \infty) \times B(x_0, r) \times B(x_0, r)$. In particular, we may assume that the exceptional set $\mathcal{N}$ in the estimates Eqs. UHKD(ϕ) and NDL(ϕ) is the empty set.

2.2 Lévy System Formula

In this section, we collect some useful facts on the Lévy system formula and positive continuous additive functionals.

Consider a μ-symmetric *Hunt process* $X = \{\Omega, \mathcal{M}, X_t, t \geq 0, \mathbb{P}^x\}$, where $\mathcal{N}$ is a properly exceptional set for $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ and $(\Omega, \mathcal{M}, \mathbb{P}^x)$. For any measure ν on M, we denote by $\mathbb{P}^\nu$ the measure $\mathbb{P}^\nu(A) = \int_M \mathbb{P}^x(A)\, d\nu(x)$. Any function f on M is extended to $M_\partial := M \cup \{\Delta\}$ by setting $f(\Delta) = 0$, where Δ denotes the cemetery state. The set M_∂ as a topological space is the one point compactification of M. Let $(\mathcal{M}_t)_{0 \leq t \leq \infty}$ denote the minimum augmented admissible filtration on Ω.

A collection of random variables $A := \{A_s : \Omega \to \mathbb{R}_+ | s \in \mathbb{R}_+\}$, is called a *positive continuous additive functional* (for short, a PCAF), if it satisfies the following conditions:

(i) $A_t(\cdot)$ is $(\mathcal{M}_t)$-measurable,
(ii) there exist a set $\Lambda \in \mathcal{M}_\infty$ and an exceptional set $\mathcal{N} \subset M$ for X such that $\mathbb{P}^x(\Lambda) = 1$ for all $x \in M \setminus \mathcal{N}$ and $\theta_t \Lambda \subset \Lambda$ for all $t > 0$, where θ_t denotes the shift map on Ω.
(iii) For any $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is continuous, non-negative with $A_0(\omega) = 0$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geq \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for any $s, t \geq 0$. Here $\zeta(\cdot)$ denotes the life time of the process.

The sets Λ and $\mathcal{N}$ are referred to as a *defining set* and *exceptional set* of the PCAF A_t respectively. If $\mathcal{N}$ can be taken to the empty set, then we say that A_t is a PCAF in the *strict sense*.

A measure ν is called the *Revuz measure* of the PCAF A, if and only if for any non-negative Borel functions h and f,

$$\mathbb{E}^{h\cdot\mu}\left(\int_0^t f(X_s(\omega))\,dA_s(\omega)\right) = \int_0^t \langle f\cdot\nu, P_s h\rangle\,ds, \tag{2.4}$$

where P_s denotes the Markov semigroup corresponding to the Hunt process. By [14, Theorem 5.1.4], the Revuz measure ν is uniquely determined by A and does not charge any set of zero capacity. In particular,

$$\nu(\mathcal{N}) = 0, \quad \text{for any properly exceptional set } \mathcal{N}. \tag{2.5}$$

Every Hunt process has a *Lévy system* (N, H) [8, Appendix A.3.4]. Recall that a pair (N, H) is a Lévy sytem for the Hunt process X if $N(x, dy)$ is a kernel on M_∂ equipped with the Borel σ-field and H is a PCAF in the strict sense satisfying the following property: for any non-negative Borel function $F : M_\partial \times M_\partial \to [0,\infty)$ such that $F(x,x) = 0$ for all $x \in M_\partial$, we have

$$\mathbb{E}^z\left[\sum_{s\le t} F(X_{s-}, X_s)\right] = \mathbb{E}^z\left[\int_0^t \int_M F(X_s, y)N(X_s, dy)dH_s\right]. \tag{2.6}$$

The property in Eq. 2.6 called the *Lévy system formula* and admits the following generalization. By [8, (A.3.33)], for any non-negative Borel function g on $(0,\infty)$, any $z \in M_\partial$, any $(\mathcal{M}_t)$-stopping time T, and any non-negative Borel function $F : M_\partial \times M_\partial \to [0,\infty)$ such that $F(x,x) = 0$ for all $x \in M_\partial$, we have

$$\mathbb{E}^z\left[\sum_{0<s\le T} g(s)F(X_{s-}, X_s)\right] = \mathbb{E}^z\left[\int_0^T g(s)\int_M F(X_s, y)N(X_s, dy)dH_s\right]. \tag{2.7}$$

By [14, (5.3.6) and Theorem 5.3.1], we know if ν is the Revuz measure of H, then

$$J(dx, dy) = \frac{1}{2}N(x, dy)\nu(dx). \tag{2.8}$$

Lemma 2.3 [8, Proposition 4.1.10] *Let H be a PCAF for the process (X_t) and let ν be the corresponding Revuz measure. For any open set D the process $(H_{t\wedge\tau_D})$ is a PCAF for the process X^D killed upon exiting D and its Revuz measure is $\nu_D(\cdot) = \nu(D\cap\cdot)$, where $\tau_D = \inf\{t > 0 : X_t \notin D\}$. In particular, we have*

$$\mathbb{E}^{h\cdot\mu}\left(\int_0^{\tau_D\wedge t} f(X_s)\,dH_s\right) = \int_0^t \langle f\cdot\nu_D, P_s^D h\rangle\,ds, \tag{2.9}$$

for all non-negative measurable functions $f, h : D \to [0,\infty)$, where P_s^D denotes the Markov semigroup corresponding to the X^D.

2.3 Existence of Jump Kernel

The following estimate on the jumping measure plays a crucial role in the proof of Theorem 1.2. This estimate can be viewed as an integrated version of the condition (UJS) considered in [9, Definition 1.18].

Lemma 2.4 *Under the assumptions of Theorem 1.2, there exist* $\delta \in (0,1), C_J > 0$ *such that for any pair of balls* $B_i = B(x_i, r_i), i = 1, 2$ *with* $d(x_1, x_2) > r_1 + r_2$ *and for any ball* $B' = B(x', r') \subset B(x_1, \delta r_1)$ *such that* $r' \le \delta r_1$*, we have the estimate*

$$J(B' \times B_2) \le C_J \frac{\mu(B')}{\mu(B_1)} J(B_1 \times B_2).$$

Proof Let $c_0 \in (0,1), C_1, C_2, C_3, C_4 > 0, C_5 > 1$ denote the constants in Eq. PHI(ϕ). Let $B_i = B(x_i, r_i), i = 1, 2$ be balls such that $r_1 + r_2 < d(x_1, x_2)$. Set $f_h(t, z) = \mathbb{1}_{(C\phi(r)-h, C\phi(r))}(t)\mathbb{1}_{B_2}(z)$, where $C = (C_1 + C_2)/2$ and $h \in (0, C\phi(r))$. Then

$$u_h(t,x) = \begin{cases} \mathbb{E}^x[f_h\left(t - \tau_{B_1}, X_{\tau_{B_1}}\right); \tau_{B_1} \le t] & \text{if } x \in M \setminus \mathcal{N}, t > 0 \\ 0 & \text{if } x \in \mathcal{N}, t > 0, \end{cases} \tag{2.10}$$

is caloric in $(0, \infty) \times B_1$, where $\mathcal{N}$ is an exceptional set for the corresponding Hunt process X and $\tau_{B_1} = \inf\{t > 0 \mid X_t \notin B_1\}$ denote the exit time from B_1. By Remark 2.2, we may assume that the heat kernel corresponding to the process killed upon exiting B_1 given by $(t, x, y) \mapsto p_t^{B_1}(x, y)$ is continuous in $(0, \infty) \times B_1 \times B_1$.

We choose a Lévy system (N, H) for the process X. Let ν denote the Revuz measure of H, where H is a PCAF in the strict sense. Set $g(x) = N(x, B_2)$.

For any $t > C\phi(r)$, for quasi-every $x \in B_1$, for any $h \in (0, C\phi(r))$, and for any $s_1 \in (0, t - C\phi(r)))$, we have

$$\begin{aligned}
u_h(t,x) &= \mathbb{E}^x\left[\sum_{s \le \tau_{B_1}} \mathbb{1}_{(t-C\phi(r), t-C\phi(r)+h)}(s)\mathbb{1}_{B_2}(y)\right] \\
&= \mathbb{E}^x\left[\int_0^{\tau_{B_1}} \int_M \mathbb{1}_{(t-C\phi(r), t-C\phi(r)+h)}(s)\mathbb{1}_{B_2}(y)\, N(X_s, dy)\, dH_s\right] \quad \text{(by Eq. 2.7)} \\
&= \mathbb{E}^x\left[\int_{(t-C\phi(r))\wedge\tau_{B_1}}^{(t-C\phi(r)+h)\wedge\tau_{B_1}} N(X_s, B_2)\, dH_s\right] = \mathbb{E}^x\left[\int_{(t-C\phi(r))\wedge\tau_{B_1}}^{(t-C\phi(r)+h)\wedge\tau_{B_1}} g(X_s)\, dH_s\right] \\
&= \mathbb{E}^{p_{s_1}^{B_1}(x,\cdot)\cdot\mu}\left[\int_{(t-C\phi(r)-s_1)\wedge\tau_{B_1}}^{(t-C\phi(r)+h-s_1)\wedge\tau_{B_1}} g(X_s)\, dH_s\right] \quad \text{(by Markov property)} \\
&= \int_{t-C\phi(r)-s_1}^{t-C\phi(r)+h-s_1} \langle g \cdot \nu_{B_1}, P_s^{B_1} p_{s_1}^{B_1}(x, \cdot)\rangle\, ds \quad \text{(by Eq. 2.9)} \\
&= \int_{t-C\phi(r)-s_1}^{t-C\phi(r)+h-s_1} \langle g \cdot \nu_{B_1}, p_{s+s_1}^{B_1}(x, \cdot)\rangle\, ds \quad \text{(by Eq. 2.1 and Eq. 2.3)} \\
&= \int_{t-C\phi(r)}^{t-C\phi(r)+h} \int_{B_1} p_s^{B_1}(x, w) N(w, B_2)\, \nu(dw)\, ds \quad \text{(since } g(\cdot) = N(\cdot, B_2)) \\
&= 2\int_{t-C\phi(r)}^{t-C\phi(r)+h} \int_{B_1} p_s^{B_1}(x, w) J(dw, B_2)\, ds \quad \text{(by Eq. 2.8).}
\end{aligned} \tag{2.11}$$

By Proposition 2.1 and Remark 2.2(b), we have that

$$\widetilde{u}_h(t,x) := 2\int_{t-C\phi(r)}^{t-C\phi(r)+h} \int_{B_1} p_s^{B_1}(x, w) J(dw, B_2)\, ds \quad \text{is continuous in } (C\phi(r), \infty) \times B_1. \tag{2.12}$$

By Eq. 1.1, we choose $A > 1$ and $\kappa \in (0, 1)$ such that

$$\phi(Ar) > 2\phi(r), \text{ and } \phi(\kappa r) < (C_2 - C_1)\phi(r)/2 \quad \text{for all } r > 0. \tag{2.13}$$

Let $\delta_N \in (0, 1)$ denote the constant in Eq. NDL(ϕ). For any ball $B' = B(x', r') \subset B_1$ such that $B\left(x', A\delta_N^{-1} r'\right) \subset B_1$, we have

$$p_s^{B_1}(x', w) \geq p_s^{B\left(x', A\delta_N^{-1} r'\right)}(x', w) \gtrsim \frac{1}{V(x', r')}, \quad \text{for all } s \in [\phi(r'), 2\phi(r')] \text{ and } w \in B(x', r'). \tag{2.14}$$

We use Eqs. NDL(ϕ), 1.1 and VD to obtain the above estimate. Set

$$\delta := \min\left(c_0, \kappa, \left(A\delta_N^{-1} + 1\right)^{-1}\right). \tag{2.15}$$

The constant $\delta \in (0, 1)$ is chosen so that for any ball $B' = B(x', r') \subset B(x_1, \delta r_1)$ with $r' \leq \delta r_1$, we have

$$(C\phi(r) + \phi(r'), x') \in (C_1\phi(r), C_2\phi(r)) \times B(x_1, c_0 r_1), \text{ and } B\left(x', A\delta_N^{-1} r'\right) \subset B(x_1, r_1). \tag{2.16}$$

For any ball $B(x', r') \subset B(x_1, \delta r_1)$ with $r' \leq \delta r_1$, we have

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x_1, c_0 r_1) \times (C_1\phi(r), C_2\phi(r))} u_{\phi(r')}(t, x) &\geq \widetilde{u}_{\phi(r')}\left(C\phi(r) + \phi(r'), x'\right) \quad \text{(by Eq. 2.11, Eq. 2.12, and Eq. 2.16)} \\ &= 2\int_{\phi(r')}^{2\phi(r')} \int_{B_1} p_s^{B_1}(x, w)\, J(dw, B_2)\, ds \quad \text{(by Eq. 2.12)} \\ &\gtrsim \phi(r') \frac{J(B(x', r') \times B_2)}{\mu(B(x', r'))} \quad \text{(by Eq. 2.14 and Eq. 2.16).} \end{aligned} \tag{2.17}$$

Set $C' = (C_3 + C_4)/2$. For any $r' \leq \delta r_1$, we have

$$\begin{aligned} \operatorname*{ess\,inf}_{B(x_1, c_0 r_1) \times (C_1\phi(r), C_2\phi(r))} u_{\phi(r')}(t, x) &\leq \widetilde{u}_{\phi(r')}\left(C'\phi(r), x_1\right) \quad \text{(by Eq. 2.11 and Eq. 2.12)} \\ &\leq \int_{(C'-C)\phi(r)}^{(C'-C)\phi(r)+\phi(r')} \int_{B_1} p_s(x, w)\, J(dw, B_2)\, ds \quad (p^{B_1} \leq p) \\ &\lesssim \phi(r') \frac{J(B_1 \times B_2)}{\mu(B_1)} \quad \text{(by Eqs. UHKD}(\phi)\text{, VD, and 1.1).} \end{aligned} \tag{2.18}$$

The conclusion follows from Eqs. 2.17, 2.18 and PHI(ϕ).

Alternate proof of Lemma 2.4. We sketch an alternate argument due to Z.-Q. Chen [7]. This alternate proof follows from modifying the argument in [9, Proof of Proposition 3.3] by replacing the use of the Lévy system formula with a more abstract one $J(dx, dy) = N(x, dy)\nu(dx)$ as done in this work. The advantage of the alternate argument is that it avoids the use of the near diagonal lower bound Eq. NDL(ϕ) directly.

By Eq. 2.9, we obtain that

$$\lim_{h \downarrow 0} \mathbb{E}^{\mu}\left[\frac{1}{h}\int_0^{\tau_D \wedge h} f(X_s)\, dH_s\right] = \int_D f(x)\nu(dx).$$

We use this expression in the proof of Proposition 3.3 in [9], where $f(x) = N(x, B_2)$. By dividing [9, (3.13)] by h on both sides, and letting $h \to 0$ we can obtain the conclusion of Lemma 2.4 (this is eight lines below equation (3.13) in [11] but using a less abstract

Lévy system formula). The notation $\varepsilon_1, \varepsilon$ in [9] corresponds to r' and r_2 respectively in our notation.

Proof of Theorem 1.2. Assume to the contrary that J is not absolutely continuous with respect to $\mu \otimes \mu$ on $(M \times M) \setminus d_M$, where $d_M = \{(x, x) : x \in M\}$ denotes the diagonal in M.

Let ρ be the metric on $M \times M$ defined by $\rho((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), d(y_1, y_2))$. It is easy to verify that the product measure $\mu \otimes \mu$ satisfies the volume doubling property Eq. VD on the product space $(M \times M, \rho)$. For $(x_1, x_2) \in M \times M$, let $B_\rho((x_1, x_2), r)$ denote the open ball of radius r in the metric ρ centered at (x_1, x_2).

By the inner regularity of J, there exists a compact subset K of $(M \times M) \setminus d_M$ such that $J(K) > 0$ and $\mu \otimes \mu(K) = 0$. Let $\delta > 0$ be the constant in the statement of Lemma 2.4. By the compactness of K, we can cover K with finitely many sets of the form $\{B(x, \delta d(x, y/4)) \times B(y, \delta d(x, y)/4) : (x, y) \in K\}$. Therefore, there exists $(x, y) \in K$ such that $\widetilde{K} = K \cap [B(x, \delta d(x, y/4)) \times B(y, \delta d(x, y)/4)]$ satisfies $J(\widetilde{K}) > 0$ and $(\mu \otimes \mu)(\widetilde{K}) = 0$.

By the regularity of $\mu \otimes \mu$, for any $\epsilon > 0$, there exists an open set $K_\epsilon \subset B(x, \delta d(x, y/4)) \times B(y, \delta d(x, y)/4)$, $\widetilde{K} \subset K_\epsilon$ such that $\mu \otimes \mu(K_\epsilon) < \epsilon$. By the 5B-covering lemma [18, Theorem 1.2], there exists balls $B_\rho((x_i, y_i), \rho_i) \subset K_\epsilon, i \in I$ such that $\rho_i \le \delta d(x_i, y_i)/4$ for all $i \in I$, $\bigcup_{i \in I} B_\rho((x_i, y_i), \rho_i) = K_\epsilon$ and $B_\rho((x_i, y_i), \rho_i)/5), i \in I$ are pairwise disjoint. Hence, we have

$$\begin{aligned}
J(\widetilde{K}) &\le J(K_\varepsilon) \le \sum_{i \in I} J(B_\rho((x_i, y_i), \rho_i)) = \sum_{i \in I} J\left(B((x_i, \rho_i) \times B(y_i, \rho_i))\right) \\
&\lesssim \sum_{i \in I} \frac{\mu(B((x_i, \rho_i))}{\mu(B(x, d(x, y)/4)} J\left(B(x, d(x, y)/4) \times B(y_i, \rho_i)\right) \quad \text{(by Lemma 2.4)} \\
&\lesssim \sum_{i \in I} \frac{\mu(B((x_i, \rho_i))\mu(B((y_i, \rho_i))}{\mu(B(x, d(x, y)/4)\mu(B(y, d(x, y)/4))} J\left(B(x, d(x, y)/4) \times B(y, d(x, y)/4)\right) \\
&\qquad \text{(by Lemma 2.4 and symmetry of } J) \\
&\lesssim \frac{J\left(B(x, d(x, y)/4) \times B(y, d(x, y)/4)\right)}{\mu(B(x, d(x, y)/4)\mu(B(y, d(x, y)/4))} \sum_{i \in I} (\mu \otimes \mu)(B_\rho((x_i, y_i), \rho_i/5)) \quad \text{(by Eq. VD)} \\
&\lesssim \frac{J\left(B(x, d(x, y)/4) \times B(y, d(x, y)/4)\right)}{\mu(B(x, d(x, y)/4)\mu(B(y, d(x, y)/4)} (\mu \otimes \mu)(K_\epsilon) \\
&\qquad \text{(since } B_\rho((x_i, y_i), \rho_i/5), i \in I \text{ are pairwise disjoint and } \textstyle\bigcup_{i \in I} B_\rho((x_i, y_i), \rho_i) = K_\epsilon) \\
&\lesssim \epsilon \frac{J\left(B(x, d(x, y)/4) \times B(y, d(x, y)/4)\right)}{\mu(B(x, d(x, y)/4)\mu(B(y, d(x, y)/4)} \quad \text{(since } (\mu \otimes \mu)(K_\epsilon) < \varepsilon).
\end{aligned}$$

By letting $\epsilon \downarrow 0$, we obtain $J(\widetilde{K}) = 0$, a contradiction.

Remark 2.5 Consider the μ-symmetric process on $\mathbb{R}^n$ whose generator is $\mathcal{L} f = \sum_{i=1}^n (-\partial_{ii} f)^{\alpha/2}$ where $\alpha \in (0, 2)$. Let d be the Euclidean metric and μ denote the Lebesgue measure. In this case the jumping measure is singular with respect to the product $\mu \times \mu$. It is known that such a jump process fails the elliptic Harnack inequality and parabolic Harnack inequality [6, Section 3] for any $n \ge 2$ and $\alpha \in (0, 2)$. More generally, one could consider jump processes whose jumping measure is comparable to that of this example. Our main result provides a new proof that parabolic Harnack inequality does not hold in such examples. We refer to [20] for sharp heat kernel bounds of such jump processes with singular jumping measure. For $n = 2$ and $\alpha \in [1, 2)$ such examples satisfy the heat kernel upper bound Eq. UHKD(ϕ) and near diagonal lower bound Eq. NDL(ϕ) but fail to satisfy Eq. PHI(ϕ). Therefore, we can not weaken the hypothesis in Theorem 1.2 by replacing Eq. PHI(ϕ) with heat kernel bounds Eqs. UHKD(ϕ) and NDL(ϕ).

In this example the upper bound Eq. UHKD(ϕ) directly follows from [20, Theorem 1.1] with $\phi(r) = r^\alpha$. Since the near diagonal lower bound is not contained in [20], we sketch the argument below.

For the remainder of the argument we assume that $n = 2$ and $\alpha \in [1, 2)$. Let $B = B(x_0, r)$ be a ball in X with $x_0 \in X, r > 0$ and let τ denote the exit time from B. Let $\delta_N \in (0, 1/2)$ be a constant whose value will be determined later in the argument. Let $p_t(\cdot,\cdot)$, $p_t^B(\cdot,\cdot)$ denote the heat kernel of the process and the process killed upon exiting B respectively. We recall that the heat kernel $(t, x, y) \mapsto p_t(x, y)$ exists and is continuous [23]. Following [17, Lemma 3.7], we use the Dynkin-Hunt formula

$$p_t^B(x, y) = p_t(x, y) - \mathbb{E}^x[p_t(X_\tau, y)1_{\{\tau \le t\}}] \ge p_t(x, y) - \sup_{0<s\le t} \sup_{z\in B^c} p_s(z, y). \tag{2.19}$$

By [20, Theorem 1.1], there exists $C > 0$ such that

$$C^{-1}t^{-n/\alpha}\Pi_{i=1}^n \left(1 \wedge \frac{t^{1/\alpha}}{|x_i - y_i|}\right)^{1+\alpha} \le p_t(x, y) \le Ct^{-n/\alpha}\Pi_{i=1}^n \left(1 \wedge \frac{t^{1/\alpha}}{|x_i - y_i|}\right)^{1+\alpha}, \tag{2.20}$$

for all $t > 0, x, y \in \mathbb{R}^n$. If $x, y \in B(x_0, \delta_N t^{1/\alpha})$, then $d(x, y) \le 2\delta_N t^{1/\alpha}$ and therefore $|x_i - y_i| \le 2\delta_N t^{1/\alpha} < t^{1/\alpha}$ for each $i = 1, 2, \ldots, n$. Hence by Eq. 2.20, we have the lower bound

$$p_t(x, y) \ge C^{-1}t^{-n/\alpha} \quad \text{for all } x, y \in B\left(x_0, \delta_N t^{1/\alpha}\right). \tag{2.21}$$

On the other hand, for all $z \in B^c$, $y \in B(x_0, \delta_N r)$, $0 < t < \phi(\delta_N r) = (\delta_N r)^\alpha$, we have

$$d(y, z) \ge d(x_0, z) - d(x_0, y) \ge r - \delta_N t^{1/\alpha} \ge \left(\delta_N^{-1} - \delta_N\right) t^{1/\alpha}.$$

Hence for any such y, z as above, there exists $i \in \{1, \ldots, n\}$ such that

$$|y_i - z_i| \ge 2^{-1/2}\left(\delta_N^{-1} - \delta_N\right) t^{1/\alpha}.$$

We choose $\delta_N \in (0, 1/2)$ small enough so that $2^{-1/2}\left(\delta_N^{-1} - \delta_N\right) \le 1$. Therefore for any $0 < s < t, z \in B^c, y \in B(x_0, \delta_N r), 0 < t < \phi(\delta_N r) = (\delta_N r)^\alpha$ we have the upper bound

$$\begin{aligned} p_s(y, z) &\le Cs^{-n/\alpha}\Pi_{i=1}^n \left(1 \wedge \frac{s^{1/\alpha}}{|y_i - z_i|}\right)^{1+\alpha} \le Cs^{-n/\alpha}\left(1 \wedge \frac{s^{1/\alpha}}{2^{-1/2}\left(\delta_N^{-1} - \delta_N\right) t^{1/\alpha}}\right)^{1+\alpha} \\ &\le Cs^{-n/\alpha}\left(\frac{s^{1/\alpha}}{2^{-1/2}\left(\delta_N^{-1} - \delta_N\right) t^{1/\alpha}}\right)^{1+\alpha} \\ &\le Ct^{-n/\alpha}\left(2^{-1/2}\left(\delta_N^{-1} - \delta_N\right) t^{1/\alpha}\right)^{-1-\alpha}. \end{aligned} \tag{2.22}$$

In the last line above, we used the fact that $1 + \alpha - n \ge 0$ and $s \le t$. Combining Eqs. 2.19, 2.21 and 2.22, we obtain

$$p_t^{B(x_0,r)}(x, y) \ge \left(C^{-1} - C\left(2^{-1/2}(\delta_N^{-1} - \delta_N)t^{1/\alpha}\right)^{-1-\alpha}\right) t^{-n/\alpha}. \tag{2.23}$$

By choosing δ_N small enough so that

$$C^{-1} - C\left(2^{-1/2}\left(\delta_N^{-1} - \delta_N\right) t^{1/\alpha}\right)^{-1-\alpha} \ge (2C)^{-1},$$

Equation 2.22 implies Eq. NDL(ϕ).

Remark 2.6 We discuss further extensions of Theorem 1.2.

(a) Theorem 1.2 generalizes under the weaker assumption that the parabolic Harnack inequality Eq. PHI(ϕ) still holds true under small enough scales $r \le r_0$ for some $r_0 > 0$. We outline the modifications needed in the proof of Theorem 1.2 under this assumption. In this case Eqs. UHKD(ϕ), NDL(ϕ), and Lemma 2.4 hold only for small enough scales $r < cr_0$ for some $c, r_0 > 0$ and small enough times $t < c\phi(r_0)$. In the proof of Theorem 1.2, we cover the compact set K with sets of the form $B_\rho((x_i, y_i), \rho_i)/5)$ with $\rho_i \le \delta(\min(cr_0, d(x_i, y_i))/4$ instead of $\rho_i \le \delta d(x_i, y_i)/4$. After these changes, the same argument yields the existence of jump kernel.

In a similar vein, we could consider a version of the parabolic Harnack inequality where the constant C_5 in Definition 1.1 depends on $r > 0$, say $C_5(r)$. Such a non-scale invariant version of parabolic Harnack inequality was considered in [3] for certain jump processes on $\mathbb{R}^n$. In this case, the argument of Lemma 2.4 gives a constant C_J that depends on r', r_1, r_2. In the proof of Theorem 1.2, the compactness of K implies that

$$0 < \inf_{(x,y)\in K} d(x, y) \le \sup_{(x,y)\in K} d(x, y) < \infty.$$

By making the sets slightly smaller if necessary, we may also assume the above property with K replaced by K_ε uniformly for all $\varepsilon > 0$. That is, there exist $0 < r_1 < r_2 < \infty$ such that

$$0 < r_1 < \inf_{(x,y)\in K_\varepsilon} d(x, y) \le \sup_{(x,y)\in K_\varepsilon} d(x, y) < r_2 < \infty, \quad \text{for all } \varepsilon > 0.$$

If we further assume that

$$\sup_{r_1<r<r_2} C_5(r) < \infty \quad \text{for any } 0 < r_1 < r_2 < \infty,$$

then the proof of Theorem 1.2 also extends to this non-scale invariant generalization.

(b) It is an interesting direction to further relax the assumption of parabolic Harnack inequality to obtain the existence of jump kernel. For example, a weaker variant of the parabolic Harnack inequality is studied in [13]. The authors do not know if this weaker version of parabolic Harnack inequality suffices to imply the existence of jump kernel or other similar properties. The weak Harnack inequality in [13] is closed related to L^p mean value inequalities considered in [10] and is sufficient to imply parabolic Hölder regularity.

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