

Stability of the elliptic Harnack inequality

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Abstract

We prove that the elliptic Harnack inequality (on a manifold, graph, or suitably regular metric measure space) is stable under bounded perturbations, as well as rough isometries.

1. Introduction

A well-known theorem of Moser [Mos61] is that an elliptic Harnack inequality (EHI) holds for solutions associated with uniformly elliptic divergence form PDE. Let \mathcal{A} be given by

$$(1.1) \quad \mathcal{A}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial f}{\partial x_j} \right),$$

where $(a_{ij}(x), x \in \mathbb{R}^d)$ is bounded, measurable and uniformly elliptic. Let h be a non-negative \mathcal{A} -harmonic function in a domain $B(x, 2R)$, and let $B = B(x, R) \subset B(x, 2R)$. Moser's theorem states that there exists a constant C_H , depending only on d and the ellipticity constant of $a_{..}(\cdot)$, such that

$$(1.2) \quad \operatorname{ess\,sup}_{B(x,R)} h \leq C_H \operatorname{ess\,inf}_{B(x,R)} h.$$

A few years later Moser [Mos64], [Mos71] extended this to obtain a parabolic Harnack inequality (PHI) for solutions $u = u(t, x)$ to the heat equation associated with \mathcal{A} :

$$(1.3) \quad \frac{\partial u}{\partial t} = \mathcal{A}u.$$

This states that if u is a non-negative solution to (1.3) in a space-time cylinder $Q = (0, T) \times B(x, 2R)$, where $R = T^2$, then writing $Q_- = (T/4, T/2) \times B(x, R)$,

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$$Q_+ = (3T/4, T) \times B(x, R),$$

$$(1.4) \quad \operatorname{ess\,sup}_{Q_-} u \leq C_P \operatorname{ess\,inf}_{Q_+} u.$$

If h is harmonic, then $u(t, x) = h(x)$ is a solution to (1.3), so the PHI implies the EHI. The methods of Moser are very robust and have been extended to manifolds, metric measure spaces, and graphs; see [BG72], [SC92a], [Stu96], [Del99], [MSC15].

The EHI and PHI have numerous applications and, in particular, give a priori regularity for solutions to (1.3). It is well known that Harnack inequality is useful beyond the linear elliptic and parabolic equations mentioned above. For instance variants of Harnack inequality apply to non-local operators, non-linear equations and geometric evolution equations including the Ricci flow and mean curvature flow; see the survey [Kas07].

S.T. Yau and his collaborators [Yau75], [CY75], [LY86] developed a completely different approach to Harnack inequalities based on gradient estimates. [Yau75] proves the Liouville property for Riemannian manifolds with non-negative Ricci curvature using gradient estimates for positive harmonic functions. A local version of these gradient estimates was given by Cheng and Yau in [CY75]. Let (M, g) be a Riemannian manifold whose Ricci curvature is bounded below by $-K$ for some $K \geq 0$. Fix $\delta \in (0, 1)$. Then there exists $C > 0$, depending only on δ and $\dim(M)$, such that any positive solution u of the Laplace equation $\Delta u = 0$ in $B(x, 2r) \subset M$ satisfies

$$|\nabla \ln(u)| \leq C(r^{-1} + \sqrt{K}) \quad \text{in } B(x, 2\delta r).$$

Integrating this estimate along geodesics immediately yields a local version of the EHI. In particular, any u above satisfies

$$u(z)/u(y) \leq \exp(C(1 + \sqrt{K}r)), \quad z, y \in B(x, 2\delta r).$$

For the case of manifolds with non-negative Ricci curvature, we have $K = 0$, and so we obtain the EHI. This gradient estimate was extended to the parabolic setting by Li and Yau [LY86]. See [SC95, p. 435] for a comparison between the gradient estimates of [Yau75], [CY75], [LY86] and the Harnack inequalities of Moser [Mos61], [Mos64].

A major advance in understanding the PHI was made in 1992 by Grigor'yan and Saloff-Coste [Gri91], [SC92a], who proved that the PHI is equivalent to two conditions: volume doubling (VD) and a family of Poincaré inequalities (PI). The context of [Gri91], [SC92a] is the Laplace-Beltrami operator on Riemannian manifolds, but the basic equivalence $\text{VD} + \text{PI} \Leftrightarrow \text{PHI}$ also holds for graphs and metric measure spaces with a Dirichlet form; see [Del99], [Stu96]. This characterization of the PHI implies that it is stable with respect to rough

isometries; see [CSC95, Th. 8.3]. For more details and a survey of the literature, see the introduction of [SC95].

One consequence of the EHI is the Liouville property — that all bounded harmonic functions are constant. However, the Liouville property is not stable under rough isometries; see [Lyo87]. See [SC04, §5] for a survey of related results and open questions.

Using the gradient estimate in [CY75, Prop. 6], Grigor'yan [Gri91, p. 340] remarks that there exists a two dimensional Riemannian manifold that satisfies the EHI but does not satisfy the PHI. In the late 1990s further examples inspired by analysis on fractals were given; see [BB99]. The essential idea behind the example in [BB99] is that if a space is roughly isometric to an infinite Sierpinski carpet, then a PHI holds, but with anomalous space-time scaling given by $R = T^\beta \sqrt{T^2}$, where $\beta > 2$. This PHI implies the EHI, but the standard PHI (with $R = T^2$) cannot then hold. (One cannot have the PHI with two asymptotically distinct space-time scaling relations.) [BB04], [BBK06] prove that the anomalous PHI(Ψ) with scaling $R = \Psi(T) = T^{\beta_1} \mathbf{1}_{(T \leq 1)} + T^{\beta_2} \mathbf{1}_{(T > 1)}$ is stable under rough isometries. These papers also prove that PHI(Ψ) is equivalent to volume doubling, a family of Poincaré inequalities with scaling Ψ , and a new inequality that controlled the energy of cutoff functions in annuli, called a *cutoff Sobolev inequality*, and denoted CS(Ψ). The papers [BB04], [BBK06] proved the PHI by Moser's argument, but the more recent papers [AB15], [GHL15] use de Giorgi's argument and a mean value inequality to obtain similar results, but with a simpler form of the cutoff Sobolev inequality. In addition, an important point for this paper, [GHL15] does not require the underlying metric space to be a length space.

A further example of weighted Laplace operators on Riemannian manifolds that satisfy EHI but not PHI is given in [GSC05, Example 6.14]. Consider the second order differential operators L_α on \mathbb{R}^n , $n \geq 2$ given by

$$L_\alpha = (1 + |x|^2)^{-\alpha/2} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left((1 + |x|^2)^{\alpha/2} \frac{\partial}{\partial x_i} \right) = \Delta + \alpha \frac{x \cdot \nabla}{1 + |x|^2}.$$

Then L_α satisfies the PHI if and only if $\alpha > -n$ but satisfies the EHI for all $\alpha \in \mathbb{R}$. Weighted Laplace operators of this kind arise naturally in the context of Schrödinger operators and conformal transformations of Riemannian metrics; see [Gri06, §§6.4, 10].

These papers left open the problem of the stability of the EHI, and also the question of finding a satisfactory characterization of the EHI. This problem is mentioned in [Gri95], [SC04, Question 12] and [Kum04]. In [GHL14], the authors write “An interesting (and obviously hard) question is the characterization of the elliptic Harnack inequality in more geometric terms — so far nothing is known, not even a conjecture.”

In [Del02], Delmotte gave an example of a graph which satisfies the EHI but for which (VD) fails; his example was to take the join of the infinite Sierpinski gasket graph with another (suitably chosen) graph. This example shows that any attempt to characterize the EHI must tackle the difficulty that different parts of the space may have different space-time scaling functions. Considerable progress on this was made by R. Bass [Bas13], but his result requires volume doubling, as well as some additional hypotheses on capacity.

As Bass remarks, all the robust proofs of the EHI, using the methods of De Giorgi, Nash or Moser, use the volume doubling property in an essential way, as well as Sobolev and Poincaré type inequalities. The starting point for this paper is the observation that a change of the symmetric measure (or equivalently a time change of the process) does not affect the sheaf of harmonic functions on bounded open sets. On the other hand properties such as volume doubling or Poincaré inequality are not in general preserved by this transformation.

Conversely, given a space satisfying the EHI, one could seek to construct a “good” measure μ such that volume doubling as well as additional Poincaré and Sobolev inequalities do hold with respect to μ ; this is indeed the approach of this paper. Our main result, [Theorem 1.3](#), is that the EHI is stable. Our methods also give a characterization of the EHI by properties that are easily seen to be stable under perturbations; see [Theorem 5.15](#).

Our main interest is the EHI for manifolds and graphs. To handle both cases at once we work in the general context of metric measure spaces. So we consider a complete, locally compact, separable, geodesic (or length) metric space (\mathcal{X}, d) with a Radon measure m that has full support, so that $m(U) > 0$ for all non-empty, open U . We call this a *metric measure space*. Let $(\mathcal{E}, \mathcal{F}^m)$ be a strongly local Dirichlet form on $L^2(\mathcal{X}, m)$; see [FOT94]. We call the quintuple $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ a *measure metric space with Dirichlet form*, or *MMD space*. We write $B(x, r) = \{y : d(x, y) < r\}$ for open balls in \mathcal{X} , and given a ball $B = B(x, r)$, we sometimes use the notation θB to denote the ball $B(x, \theta r)$. We assume (\mathcal{X}, d) has infinite radius, so that $\mathcal{X} - B(x, R) \neq \emptyset$ for all $R > 0$. See [Section 2](#) for more details of these spaces and the definitions of harmonic functions and capacities in this context.

Our two fundamental examples are Riemannian manifolds and the cable systems of graphs. If (\mathcal{M}, g) is a Riemannian manifold, we take d and m to be the Riemannian distance and measure respectively, and we define the Dirichlet form to be the closure of the symmetric bilinear form

$$\mathcal{E}(f, f) = \int_{\mathcal{X}} |\nabla_g f|^2 dm, \quad f \in C_0^\infty(\mathcal{M}).$$

Given a graph $\mathbb{G} = (\mathbb{V}, E)$, the *cable system* of \mathbb{G} is the metric space obtained by replacing each edge by a copy of the unit interval, glued together

in the obvious way. For a graph with uniformly bounded vertex degree, the EHI for the graph is equivalent to the EHI for its cable system, and so our theorem also implies stability of the EHI for graphs. See Section 6 for more details of both these examples.

Since our main spaces of interest are regular at small length scales, we will avoid a number of technical issues that could arise for general MMD spaces by making two assumptions of local regularity: Assumptions 2.3 and 2.5. Both our main examples satisfy these assumptions; see Section 6.

The hypothesis of volume doubling plays an important role in the study of heat kernel bounds for the process X , and as mentioned above it is a necessary condition for the PHI.

Definition 1.1 (Volume doubling property). We say that a Borel measure μ on a metric space (\mathcal{X}, d) satisfies the *volume doubling property* if μ is non-zero and there exists a constant $C_V < \infty$ such that

$$(1.5) \quad \mu(B(x, 2r)) \leq C_V \mu(B(x, r))$$

for all $x \in \mathcal{X}$ and for all $r > 0$.

Definition 1.2. We say that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ satisfies the *elliptic Harnack inequality* (EHI) if there exist constants $1 < A, C_H < \infty$ such that for any $x \in \mathcal{X}$ and $R > 0$, for any non-negative harmonic function h on a ball $B(x, AR)$, one has

$$(1.6) \quad \operatorname{ess\,sup}_{B(x,R)} h \leq C_H \operatorname{ess\,inf}_{B(x,R)} h.$$

If (\mathcal{X}, d) is a geodesic metric space and the above inequality holds for some value of $A > 1$, then it holds for any other $A' > 1$ with a constant $C_H(A')$. If the EHI holds, then iterating condition (1.6) gives a.e. Hölder continuity of harmonic functions, and it follows that any harmonic function has a continuous modification.

Our first main theorem is

THEOREM 1.3. *Let (\mathcal{X}, d, m) be a length metric measure space, and let $(\mathcal{E}, \mathcal{F})$ be a strongly local Dirichlet form on $L^2(\mathcal{X}, m)$. Suppose that Assumptions 2.3 and 2.5 hold. Let $(\mathcal{E}', \mathcal{F})$ be a strongly local Dirichlet form on $L^2(\mathcal{X}, m')$ that is equivalent to \mathcal{E} , so that there exists $C < \infty$ such that*

$$\begin{aligned} C^{-1}\mathcal{E}(f, f) &\leq \mathcal{E}'(f, f) \leq C\mathcal{E}(f, f) && \text{for all } f \in \mathcal{F}, \\ C^{-1}m(A) &\leq m'(A) \leq Cm(A) && \text{for all measurable sets } A. \end{aligned}$$

Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the elliptic Harnack inequality. Then the EHI holds for $(\mathcal{X}, d, m', \mathcal{E}', \mathcal{F})$.

We now state some consequences of [Theorem 1.3](#) for Riemannian manifolds and graphs. We say that two Riemannian manifolds (M, g) and (M', g') are *quasi-isometric* if there exist a diffeomorphism $\phi : (M, g) \rightarrow (M', g')$ and a constant $K \geq 1$ such that

$$K^{-1}g(\xi, \xi) \leq g'(d\phi(\xi), d\phi(\xi)) \leq Kg(\xi, \xi) \quad \text{for all } \xi \in TM.$$

Let (M, g) be a Riemannian manifold, and let $\text{Sym}(TM)$ denote the bundle of symmetric endomorphisms of the tangent bundle TM . We say that \mathcal{A} is a *uniformly elliptic operator in divergence form* if there exists $A : M \rightarrow \text{Sym}(TM)$ a measurable section of $\text{Sym}(TM)$ and a constant $K \geq 1$ such that

$$K^{-1}g(\xi, \xi) \leq g(A\xi, \xi) \leq Kg(\xi, \xi) \quad \text{for all } \xi \in TM,$$

such that $\mathcal{A}(\cdot) = \text{div}(A\nabla(\cdot))$. Here div and ∇ denote the Riemannian divergence and gradient respectively.

THEOREM 1.4.

- (a) *Let (M, g) be a Riemannian manifold that is quasi-isometric to a manifold whose Ricci curvature is bounded below, and let Δ denote the corresponding Laplace-Beltrami operator. If (M, g) satisfies the EHI for non-negative solutions of $\Delta u = 0$, then it satisfies the EHI for non-negative solutions of $\mathcal{A}u = 0$, where \mathcal{A} is any uniformly elliptic operator in divergence form.*
- (b) *Let (M, g) and (M', g') be two Riemannian manifolds that are quasi-isometric to a manifold whose Ricci curvature is bounded below. Let Δ and Δ' denote the corresponding Laplace-Beltrami operators. Then non-negative Δ -harmonic functions satisfy the EHI if and only if non-negative Δ' -harmonic functions satisfy the EHI.*

THEOREM 1.5. *Let $\mathbb{G} = (\mathbb{V}, E)$ and $\mathbb{G}' = (\mathbb{V}', E')$ be bounded degree graphs that are roughly isometric. Then the EHI holds for \mathbb{G}' if and only if it holds for \mathbb{G} .*

Remark 1.6. (1) [Theorem 1.4\(a\)](#) is a generalization of Moser's elliptic Harnack inequality [[Mos61](#)]. The parabolic versions of (a) and (b) are due to [[SC92b](#)]. For (b), note that the manifold (M, g) might not have Ricci curvature bounded below and hence the methods of [[Yau75](#)], [[CY75](#)] will not apply. A parabolic version of [Theorem 1.5](#) is essentially due to [[Del99](#)].

(2) As proved in [[Lyo87](#)], the Liouville property is not stable under rough isometries.

The outline of our argument is as follows. In [Section 3](#) using the tools of potential theory we prove that the EHI implies certain regularity properties for Green's functions and capacities. The main result of this section ([Theorem 3.11](#)) is that the EHI implies that (\mathcal{X}, d) has the metric doubling property.

Definition 1.7. The space (\mathcal{X}, d) satisfies the *metric doubling property* (MD) if there exists $M < \infty$ such that any ball $B(x, R)$ can be covered by M balls of radius $R/2$.

An equivalent definition is that there exists $M' < \infty$ such that any ball $B(x, R)$ contains at most M' points that are all a distance of at least $R/2$ from each other. We will frequently use the fact that (MD) holds for (\mathcal{X}, d) if and only if (\mathcal{X}, d) has finite Assouad dimension. Recall that the Assouad dimension is the infimum of all numbers $\beta > 0$ with the property that every ball of radius $r > 0$ has at most $C\varepsilon^{-\beta}$ disjoint points of mutual distance at least εr for some $C \geq 1$ independent of the ball; see [Hei01, Exercise 10.17]. Equivalently, this is the infimum of all numbers $\beta > 0$ with the property that every ball of radius $r > 0$ can be covered by at most $C\varepsilon^{-\beta}$ balls of radius εr for some $C \geq 1$ independent of the ball.

It is well known that volume doubling implies metric doubling. A partial converse also holds: if (\mathcal{X}, d) satisfies (MD), then there exists a Radon measure μ on \mathcal{X} such that (\mathcal{X}, d, μ) satisfies (VD). This is a classical result due to Vol'berg and Konyagin [VK87] in the case of compact spaces, and Luukkainen and Saksman [LS98] in the case of general complete spaces. For other proofs, see [Wu98] and [Hei01, Ch. 13], and also [Hei01, Ch. 10] for a survey of some conditions equivalent to (MD).

The measures constructed in these papers are very far from being unique. In Section 4, using the approach of [VK87], we show that if \mathcal{X} satisfies the EHI and Assumptions 2.3 and 2.5, then we can construct a “good” doubling measure μ that is absolutely continuous with respect to m and connects capacities with the measures of balls in a suitable fashion; see Definition 4.1 and Theorem 4.2.

At this point we could use some extensions of the methods of [Bas13], [GHL15] to prove the stability of the EHI. However, a quicker approach, which we follow in Section 5, is to use ideas from the theory of quasisymmetric transformations of metric spaces. (See [Hei01] for an introduction to this theory and [Kig12] for applications to heat kernels.) These transformations do not distort annuli too much, and therefore they preserve the EHI; see Lemma 5.3. In Section 5 we prove that there exist a new metric d_Ψ on \mathcal{X} and a constant $\beta > 0$ such that the new space $(\mathcal{X}, d_\Psi, \mu)$ satisfies Poincaré and cutoff energy inequalities with respect to a global space-time scaling relation of the form $R = T^\beta$; see Theorem 5.14. These inequalities are stable with respect to bounded perturbations of the Dirichlet form. While the metric d_Ψ is not geodesic, the main theorem of [GHL15] does apply in this context, and it gives that the PI and CS inequalities in Theorem 5.14 imply the EHI. This gives the stability of the EHI, as well as a stable characterization; see Theorem 5.15.

In Section 6 we return to our two main classes of examples, weighted Riemannian manifolds and weighted graphs. We show that they both satisfy our

local regularity hypotheses [Assumptions 2.3](#) and [2.5](#), and we give the (short) proof of [Theorem 1.4](#).

The final [Section 7](#) formulates the class of rough isometries that we consider and states our result on the stability of the EHI under rough isometries. Since rough isometries only relate spaces at large scales, and the EHI is a statement that holds at all length scales, any statement of stability under rough isometries requires that the family of spaces under consideration satisfies suitable local regularity hypotheses.

A characterization of the EHI in terms of effective resistance (equivalently capacity) was suggested in [\[Bar05\]](#). G. Kozma [\[Koz05\]](#) gave an illuminating counterexample — a spherically symmetric tree. This example does not satisfy (MD), and at the end of [Section 7](#) we suggest a modified characterization, which is the “dumbbell condition” of [\[Bar05\]](#) together with (MD).

We use c, c', C, C' for strictly positive constants, which may change value from line to line. Constants with numerical subscripts will keep the same value in each argument, while those with letter subscripts will be regarded as constant throughout the paper. The notation $C_0 = C_0(a, b)$ means that the constant C_0 depends only on the constants a and b .

2. Metric measure spaces with Dirichlet form

In this section we give some background on MMD spaces, and we give our two assumptions of local regularity. We take (\mathcal{X}, d) to be a locally compact metric space with infinite radius and m to be a Radon measure on (\mathcal{X}, d) with full support. Let $(\mathcal{E}, \mathcal{F}^m)$ be a strongly local Dirichlet form on $L^2(\mathcal{X}, m)$. We call $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ a *measure metric space with Dirichlet form*, or MMD space. Except in [Section 5](#) we will assume also that (\mathcal{X}, d) is a length space.

In the context of MMD spaces, Poincaré and Sobolev inequalities involve integrals with respect to the *energy measures* $d\Gamma(f, f)$; formally these can be regarded as $|\nabla f|^2 dm$. For bounded $f \in \mathcal{F}^m$, the measure $d\Gamma(f, f)$ is defined to be the unique measure such that for all bounded $g \in \mathcal{F}^m$, we have

$$\int g d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g).$$

We have

$$\mathcal{E}(f, f) = \int_{\mathcal{X}} d\Gamma(f, f)$$

for a Riemannian manifold $d\Gamma(f, f) = |\nabla_g f|^2 dm$.

Associated with $(\mathcal{E}, \mathcal{F}^m)$ is a semigroup (P_t) and its infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. The operator \mathcal{L} satisfies

$$(2.1) \quad - \int (f \mathcal{L}g) dm = \mathcal{E}(f, g), \quad f \in \mathcal{F}^m, g \in \mathcal{D}(\mathcal{L});$$

in the case of a Riemannian manifold, \mathcal{L} is the Laplace-Beltrami operator. (P_t) is the semigroup of a continuous Hunt process $X = (X_t, t \in [0, \infty), \mathbb{P}^x, x \in \mathcal{X})$.

We define capacities for $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ as follows. For a non-empty open subset $D \subset \mathcal{X}$, let $\mathcal{C}_0(D)$ denote the space of all continuous functions with compact support in D . Let \mathcal{F}_D denote the closure of $\mathcal{F}^m \cap \mathcal{C}_0(D)$ with respect to the $\sqrt{\mathcal{E}(\cdot, \cdot) + \|\cdot\|_2^2}$ -norm. By $A \Subset D$, we mean that the closure of A is a compact subset of D . For $A \Subset D$, we set

$$(2.2) \quad \text{Cap}_D(A) = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}_D \text{ and } f \geq 1 \text{ in a neighborhood of } A\}.$$

It is clear from the definition that if $A_1 \subset A_2 \Subset D_1 \subset D_2$, then

$$(2.3) \quad \text{Cap}_{D_2}(A_1) \leq \text{Cap}_{D_1}(A_2).$$

We can consider $\text{Cap}_D(A)$ to be the effective conductance between the sets A and D^c if we regard \mathcal{X} as an electrical network and $\mathcal{E}(f, f)$ as the energy of the function f . A statement depending on $x \in B$ is said to hold quasi-everywhere on B (abbreviated as q.e. on B) if there exists a set $N \subset B$ of zero capacity such that the statement is true for every $x \in B \setminus N$. It is known that $(\mathcal{E}, \mathcal{F}_D)$ is a regular Dirichlet form on $L^2(D, m)$ and

$$(2.4) \quad \mathcal{F}_D = \{f \in \mathcal{F}^m : \tilde{f} = 0 \text{ q.e. on } D^c\},$$

where \tilde{f} is any quasi-continuous representative of f ; see [FOT94, Cor. 2.3.1, Th. 4.4.3]. Functions in the extended Dirichlet space will always be represented by their quasi-continuous version (cf. [FOT94, Th. 2.1.7]), so that expressions like $\int f^2 d\Gamma(\varphi, \varphi)$ are well defined.

Given an open set $U \subset \mathcal{X}$, we set

$$\begin{aligned} \mathcal{F}_{\text{loc}}(U) = \{h \in L^2_{\text{loc}}(U) : \text{for all relatively compact } V \subset U, \\ \text{there exists } h^\# \in \mathcal{F}^m, \text{ s.t. } h\mathbb{1}_V = h^\#\mathbb{1}_V \text{ } m\text{-a.e.}\}. \end{aligned}$$

Definition 2.1. If $h \in \mathcal{F}_{\text{loc}}(U)$ and $f \in \mathcal{F}^m \cap C_0(U)$, we define $\mathcal{E}(f, h) = \mathcal{E}(f, h^\#)$, where $h^\#$ is any function in \mathcal{F}^m with $h = h^\#$ on a precompact open set containing $\text{supp}(f)$. We say $h \in \mathcal{F}_{\text{loc}}(U)$ is harmonic if $\mathcal{E}(f, h) = 0$ for all $f \in \mathcal{F}^m \cap C_0(U)$.

This definition implies that $\mathcal{L}h = 0$ in D provided that h is in the domain of \mathcal{L}_D .

Next, we define the Green’s operator and Green’s function.

Definition 2.2. Let D be a bounded open subset of \mathcal{X} . Let \mathcal{L}_D denote the generator of the Dirichlet form $(\mathcal{E}, \mathcal{F}_D, L^2(D, m))$ and assume that

$$(2.5) \quad \lambda_{\min}(D) = \inf_{f \in \mathcal{F}_D \setminus \{0\}} \frac{\mathcal{E}(f, f)}{\|f\|_2^2} > 0.$$

We define the inverse of $-\mathcal{L}_D$ as the *Green operator* $G_D = (-\mathcal{L}_D)^{-1} : L^2(D, m) \rightarrow L^2(D, m)$. We say a jointly measurable function $g_D(\cdot, \cdot) : D \times D \rightarrow \mathbb{R}$ is the *Green function* for D if

$$G_D f(x) = \int_D g_D(x, y) f(y) m(dy) \text{ for all } f \in L^2(D, m) \text{ and for } m \text{ a.e. } x \in D.$$

Assumption 2.3 (Existence of Green function). For any bounded, non-empty open set $D \subset \mathcal{X}$, we assume that $\lambda_{\min}(D) > 0$ and that there exists a Green function $g_D(x, y)$ for D defined for $(x, y) \in D \times D$ with the following properties:

- (i) (Symmetry). $g_D(x, y) = g_D(y, x) \geq 0$ for all $(x, y) \in D \times D \setminus \text{diag}$.
- (ii) (Continuity). $g_D(x, y)$ is jointly continuous in $(x, y) \in D \times D \setminus \text{diag}$.
- (iii) (Maximum principles). If $x_0 \in U \Subset D$, then

$$\inf_{U \setminus \{x_0\}} g_D(x_0, \cdot) = \inf_{\partial U} g_D(x_0, \cdot), \quad \sup_{D \setminus U} g_D(x_0, \cdot) = \sup_{\partial U} g_D(x_0, \cdot).$$

- (iv) (Harmonic). For any fixed $x \in D$, the function $y \mapsto g_D(x, y)$ is in $\mathcal{F}_{\text{loc}}(D \setminus \{x\})$ and is harmonic in $D \setminus \{x\}$.

Here diag denotes the diagonal in $D \times D$.

Remark 2.4. Note that changing the measure m to an equivalent Radon measure m' does not affect either the capacity of bounded sets or the class of harmonic functions. Further, if $f_1, f_2 \in C(\mathcal{X}) \cap \mathcal{F}_D$, then writing $\langle \cdot, \cdot \rangle_m$ for the inner product in $L^2(m)$,

$$(2.6) \quad \mathcal{E}(G_D f_1, f_2) = \langle f_1, f_2 \rangle_m,$$

and it follows that $g_D(\cdot, \cdot)$ is also not affected by this change of measure.

Our second key local regularity assumption is as follows.

Assumption 2.5 (Bounded geometry or BG). We say that a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ satisfies (BG) if there exist $r_0 \in (0, \infty]$ and $C_L < \infty$ such that the following hold:

- (i) (Volume doubling property at small scales). For all $x \in \mathcal{X}$ and for all $r \in (0, r_0]$, we have

$$(2.7) \quad \frac{m(B(x, 2r))}{m(B(x, r))} \leq C_L.$$

- (ii) (Expected occupation time growth at small scales). There exists $\gamma_2 > 0$ such that for all $x_0 \in \mathcal{X}$ and for all $0 < s \leq r \leq r_0$, we have

$$(2.8) \quad \frac{m(B(x, s))}{\text{Cap}_{B(x, 8s)}(B(x, s))} \frac{\text{Cap}_{B(x, 8r)}(B(x, r))}{m(B(x, r))} \leq C_L \left(\frac{s}{r}\right)^{\gamma_2}.$$

See [Section 6](#) for the verification of [Assumptions 2.5](#) and [2.3](#) for our two main cases of interest, weighted Riemannian manifolds with Ricci curvature bounded below, and the cable system of graphs with uniformly bounded vertex degree. The condition (BG) is a robust one because under mild conditions, it is preserved under bounded perturbation of conductance in a weighted graph, and quasi-isometries of weighted manifolds.

3. Consequences of EHI

Throughout this section we assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ is a MMD space that satisfies [Assumption 2.3](#), as well as the EHI with constant C_H . In addition we assume that (\mathcal{X}, d) is a length space and will write $\gamma(x, y)$ for a geodesic between x and y . Recall that (X_t) is the Hunt process associated with $(\mathcal{E}, \mathcal{F}^m)$, and for $F \subset \mathcal{X}$, write

$$(3.1) \quad T_F = \inf\{t \geq 0 : X_t \in F\}, \tau_F = T_{F^c}.$$

THEOREM 3.1. *Let (\mathcal{X}, d) satisfy the EHI. Then there exists a constant $C_G = C_G(C_H)$ such that if $B(x_0, 2R) \subset D$, then*

$$(3.2) \quad g_D(x_0, y) \leq C_G g_D(x_0, z) \text{ if } d(x_0, y) = d(x_0, z) = R.$$

Proof. The proof of [[Bar05](#), Th. 2] carries over to this situation with essentially no change. (In fact, it is slightly simpler, since there is no need to make corrections at small length scales.) Note that since $g_D(\cdot, \cdot)$ is continuous off the diagonal, we can use the EHI with sup and inf instead of ess sup and ess inf. \square

COROLLARY 3.2. *Let $B(x_0, 2R) \subset D$. Let $A \geq 2$. Then there exists a constant $C_1 = C_1(C_H, A)$ such that*

$$g_D(x_0, x) \leq C_1 g_D(x_0, y) \quad \text{for } x, y \in B(x_0, R) \setminus B(x_0, R/A).$$

Proof. We can assume $d(x, x_0) \geq d(x_0, y)$. Let z be the point on $\gamma(x_0, x)$ with $d(x_0, z) = d(x_0, y)$. Then we can compare $g_D(x_0, y)$ and $g_D(x_0, z)$ by [Theorem 3.1](#), and $g_D(x_0, z)$ and $g_D(x_0, x)$ by using a chain of balls with centers in $\gamma(z, x)$. (The number of balls needed will depend on A .) \square

LEMMA 3.3. *Let $x_0 \in \mathcal{X}$, $R > 0$, and let $B(x_0, 2R) \subset D$. There exists a constant $C_0 = C_0(C_H)$ such that if $x_1, x_2, y_1, y_2 \in B(x_0, R)$ with $d(x_j, y_j) \geq R/4$, then*

$$(3.3) \quad g_D(x_1, y_1) \leq C_0 g_D(x_2, y_2).$$

Proof. A counting argument shows there exists a ball $B(z, R/9) \subset B(x_0, R)$ that contains none of the points x_1, x_2, y_1, y_2 . Using [Corollary 3.2](#) we have $g_D(x_1, y_1) \leq c g_D(z, x_1)$, $g_D(z, x_1) \leq c g_D(z, x_2)$, and $g_D(z, x_2) \leq g_D(x_2, y_2)$, and combining these comparisons gives the required bound. \square

Definition 3.4. Set

$$(3.4) \quad g_D(x, r) = \inf_{y:d(x,y)=r} g_D(x, y).$$

The maximum principle implies that $g_D(x, r)$ is non-increasing in r . An easy argument gives that if $d(x, y) = r$ and $B(x, 2r) \cup B(y, 2r) \subset D$, then

$$(3.5) \quad g_D(x, r) \leq C_G g_D(y, r).$$

Let D be a bounded domain in \mathcal{X} , A be Borel set, $A \Subset D \subset \mathcal{X}$, and recall from (2.2) the definition of $\text{Cap}_D(A)$. By [FOT94, Th. 4.3.3], [GH14, Prop. A.2] there exists a function $h_{A,D} \in \mathcal{F}_D$ called the *equilibrium potential* such that $h_{A,D} = 1$ quasi-everywhere in A and $\mathcal{E}(h_{A,D}, h_{A,D}) = \text{Cap}_D(A)$. The function $h_{A,D}(\cdot)$ is the hitting probability of the set A :

$$(3.6) \quad h_{A,D}(x) = \mathbb{P}^x(T_A < \tau_D) \quad \text{for } x \in D \text{ quasi-everywhere.}$$

Further

$$(3.7) \quad h_{A,D}(x) = 1 \quad \text{quasi-everywhere on } A.$$

There exists a Radon measure $\nu_{A,D}$ called the *capacitary measure* or *equilibrium measure* that does not charge any set of zero capacity, supported on ∂A such that $\nu_{A,D}(\partial A) = \text{Cap}_D(A)$, and satisfies (cf. [FOT94, Lemma 2.2.10, Th. 2.2.5] and [GH14, Lemma 6.5]):

$$(3.8) \quad \mathcal{E}(h_{A,D}, v) = \int_{\partial A} \tilde{v} d\nu_{A,D} = \int_D \tilde{v} d\nu_{A,D} \quad \text{for all } v \in \mathcal{F}_D,$$

$$(3.9) \quad h_{A,D}(y) = \nu_{A,D}G_D(y) = \int_{\partial A} \nu_{A,D}(dx)g_D(x, y) \quad \text{for all } y \in D \setminus \partial A.$$

Here \tilde{v} in (3.8) denotes a quasi-continuous version of v . By [FOT94, Th. 2.1.5 and p. 71], $\text{Cap}_D(A)$ can be expressed as

$$(3.10) \quad \text{Cap}_D(A) = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}_D, f \geq 1 \text{ quasi-everywhere on } A\}.$$

LEMMA 3.5. *Let $B(x_0, 2r) \subset D$. Then*

$$(3.11) \quad g_D(x_0, r) \leq \text{Cap}_D(B(x_0, r))^{-1} \leq C_G g_D(x_0, r).$$

Proof. Let ν be the capacitary measure for $B(x_0, r)$ with respect to G_D . Then ν is supported by $\partial B(x, r)$ and by (3.9),

$$1 = \nu G(x_0) = \int_{\partial B} g_D(x_0, z)\nu(dz).$$

Hence

$$\begin{aligned} \nu(B(x_0, r))g_D(x_0, r) &\leq 1 \leq \nu(B(x_0, r)) \sup_{z \in \partial B} g_D(x_0, z) \\ &\leq C_G \nu(B(x_0, r))g_D(x_0, r). \end{aligned} \quad \square$$

Remark 3.6. The assumption $B(x_0, 2r) \subset D$ in [Theorem 3.1](#), [Corollary 3.2](#), and [Lemmas 3.3](#) and [3.5](#) can be replaced with the assumption $B(x_0, Kr) \subset D$ for any fixed $K > 1$.

LEMMA 3.7. *Let $B = B(x_0, R) \subset \mathcal{X}$, and let $x_1 \in B(x_0, R/2)$, $B_1 = B(x_1, R/4)$. There exists $p_0 = p_0(C_H)$ such that*

$$(3.12) \quad \mathbb{P}^y(T_{B_1} < \tau_B) \geq p_0 > 0 \text{ for } y \in B(x_0, 7R/8).$$

Proof. Let ν be the capacity measure for B_1 with respect to G_B , and let $h(x) = \nu G_B(x) = \mathbb{P}^x(T_{B_1} < \tau_B)$. Then h is 1 on B_1 , so by the maximum principle it is enough to prove (3.12) for $y \in B(x_0, 7R/8)$ with $d(y, B_1) \geq R/16$.

By [Corollary 3.2](#) (applied in a chain of balls if necessary) there exists $p_0 > 0$ depending only on C_H such that $g_B(y, z) \geq p_0 g_B(x_1, z)$ for $z \in \partial B_1$. Thus

$$h(y) \geq p_0 \int_{\partial B_1} g_B(x_1, z) \nu(dz) = p_0 \nu G_B(x_1) = p_0. \quad \square$$

COROLLARY 3.8. *Let $B(x_0, 2R) \subset D$. Then there exists $\theta = \theta(C_H) > 0$ such that if $0 < s < r < R/2$ and $x \in B(x_0, R)$, then*

$$(3.13) \quad \frac{g_D(x, r)}{g_D(x, s)} \geq c \left(\frac{s}{r}\right)^\theta.$$

Proof. Let $w \in \partial B(x, 2s)$, and let $z \in \gamma(x, w) \cap \partial B(x, s)$. Applying the EHI on a chain of balls on $\gamma(z, w)$ gives $g_D(x, w) \geq c_1 g_D(x, z)$, and it follows that

$$g_D(x, s) \leq C_1 g_D(x, 2s).$$

Iterating this estimate then gives (3.13) with $\theta = \log_2 C_1$. □

Remark 3.9. The example of \mathbb{R}^2 shows that we cannot expect a corresponding upper bound on $g_D(x, r)/g_D(x, s)$.

The key estimate in this section is the following geometric consequence of the EHI. A weaker result proved with some of the same ideas, and in the graph case only, is given in [\[Bar05, Th. 1\]](#).

LEMMA 3.10. *Let $B = B(x_0, R) \subset \mathcal{X}$. Let $\lambda \in [\frac{1}{4}, 1]$, $0 < \delta \leq 1/32$, and let $B_i = B(z_i, \delta R)$, $i = 1, \dots, n$ satisfy*

- (1) $B_i \cap \partial B(x_0, \lambda R) \neq \emptyset$;
- (2) $B_i^* = B(z_i, 8\delta R)$ are disjoint.

Then there exists a constant $C_1 = C_1(C_H, \delta)$ such that $n \leq C_1$.

Proof. Let y_i and w_i be points on $\gamma(x_0, z_i)$ with $d(z_i, y_i) = 3\delta R$ and $d(z_i, w_i) = 5\delta R$. Let $A_i = \overline{B}(y_i, \delta)$. By [Lemma 3.7](#),

$$(3.14) \quad \mathbb{P}^x(T_{B_i} < \tau_{B_i^*}) \geq p_1 > 0 \quad \text{for all } x \in A_i.$$

Now let $D = B - \cup_i B_i$, and let N be the number of distinct balls A_i hit by X before τ_D . Write $S_1 < S_2 < \dots < S_N$ for the hitting times of these balls. Let $G_{ki} = \{N \geq k, X_{S_k} \in A_i\}$. On the event G_{ki} , if X then hits B_i before leaving B_i^* , we will have $N = k$. So using (3.14), for $k \geq 1$,

$$\mathbb{P}^x(N = k | N \geq k) \geq p_1,$$

and thus N is dominated by a geometric random variable with mean $1/p_1$. Hence,

$$(3.15) \quad \mathbb{E}^{x_0} N \leq 1/p_1.$$

Now set

$$h_i(x) = \mathbb{P}^x(T_{A_i} < \tau_D).$$

Then $h_i(y_i) = 1$ and by Lemma 3.7 $h_i(w_i) \geq p_1$. Using the EHI in a chain of balls we have $h_i(x_0) \geq p_2 = p_2(\delta) > 0$. Thus

$$p_1^{-1} \geq \mathbb{E}^{x_0} N = \sum_{i=1}^n h_i(x_0) \geq np_2,$$

which gives an upper bound for n . □

THEOREM 3.11. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ satisfy EHI. Then (\mathcal{X}, d) satisfies the metric doubling property (MD).*

Proof. Let $\delta = 1/32$, $x_0 \in \mathcal{X}$, $R > 0$. It is sufficient to show that there exists M (depending only on C_H) such that if $B(z_i, 8\delta R)$, $1 \leq i \leq n$ are disjoint balls with centers in $B(x_0, R) - B(x_0, R/4)$, then $n \leq M$. So let $B(z_i, 8\delta R)$, $i = 1, \dots, n$ satisfy these conditions.

Let $B_k = B(x_0, \frac{1}{2}k\delta R)$ for $1/(2\delta) \leq k \leq 2/\delta$, and let n_k be the number of balls $B(z_i, \delta R)$ that intersect ∂B_k . Since each $B(z_i, \delta R)$ must intersect at least one of the sets ∂B_k , we have $n \leq \sum_k n_k$. The previous lemma gives $n_k \leq C_1$, and thus $n \leq 2C_1/\delta$. □

We now compare g_D in two domains.

LEMMA 3.12. *There exists a constant C_0 such that if $B = B(x_0, R)$ and $2B = B(x_0, 2R)$, then*

$$(3.16) \quad g_{2B}(x, y) \leq C_0 g_B(x, y) \quad \text{for } x, y \in B(x_0, R/4).$$

Proof. Let $B' = B(x_0, R/2)$ and $y \in B(x_0, R/4)$. Choose $x_1 \in \partial B'$ to maximize $g_{2B}(x_1, y)$. Let γ be a geodesic path from x_0 to $\partial B(x_0, 3R)$, let z_0 be the point on $\gamma \cap \partial B$, and let $A = B(z_0, R/4)$.

Using Lemma 3.7 there exists $p_1 > 0$ such that

$$p_A(w) = \mathbb{P}^w(X_{\tau_B} \in A) \geq p_1, \quad w \in B',$$

$$\mathbb{P}^z(\tau_{2B} < T_{B'}) \geq p_1, \quad z \in A.$$

Then

$$\begin{aligned} g_{2B}(x_1, y) &= g_B(x_1, y) + \mathbb{E}^{x_1} g_{2B}(X_{\tau_B}, y) \\ &= g_B(x_1, y) + \mathbb{E}^{x_1} \mathbb{1}_{(X_{\tau_B} \in A)} g_{2B}(X_{\tau_B}, y) + \mathbb{E}^{x_1} \mathbb{1}_{(X_{\tau_B} \notin A)} g_{2B}(X_{\tau_B}, y) \\ &\leq g_B(x_1, y) + p_A(x_1) \sup_{w \in A \cap \partial B} g_{2B}(w, y) + (1 - p_A(x_1)) \sup_{z \in \partial B} g_{2B}(z, y) \\ &\leq g_B(x_1, y) + p_1 \sup_{w \in A \cap \partial B} g_{2B}(w, y) + (1 - p_1) \sup_{z \in \partial B} g_{2B}(z, y). \end{aligned}$$

If $w \in A$, then

$$\begin{aligned} g_{2B}(w, y) &= \mathbb{E}^w \mathbb{1}_{(T_{B'} < \tau_{2B})} g_{2B}(X_{T_{B'}}, y) \\ &\leq (1 - p_1) \sup_{z \in \partial B'} g_{2B}(z, y) \leq (1 - p_1) g_{2B}(x_1, y). \end{aligned}$$

The maximum principle implies that $g_{2B}(z, y) \leq g_{2B}(x_1, y)$ for all $z \in \partial B$. Combining the inequalities above gives

$$g_{2B}(x_1, y) \leq g_B(x_1, y) + p_1(1 - p_1)g_{2B}(x_1, y) + (1 - p_1)g_{2B}(x_1, y),$$

which implies that

$$(3.17) \quad g_{2B}(x_1, y) \leq p_1^{-2} g_B(x_1, y).$$

Now let $x \in B(x_0, R/4)$. By [Corollary 3.2](#),

$$g_{2B}(x_1, y) \leq p_1^{-2} g_B(x_1, y) \leq C g_B(x, y).$$

Hence

$$\begin{aligned} g_{2B}(x, y) &= g_{B'}(x, y) + \mathbb{E}^x g_{2B}(X_{\tau_{B'}}, y) \\ &\leq g_{B'}(x, y) + g_{2B}(x_1, y) \leq (1 + C)g_B(x, y). \end{aligned} \quad \square$$

COROLLARY 3.13. *Let $A \geq 4$. There exists $C_0 = C_0(C_H, A)$ such that for $x \in \mathcal{X}$, $r > 0$,*

$$(3.18) \quad \text{Cap}_{B(x, 2Ar)}(B(x, r)) \leq \text{Cap}_{B(x, Ar)}(B(x, r)) \leq C_0 \text{Cap}_{B(x, 2Ar)}(B(x, r)).$$

Proof. The first inequality is immediate from the monotonicity of capacity, and the second one follows immediately from [Lemmas 3.5](#) and [3.12](#). \square

LEMMA 3.14. *Let $A \geq 8$, and let D be a bounded domain in \mathcal{X} . Let $x \in \mathcal{X}$ and $r > 0$ be such that $B(x, 4r) \subset D$.*

(a) *There exists $C_0 = C_0(C_H)$ such that*

$$\text{Cap}_D(B(x, r)) \leq C_0 \text{Cap}_D(B(y, r)) \quad \text{for } y \in B(x, r).$$

(b) *There exists $C_1 = C_1(A, C_H)$ such that*

$$(3.19) \quad \text{Cap}_{B(x, Ar)}(B(x, r)) \leq C_1 \text{Cap}_{B(y, Ar)}(B(y, r)) \quad \text{for } y \in B(x, r).$$

Proof. (a) follows easily from [Lemmas 3.5](#) and [3.12](#).

(b) We have

$$\begin{aligned} \text{Cap}_{B(x,Ar)}(B(x,r)) &\leq C_2 \text{Cap}_{B(x,Ar)}(B(y,r)) \\ &\leq C_3 \text{Cap}_{B(x,2Ar)}(B(y,r)) \leq C_1 \text{Cap}_{B(y,Ar)}(B(y,r)). \quad \square \end{aligned}$$

We conclude this section with a capacity estimate that will play a key role in our construction of a well-behaved doubling measure.

PROPOSITION 3.15. *Let $D \subset \mathcal{X}$ be a bounded open domain, and let $B(x_0, 8R) \subset D$. Let $F \subset B(x_0, R)$. Let $b \geq 4$, and suppose there exist disjoint Borel sets $(Q_i, 1 \leq i \leq n)$, with $n \geq 2$, such that*

$$F = \cup_{i=1}^n Q_i,$$

and for each i , there exists $z_i \in Q_i$ such that $B(z_i, R/6b) \subset Q_i$. Then there exists $\delta = \delta(b, C_H) > 0$ such that

$$\text{Cap}_D(F) \leq (1 - \delta) \sum_{i=1}^n \text{Cap}_D(Q_i).$$

Proof. Let ν_i and h_i be the equilibrium measure and equilibrium potential respectively for Q_i , so that $h_i = 1$ q.e. on Q_i . Then

$$\text{Cap}_D(Q_i) = \nu_i(\partial Q_i) = \nu_i(D).$$

By [\(3.6\)](#) and [Lemma 3.7](#), there exists $c_1 > 0$ such that

$$h_i(y) \geq c_1 \quad \text{for } y \in B(x_0, R) \text{ q.e.}$$

Let $h = \sum_{i=1}^n h_i$. Let $y \in F$, so that there exists i such that $y \in Q_i$. Then since $n \geq 2$,

$$h(y) = \sum_{i=1}^n h_i(y) \geq 1 + \sum_{j \neq i} c_1 \geq 1 + c_1 \quad \text{for } y \in F \text{ q.e.}$$

Consequently, if $h' = [(1 + c_1)^{-1}h] \wedge 1$, then $h' = 1$ quasi-everywhere on F . It follows that

$$\begin{aligned} \text{Cap}_D(F) &\leq \mathcal{E}(h', h') \leq \mathcal{E}(h', (1 + c_1)^{-1}h) = (1 + c_1)^{-1} \sum_{i=1}^n \int_D h' d\nu_i \\ &\leq (1 + c_1)^{-1} \sum_{i=1}^n \nu_i(D) = (1 + c_1)^{-1} \sum_{i=1}^n \text{Cap}_D(Q_i). \end{aligned}$$

The first inequality above follows from [\(3.10\)](#), the second inequality follows from the fact that h' is a potential (see [\[FOT94, Cor. 2.2.2 and Lemma 2.2.10\]](#)), the third equality follows from [\(3.8\)](#) and the fourth inequality holds since $h' \leq 1$. □

Remark 3.16. All the results in this section can be localized in the following sense: if we assume the EHI holds at small scales (i.e., for radii less than some R_1), then the conclusions of the results in this section also hold at sufficiently small scales.

4. Construction of good doubling measures

We continue to consider a length metric measure space with Dirichlet form $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ that satisfies the EHI and [Assumptions 2.3](#) and [2.5](#). The space (\mathcal{X}, d) satisfies metric doubling by [Theorem 3.11](#), and therefore by [\[VK87\]](#), [\[LS98\]](#) there exists a doubling measure μ on (\mathcal{X}, d) . However, this measure might be somewhat pathological (see [\[Wu98, Th. 2\]](#)), and to prove the EHI we will require some additional regularity properties of μ . In this section we adapt the argument of [\[VK87\]](#) to obtain a “good” doubling measure — that is, one that connects measures and capacities of balls in a satisfactory fashion.

Definition 4.1. Let D be either a ball $B(x_0, R) \subset \mathcal{X}$ or the whole space \mathcal{X} . If $D = \mathcal{X}$, fix $x_0 \in \mathcal{X}$. Let $C_0 < \infty$ and $0 < \beta_1 \leq \beta_2$. We say a measure ν on D is (C_0, β_1, β_2) -capacity good if the following holds:

(a) the measure ν is doubling on all balls contained in D ; that is,

$$(4.1) \quad \frac{\nu(B(x, 2s))}{\nu(B(x, s))} \leq C_0 \text{ whenever } B(x, 2s) \subset D;$$

(b) for all $x \in D$ and $0 < s_1 < s_2$ such that $B(x, s_2) \subset D$,

$$(4.2) \quad C_0^{-1} \left(\frac{s_2}{s_1}\right)^{\beta_1} \leq \frac{\nu(B(x, s_2)) \text{Cap}_{B(x, 8s_1)}(B(x, s_1))}{\nu(B(x, s_1)) \text{Cap}_{B(x, 8s_2)}(B(x, s_2))} \leq C_0 \left(\frac{s_2}{s_1}\right)^{\beta_2};$$

(c) the measure ν is absolutely continuous with respect to m , and we have

$$(4.3) \quad \text{ess sup}_{y \in B(x, 1)} \frac{d\nu}{dm}(y) \leq C_0 \text{ess inf}_{y \in B(x, 1)} \frac{d\nu}{dm}(y) \text{ whenever } B(x, 1) \subset D,$$

$$(4.4) \quad C_0^{-1-d(x_0, y)} \leq \frac{d\nu}{dm}(y) \leq C_0^{1+d(x_0, y)} \text{ for } m\text{-almost every } y \in D.$$

The following is the main result of this section.

THEOREM 4.2 (Construction of a doubling measure). *Let (\mathcal{X}, d) be a complete, locally compact, length metric space with a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F}^m)$ on $L^2(\mathcal{X}, m)$ that satisfies [Assumptions 2.3](#) and [2.5](#) and the EHI. Then there exist constants $C_0 > 1$, $0 < \beta_1 \leq \beta_2$ and a measure μ on \mathcal{X} that is (C_0, β_1, β_2) -capacity good.*

We begin by adapting the argument in [\[VK87\]](#) to construct measure with the desired properties in a ball $B(x_0, R)$. We then follow [\[LS98\]](#) and obtain μ as a weak* limit of measures defined on an increasing family of balls.

PROPOSITION 4.3 (Measure in a ball). *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ be as in the previous theorem. There exist $C_0 > 1$, $0 < \beta_1 \leq \beta_2$ such that for any ball $B_0 = B(x_0, r) \subset \mathcal{X}$ with $r \geq r_0$, there exists a measure $\nu = \nu_{x_0, r}$ on B_0 that is (C_0, β_1, β_2) -capacity good.*

The proof uses a family of generalized dyadic cubes, which provide a family of nested partitions of a space.

LEMMA 4.4 ([KRS12, Th. 2.1]). *Let (\mathcal{X}, d) be a complete, length metric space satisfying (MD), and let $A \geq 4$ and $c_A = \frac{1}{2} - \frac{1}{A-1}$. Let $B_0 = B(x_0, r)$ denote a closed ball in (\mathcal{X}, d) . Then there exists a collection*

$$\{Q_{k,i} : k \in \mathbb{Z}_+, i \in I_k \subset \mathbb{Z}_+\}$$

of Borel sets satisfying the following properties:

- (a) $B_0 = \cup_{i \in I_k} Q_{k,i}$ for all $k \in \mathbb{Z}_+$.
- (b) If $m \leq n$ and $i \in I_n, j \in I_m$, then either $Q_{n,i} \cap Q_{m,j} = \emptyset$ or else $Q_{n,i} \subset Q_{m,j}$.
- (c) For every $k \in \mathbb{Z}_+, i \in I_k$, there exists $x_{k,i}$ such that

$$B(x_{k,i}, c_A r A^{-k}) \cap B_0 \subset Q_{k,i} \subset B(x_{k,i}, A^{-k} r).$$

- (d) The sets $N_k = \{x_{k,i} : i \in I_k\}$, where $x_{k,i}$ as defined in (c) are increasing, $N_0 = \{x_0\}$, and $Q_{0,0} = B_0$. Moreover for each $k \in \mathbb{Z}_+, N_k$ is a maximal rA^{-k} -separated subset (rA^{-k} -net) of B_0 .
- (e) Property (b) defines a partial order \prec on $\mathcal{I} = \{(k, i) : k \in \mathbb{Z}_+, i \in I_k\}$ by inclusion, where $(k, i) \prec (m, j)$ whenever $Q_{k,i} \subset Q_{m,j}$.
- (f) There exists $C_M > 0$ such that, for all $k \in \mathbb{Z}_+$ and for all $x_{k,i} \in N_k$, the “successors”

$$S_k(x_{k,i}) = \{x_{k+1,j} : (k+1, j) \prec (k, i)\}$$

satisfy

$$(4.5) \quad C_M \geq |S_k(x_{k,i})| \geq 2.$$

Moreover, by property (c), we have $d(x_{k,i}, y) \leq A^{-k} r$ for all $y \in S_k(x_{k,i})$.

We now set $A = 8$, and until the end of the proof of Proposition 4.3 we fix a ball $B_0 = B(x_0, r)$. We remark that the constants in the rest of the section do not depend on the ball B_0 ; they depend only on the constants in EHI and (MD).

We fix a family

$$\{Q_{k,i} : k \in \mathbb{Z}_+, i \in I_k \subset \mathbb{Z}_+\}$$

of generalized dyadic cubes as given by Lemma 4.4, and we define the nets N_k and successors $S_k(x)$ as in the lemma. For $k \geq 1$, we define the predecessor $P_k(x)$ of $x \in N_k$ to be the unique element of N_{k-1} such that $x \in S_{k-1}(P_k(x))$.

Note that for $x \in N_k$, $S_k(x) \subset N_{k+1}$ whereas $P_k(x) \in N_{k-1}$. For $x \in B_0$, we denote by $Q_k(x)$ the unique $Q_{k,i}$ such that $x \in Q_{k,i}$. For $x \in N_k$, we denote by c_k the capacity

$$c_k(x) = \text{Cap}_{B(x,A^{-k+1}r)}(Q_k(x)).$$

The following lemma provides useful estimates on c_k .

LEMMA 4.5 (Capacity estimates for generalized dyadic cubes). *There exists $C_1 > 1$ such that the following hold:*

(a) *for all $k \in \mathbb{Z}_+$ and for all $x, y \in N_k$ such that $d(x, y) \leq 4rA^{-k}$, we have*

$$(4.6) \quad C_1^{-1}c_k(y) \leq c_k(x) \leq C_1c_k(y);$$

(b) *for all $k \in \mathbb{Z}_+$, for all $x \in N_k$, for all $y \in S_k(x)$, we have*

$$(4.7) \quad C_1^{-1}c_k(x) \leq c_{k+1}(y) \leq C_1c_k(x).$$

Proof. First, we observe that there is $C > 1$ such that

$$(4.8) \quad C^{-1} \left(g_{B(x,A^{-k+1}r)}(x, A^{-k}r) \right)^{-1} \leq c_k(x) \leq C \left(g_{B(x,A^{-k+1}r)}(x, A^{-k}r) \right)^{-1}$$

for all $x \in B(x_0, r)$. The upper bound in (4.8) follows from Lemma 4.4(c), domain monotonicity of capacity and Lemma 3.5. For the lower bound, we again use Lemma 4.4(c) to choose a point $z \in \gamma(x_0, x) \cap B_0$ such that $d(x, z) = crA^{-k}/2$, where c is as given by Lemma 4.4(c). By the triangle inequality, $Q_k(x) \supset B(z, crA^{-k}/2)$. The lower bound again follows from domain monotonicity, Lemma 3.5 and standard chaining arguments using EHI. The estimates (4.6) and (4.7) then follow from (4.8), domain monotonicity of capacity and Lemma 3.12. \square

We record one more estimate regarding the subadditivity of c_k , which will play an essential role in ensuring (4.2).

LEMMA 4.6 (Enhanced subadditivity estimate). *There exists $\delta \in (0, 1)$ such that for all $k \in \mathbb{Z}_+$, for all $x \in N_k$, we have*

$$c_k(x) \leq (1 - \delta) \sum_{y \in S_k(x)} c_{k+1}(y).$$

Proof. By the triangle inequality, $B(y, A^{-k}r) \subset B(x, A^{-k+1}r)$ for all $k \in \mathbb{Z}_+$, $x \in N_k$, $y \in S_k(x)$. The lemma now follows from Proposition 3.15 and domain monotonicity of capacity. \square

We now follow the Vol'berg-Konyagin construction closely, but with some essential changes. Recall that we want to construct a doubling measure μ on B_0 satisfying the estimates in Definition 4.1.

LEMMA 4.7 (see [VK87, Lemma, p. 631]). *Let $B_0 = B(x_0, r)$, and let c_k denote the capacities of the corresponding generalized dyadic cubes as defined*

above. There exists $C_2 \geq 1$ satisfying the following. Let μ_k be a probability measure on N_k such that

$$(4.9) \quad \frac{\mu_k(e')}{c_k(e')} \leq C_2^2 \frac{\mu_k(e'')}{c_k(e'')} \quad \text{for all } e', e'' \in N_k \text{ with } d(e', e'') \leq 4A^{-k}r.$$

Then there exists a probability measure μ_{k+1} on N_{k+1} such that

(1) For all $g', g'' \in N_{k+1}$ with $d(g', g'') \leq 4A^{-k-1}r$, we have

$$(4.10) \quad \frac{\mu_{k+1}(g')}{c_{k+1}(g')} \leq C_2^2 \frac{\mu_{k+1}(g'')}{c_{k+1}(g'')}.$$

(2) Let $\delta \in (0, 1)$ be the constant in Lemma 4.6. For all points $e \in N_k$ and $g \in S_k(e)$,

$$(4.11) \quad C_2^{-1} \frac{\mu_k(e)}{c_k(e)} \leq \frac{\mu_{k+1}(g)}{c_{k+1}(g)} \leq (1 - \delta) \frac{\mu_k(e)}{c_k(e)}.$$

(3) The construction of the measure μ_{k+1} from the measure μ_k can be regarded as the transfer of masses from the points N_k to those of N_{k+1} , with no mass transferred over a distance greater than $(1 + 4/A)A^{-k}r$.

Remark 4.8. The key differences from the lemma in [VK87] are, first, that we require the relations (4.9), (4.10) and (4.11) for the ratios μ_k/c_k rather than just for μ_k , and second, the presence of the term $1 - \delta$ in the right-hand inequality in (4.11).

Proof. By Lemma 4.4(f) we have $|S_k(x)| \leq C_M$ for all x, k . Set

$$C_2 = C_1 C_M,$$

where C_1 is the constant in (4.6). Let μ_k be a probability measure N_k satisfying (4.9).

Let $e \in N_k$; we will construct $\mu_{k+1}(g)$ for $g \in S_k(e)$ by mass transfer. Initially we distribute the mass $\mu_k(e)$ to $g \in S_k(e)$ so that the mass of $g \in S_{k+1}(e)$ is proportional to $c_{k+1}(g)$. We therefore set

$$f_0(g) = \frac{c_{k+1}(g)}{\sum_{g' \in S_k(e)} c_{k+1}(g')} \mu_k(e) \quad \text{for all } e \in N_k \text{ and } g \in S_k(e).$$

By (4.5) and Lemmas 4.5 and 4.6, we have

$$(4.12) \quad C_2^{-1} \frac{\mu_k(e)}{c_k(e)} \leq \frac{f_0(g)}{c_{k+1}(g)} \leq (1 - \delta) \frac{\mu_k(e)}{c_k(e)}$$

for all points $e \in N_k$ and $g \in S_k(e)$. If the measure f_0 on N_{k+1} satisfies condition (1) of the lemma, we set $\mu_{k+1} = f_0$. Condition (2) is satisfied by (4.12), and (3) is obviously satisfied by Lemma 4.4(c).

If f_0 does not satisfy condition (1) of the lemma, then we proceed to adjust the masses of the points in N_{k+1} in the following fashion. Let p_1, \dots, p_T be

the pairs of points $\{g', g''\}$ with $g', g'' \in N_{k+1}$ with $0 < d(g', g'') \leq 4A^{-k-1}r$. We begin with the pair $p_1 = \{g'_1, g''_1\}$. If

$$\frac{f_0(g'_1)}{c_{k+1}(g'_1)} \leq C_2^2 \frac{f_0(g''_1)}{c_{k+1}(g''_1)} \quad \text{and} \quad \frac{f_0(g''_1)}{c_{k+1}(g''_1)} \leq C_2^2 \frac{f_0(g'_1)}{c_{k+1}(g'_1)},$$

then we set $f_1 = f_0$. If one of the inequalities is violated, say the first, then we define the measure f_1 by a suitable transfer of mass from g'_1 to g''_1 . We set $f_1(g) = f_0(g)$ for $g \neq g'_1, g''_1$ and set $f_1(g'_1) = f_0(g'_1) - \alpha_1$, $f_1(g''_1) = f_0(g''_1) + \alpha_1$, where $\alpha_1 > 0$ is chosen such that

$$\frac{f_1(g'_1)}{c_{k+1}(g'_1)} = C_2^2 \frac{f_1(g''_1)}{c_{k+1}(g''_1)}.$$

We then consider the pair p_2 and construct the measure f_2 from f_1 in exactly the same way, by a suitable mass transfer between the points in the pair if this is necessary. Continuing we obtain a sequence of measures f_j , and we find that $\mu_{k+1} = f_T$ is the desired measure in the lemma.

The proof that μ_{k+1} satisfies properties (1)–(3) is almost the same as in [VK87]. We note that a key property of the construction is that we cannot have chains of mass transfers: as in [VK87], there are no pairs $p_j = (g_1, g_2)$, $p_{j+i} = (g_2, g_3)$ such that at step j mass is transferred from g_1 to g_2 , and then at a later step $j + i$ mass is transferred from g_2 to g_3 ; see [VK87, p. 633]. \square

To construct the doubling measure in Proposition 4.3 we use Lemma 4.7 for large scales and rely on (BG) for small scales.

Proof of Proposition 4.3. Recall that $A = 8$. Let μ_0 be the probability measure on $N_0 = \{x_0\}$. We use Lemma 4.7 to inductively construct probability measures μ_k on N_k . For $x \in B_0$, by $E_k(x)$ we denote the unique $y \in N_k$ such that $Q_k(x) = Q_k(y)$. Note that by the construction,

$$(4.13) \quad d(x, E_k(x)) < A^{-k}r, \quad P_{k+1}(E_{k+1}(x)) = E_k(x)$$

for all $x \in B_0$ and for all $k \in \mathbb{Z}_+$. Let l denote the smallest non-negative integer such that $A^{-l}r \leq r_0/A^2$; since $r \geq r_0$, we have $l \geq 2$. The desired measure $\nu = \nu_{x_0, r}$ is given by

$$f(z) = \alpha \sum_{y \in N_l} \frac{\mu_l(y)}{m(Q_l(y))} \mathbf{1}_{Q_l(y)}(z), \quad \nu(dz) = f(z) m(dz),$$

where $\alpha > 0$ is chosen so that $f(x_0) = 1$. Note that we have $\mu_l(x) = \alpha^{-1}\nu(Q_l(x))$ for all $x \in N_l$.

First, we show (4.3) and (4.4). By the argument in [Kan85, Lemma 2.5] there exists $C_3 > 1$ (which does not depend on r) such for any pair of points $x, y \in B_0$ that can be connected by a geodesic that stays within B_0 , there exists sequence of points $E_l(x) = y_0, y_1, \dots, y_{n-1}, y_n = E_l(y)$ in N_l , with $n \leq C_3(1 + d(x, y))$ and $d(y_i, y_{i+1}) \leq 4A^{-l}r$ for all $i = 0, 1, \dots, n-1$. By comparing

successive $\mu_l(y_i)$ using Lemma 4.7(3) and by comparing successive $m(Q_l(y))$ using the volume doubling property of m at small scales (2.7), we obtain (4.3) and (4.4).

For rest of the proof we can without loss of generality assume that $\alpha = 1$ in the definition of ν . If $x \in N_k$, then in the mass transfer from μ_k to μ_l each piece of mass moves a distance at most $(1 + 4A^{-1}) \sum_{i=k}^l A^{-i}r$. An additional distance of at most $A^{-l}r$ is then travelled in the transfer from μ_l to ν . Since $A \geq 8$, the mass $\mu_k(x)$ from $x \in N_k$ travels a distance of at most

$$(4.14) \quad (1 + 4A^{-1}) \sum_{i=k}^l A^{-i}r + A^{-l}r < 2A^{-k}r.$$

Next we show that there exists C_4 such that

$$(4.15) \quad \mu_{M+1}(E_{M+1}(x)) \leq \nu(B(x, s)) \leq C_4\mu_{M+1}(E_{M+1}(x))$$

for all $x \in B_0$ and for all $A^{-l+1}r < s < r$. Here $M = M(s) \in \mathbb{Z}_+$ is the unique integer such that $s/A \leq A^{-M}r < s$. Note that $M \leq l - 1$.

By (4.14), the mass transfer of $\mu_{M+1}(E_{M+1}(x))$ from the point $E_{M+1}(x)$ takes place over a distance at most $2A^{-M-1}r \leq 2s/8$. Since $d(x, E_{M+1}(x)) \leq A^{-M-1}r < s/8$, the triangle inequality gives the lower bound in (4.15).

To prove the upper bound, recall from (4.14) that none of the mass in $N_{M-1} \setminus B(x, s + 2A^{-M+1}r)$ of μ_{M-1} falls in $B(x, s)$. This implies that

$$(4.16) \quad \nu(B(x, s)) \leq \mu_{M-1}(N_{M-1} \cap B(x, s + 2A^{-M+1}r)).$$

Since $s \leq A^{-M+1}r$ and N_{M-1} is an $A^{-M+1}r$ -net, by (MD) there exists $C_5 > 1$ such that

$$(4.17) \quad |N_{M-1} \cap B(x, s + 2A^{-M+1}r)| \leq C_5.$$

By the triangle inequality, $d(x, E_{M-1}(x), y) < 4A^{-M+1}r$ for all $y \in N_{M-1} \cap B(x, s + 2A^{-M+1}r)$. Therefore by (4.16), (4.17) and Lemmas 4.7 and 4.5, there exists $C_6 > 1$ such that

$$(4.18) \quad \nu(B(x, s)) \leq C_6\mu_{M-1}(E_{M-1}(x)).$$

Combining (4.18) along with (4.13) and Lemma 4.5, we obtain the desired upper bound in (4.15).

For small balls we rely on (BG) as follows. If $B(x, s) \subset B_0$, $s \leq A^{-l+2}r$, $y \in B(x, s)$, there exists $C_7 > 1$ such that $E_l(x)$ and $E_l(y)$ can be connected by a chain of points in N_l given by $E_l(x) = z_0, z_1, \dots, z_{N-1}, z_N = E_l(y)$ with $N \leq C_7$. This can be shown by essentially the same argument as [Kan85, Lemma 2.5] or [MSC15, Prop. 2.16(d)]. Combining this with (2.7) and Lemmas 4.7 and 4.5, we obtain that there exists $C_8 > 1$ such that

$$C_8^{-1}f(x) \leq f(y) \leq C_8f(x).$$

Therefore for all balls $B(x, s) \subset B_0$ with $s < A^{-l+2}r$, we have

$$(4.19) \quad C_8^{-1}f(x)m(B(x, s)) \leq \nu(B(x, s)) \leq C_8f(x)m(B(x, s)).$$

Combining (4.15) with Lemmas 4.7 and 4.5, we obtain the volume doubling property for ν for all balls whose radius s satisfies $A^{-l+1}r < s < r$. The estimate (4.19) and (BG) for the measure m implies the volume doubling property for balls $B(x, s)$ with $s \leq A^{-l+1}r$ and $B(x, 2s) \subset B_0$. This completes the proof of the doubling property given in (4.1).

It remains to verify (4.2). By an application of EHI, (4.7), (4.8) along with Lemmas 3.5, 3.12 and 3.3, there exists $C_9 > 1$ such that

$$(4.20) \quad C_9^{-1}c_{M+1}(E_{M+1}(x)) \leq \text{Cap}_{B(x,As)}(B(x, s)) \leq C_9c_{M+1}(E_{M+1}(x))$$

for all $x \in B_0$, for all $A^{-l+1}r < s \leq r$, where $M = M(s)$ is as above.

Equations (4.15), (4.19) and (4.20) link the ν -measure and capacity of balls with those of the generalized cubes $Q_{k,i}$, while Lemmas 4.5 and 4.7 link ν -measures and capacities of the $Q_{k,i}$ with their successors and neighbors. Using these links, as well as the regularity on small scales given by Assumption 2.5, (4.2) follows by a straightforward argument. \square

Proof of Theorem 4.2. Fix $x_0 \in \mathcal{X}$. For $n \geq 1 \vee r_0$, let $\nu_{x_0,n}$ be the measure given by Proposition 4.3, and let

$$f_n := \frac{d\nu_{x_0,n}}{dm}.$$

Then $f_n \in L^\infty(B(x_0, n), m)$ and by (4.3), we have for $1 \leq k \leq n$ that

$$(4.21) \quad C_0^{-1-k} \leq \text{ess inf}_{B(x_0,k)} f_n \leq \text{ess sup}_{B(x_0,k)} f_n \leq C_0^{1+k}.$$

A compactness argument similar to that in [LS98] yields the existence of a subsequence n_k and a measurable function f , bounded on compacts, such that

$$(4.22) \quad \int_{\mathcal{X}} hf \, dm = \lim_{k \rightarrow \infty} \int_{\mathcal{X}} hf_{n_k} \, dm$$

for all $h \in L^1(\mathcal{X}, m)$ with compact support. Taking $d\mu = f dm$ then yields the required measure. \square

5. Quasi-symmetry, time change, and the EHI

We recall the definition of quasisymmetry; see [Hei01] for a nice exposition to this theory. For many results in this section, we do not require that the metric space (\mathcal{X}, d) be a length space.

Definition 5.1. A *distortion function* is a homeomorphism of $[0, \infty)$ onto itself. Let η be a distortion function. A map $f : (\mathcal{X}_1, d_1) \rightarrow (\mathcal{X}_2, d_2)$ between metric spaces is said to be η -*quasisymmetric* if f is a homeomorphism and

$$\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \leq \eta \left(\frac{d_1(x, a)}{d_1(x, b)} \right)$$

for all triples of points $x, a, b \in \mathcal{X}_1, x \neq b$. We say f is a *quasisymmetry* if it is η -quasisymmetric for some distortion function η . We say that metric spaces (\mathcal{X}_1, d_1) and (\mathcal{X}_2, d_2) are *quasisymmetric* if there exists a quasisymmetry $f : (\mathcal{X}_1, d_1) \rightarrow (\mathcal{X}_2, d_2)$. We say that metrics d_1 and d_2 on \mathcal{X} are *quasisymmetric* if the identity map $\text{Id} : (\mathcal{X}, d_1) \rightarrow (\mathcal{X}, d_2)$ is a quasisymmetry.

If $f : (\mathcal{X}_1, d_1) \rightarrow (\mathcal{X}_2, d_2)$ is η -quasisymmetric, then $f^{-1} : (\mathcal{X}_2, d_2) \rightarrow (\mathcal{X}_1, d_1)$ is ζ -quasisymmetric, where $\zeta(t) = 1/\eta^{-1}(1/t)$. Quasi-symmetry is an equivalence relation among metric spaces [Hei01, Prop. 10.6]. The following comparison of annuli follows readily from the definition.

LEMMA 5.2 (see [MT10, Lemma 1.2.18]). *Let the identity map $\text{Id} : (\mathcal{X}, d_1) \rightarrow (\mathcal{X}, d_2)$ be an η -quasisymmetry for some distortion function η . Then for all $A > 1, x \in \mathcal{X}, r > 0$, there exists $s > 0$ such that, writing B_i for balls in (\mathcal{X}, d_i) ,*

$$(5.1) \quad B_2(x, s) \subset B_1(x, r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s).$$

Moreover, for all $A > 1, x \in \mathcal{X}, r > 0$, there exists $s > 0$ such that

$$(5.2) \quad B_1(x, r) \subset B_2(x, s) \subset B_2(x, As) \subset B_1(x, A_1r),$$

where $A_1 = 1/\eta^{-1}(A^{-1})$.

The property of being a length metric space is not preserved under a quasisymmetric change of metric. Nevertheless, many properties that are relevant to heat kernel estimates and Harnack inequalities are preserved under such transformations. For instance, it is well known that the metric doubling property (MD) is a quasisymmetry invariant [Hei01, Th. 10.18]. It follows easily from Lemma 5.2 that quasisymmetric metrics have the same doubling measures. The EHI is also a quasisymmetry invariant as shown in the following easy but important lemma.

LEMMA 5.3. *Let $(\mathcal{X}, d_i, \mu, \mathcal{E}, \mathcal{F}^\mu), i = 1, 2$ be two MMD spaces such that d_1 and d_2 are quasisymmetric. If $(\mathcal{X}, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies EHI, then so does $(\mathcal{X}, d_1, \mu, \mathcal{E}, \mathcal{F}^\mu)$.*

Proof. Let $C_H, A > 1$ be constants corresponding to EHI for $(\mathcal{X}, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$. Then by (5.2), we have EHI for $(\mathcal{X}, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ with constants $C_H, A_1 > 1$, where A_1 is as given in Lemma 5.2. □

We introduce the notion of a regular scale function.

Definition 5.4. We say that a function $\Psi : \mathcal{X} \times [0, \infty) \rightarrow [0, \infty)$ on a metric space (\mathcal{X}, d) is a *regular scale function* if $\Psi(x, 0) = 0$ for all x , and there exist $C_1, \beta_1, \beta_2 > 0$ such that, for all $x, y \in \mathcal{X}$, $0 < s \leq r$ we have, writing $d(x, y) = R$,

$$(5.3) \quad C_1^{-1} \left(\frac{r}{R \vee r} \right)^{\beta_2} \left(\frac{R \vee r}{s} \right)^{\beta_1} \leq \frac{\Psi(x, r)}{\Psi(y, s)} \leq C_1 \left(\frac{r}{R \vee r} \right)^{\beta_1} \left(\frac{R \vee r}{s} \right)^{\beta_2}.$$

We now recall the notion of *uniform perfectness* [Hei01, §11.1].

Definition 5.5. (1) A metric space (\mathcal{X}, d) is *uniformly perfect* if there exists $C > 1$ so that for each $x \in \mathcal{X}$, and for each $r > 0$, the set $B(x, r) \setminus B(x, r/C)$ is non-empty whenever $\mathcal{X} \setminus B(x, r)$ is non-empty.

(2) A measure μ satisfies the *reverse doubling property* (RVD) if there exist constants C_0 and $\alpha > 0$ such that

$$(5.4) \quad \frac{\mu(B(x, r))}{\mu(B(x, s))} \geq C_0 (r/s)^\alpha \quad \text{for } x \in \mathcal{X}, 0 < s \leq r.$$

Remark 5.6. Every connected metric space is uniformly perfect. Uniform perfectness is a quasisymmetry invariant; see [Hei01, Exercise 11.2]. If μ satisfies (VD) and (\mathcal{X}, d) is uniformly perfect, then μ satisfies (RVD); see [Hei01, Exercise 13.1].

Next, we associate a quasisymmetric metric d_Ψ to any regular scale function Ψ on (\mathcal{X}, d) , such that d_Ψ relates nicely to Ψ .

PROPOSITION 5.7. *Let Ψ be a regular scale function on a metric space (\mathcal{X}, d) . There exists a metric $d_\Psi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ satisfying the following properties:*

(a) *There exist $C, \beta > 0$ such that for all $x, y \in \mathcal{X}$, we have*

$$(5.5) \quad C^{-1} \Psi(x, d(x, y)) \leq d_\Psi(x, y)^\beta \leq C \Psi(x, d(x, y)).$$

(b) *d and d_Ψ are quasisymmetric.*

(c) *Assume in addition that (\mathcal{X}, d) (or equivalently (\mathcal{X}, d_Ψ)) has infinite diameter and is uniformly perfect. Fix $A > 1$. Let B_Ψ and B denote metric balls in (\mathcal{X}, d_Ψ) and (\mathcal{X}, d) respectively. If either $B_\Psi(x, s) \subset B(x, r) \subset B_\Psi(x, As)$ or $B(x, r) \subset B_\Psi(x, s) \subset B(x, Ar)$ holds for some $x \in \mathcal{X}, r > 0, s > 0$, then there is a constant $C_1 > 1$ (which does not depend on $x \in \mathcal{X}, r > 0, s > 0$) such that*

$$(5.6) \quad C_1^{-1} s^\beta \leq \Psi(x, r) \leq C_1 s^\beta,$$

where $\beta > 0$ is as given by (5.5).

Proof. (a) Let $D(x, y) = \Psi(x, d(x, y)) + \Psi(y, d(x, y))$. Using (5.3) it is straightforward to check that there exists $K \geq 1$ such that $D(x, y) \leq$

$K(D(x, z) + D(z, y))$ for all $x, y, z \in \mathcal{X}$. Therefore by [Hei01, Prop. 14.5] and (5.3), there exists a metric d_Ψ on \mathcal{X} and $\beta > 0$ that satisfies (5.5).

(b) By (5.3) and (5.5), there exists $C_1 > 0, 0 < \gamma_1 \leq \gamma_2$ such that

$$C_1^{-1} \left(\frac{d(x, a)}{d(x, b)} \right)^{\gamma_1} \leq \frac{d_\Psi(x, a)}{d_\Psi(x, b)} \leq C_1 \left(\frac{d(x, a)}{d(x, b)} \right)^{\gamma_2}$$

for all $x, a, b \in \mathcal{X}$ that satisfy $d(x, a) \geq d(x, b) > 0$. (We can take $\gamma_i = \beta_i/\beta$.) Therefore the identity map $\text{Id} : (\mathcal{X}, d) \rightarrow (\mathcal{X}, d_\Psi)$ is quasimetric where the homeomorphism η in Definition 5.1 can be chosen as $\eta(t) = C_1 \max(t^{\gamma_1}, t^{\gamma_2})$.

(c) As the two cases are very similar, we just treat the case $B_\Psi(x, s) \subset B(x, r) \subset B_\Psi(x, As)$. By uniform perfectness, there exists $C_2 > 1$ such that there are points $y_1 \in B_\Psi(x, s) \setminus B_\Psi(x, s/C_2)$ and $y_2 \in B_\Psi(x, C_2As) \setminus B_\Psi(x, s)$. The upper bound in (5.6) follows from

$$\Psi(x, r) \leq c\Psi(x, d(x, y_2)) \leq c'd_\Psi(x, y_2)^\beta \leq c''s^\beta,$$

where we used (5.3) and $d(x, y_2) \geq r$ in the first estimate, (5.5) in the second, and $y_2 \in B_\Psi(x, C_2As)$ in the final estimate. Similarly, the lower bound in (5.6) follows from

$$\Psi(x, r) \geq c\Psi(x, d(x, y_1)) \geq c'd_\Psi(x, y_1)^\beta \geq c''s^\beta,$$

where we used (5.3) and $d(x, y_1) \leq r$ in the first estimate, (5.5) in the second, and $y_1 \notin B_\Psi(x, s/C_2)$ in the final estimate. □

We now introduce Poincaré and cutoff energy inequalities with respect to a regular scale function Ψ . Recall that a *cutoff function* φ for $B_1 \subset B_2$ is any function $\varphi \in \mathcal{F}^\mu$ such that $0 \leq \varphi \leq 1$ in \mathcal{X} , $\varphi \equiv 1$ in an open neighborhood of $\overline{B_1}$, and $\text{supp } \varphi \Subset B_2$.

Definition 5.8. Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a MMD space, and let Ψ be a regular scale function. We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *Poincaré inequality* $\text{PI}(\Psi)$ if there exist constants $C, A \geq 1$ such that for all $x \in \mathcal{X}, R > 0$ and $f \in \mathcal{F}^\mu$,

$$\text{PI}(\Psi) \quad \int_{B(x, R)} (f - \bar{f})^2 d\mu \leq C\Psi(x, R) \int_{B(x, AR)} d\Gamma(f, f),$$

where $\bar{f} = \mu(B(x, R))^{-1} \int_{B(x, R)} f d\mu$.

We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *cutoff energy inequality* $\text{CS}(\Psi)$ if there exist $C_1, C_2 > 0, A > 1$ such that the following holds. For all $R > 0, x \in \mathcal{X}$ with $B_1 = B(x, R), B_2 = B(x, AR)$, there exists a cutoff function φ for $B_1 \subset B_2$ such that for any $u \in \mathcal{F}^\mu \cap L^\infty$,

$$\text{CS}(\Psi) \quad \int_{B_2 \setminus B_1} u^2 d\Gamma(\varphi, \varphi) \leq C_1 \int_{B_2 \setminus B_1} d\Gamma(u, u) + \frac{C_2}{\Psi(x, R)} \int_{B_2 \setminus B_1} u^2 d\mu.$$

If there exists $\beta > 0$ such that $\Psi(x, r) = r^\beta$ for all $x \in \mathcal{X}, r > 0$, we denote the properties $\text{PI}(\Psi)$ and $\text{CS}(\Psi)$ by $\text{PI}(\beta)$ and $\text{CS}(\beta)$ respectively.

The following lemma shows that the Poincaré and cutoff energy inequalities take a much simpler form with respect to the metric d_Ψ .

LEMMA 5.9. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be an unbounded, uniformly perfect MMD space, and let Ψ be a regular scale function. Let d_Ψ be the metric constructed in Proposition 5.7 with $\beta > 0$ as given in (5.5). Then*

- (a) $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\text{PI}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\text{PI}(\beta)$;
- (b) $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\text{CS}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\text{CS}(\beta)$.

Proof. As before, we denote balls in the d_Ψ and d metrics by B_Ψ and B respectively.

(a) Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfy $\text{PI}(\Psi)$ with constants $C, A \geq 1$. By (5.2), there exists $A' > 1$ such that for all $x \in \mathcal{X}, r > 0$, there exists $r' = r'(r) > 0$ such that

$$(5.7) \quad B_\Psi(x, r) \subset B(x, r') \subset B(x, Ar') \subset B_\Psi(x, A'r).$$

Let $x \in X, r > 0$ be arbitrary, and let $r' > 0, A' > 1$ be given as above. By $\text{PI}(\Psi)$, (5.7), and Proposition 5.7(c), there exists $C' > 1$ such that

$$(5.8) \quad \begin{aligned} \int_{B(x, r')} |f - f_{B(x, r')}|^2 d\mu &\leq C\Psi(x, r') \int_{B(x, Ar')} d\Gamma(f, f) \\ &\leq C'r^{\beta} \int_{B_\Psi(x, A'r)} d\Gamma(f, f) \end{aligned}$$

for all $f \in \mathcal{F}^\mu$. Further, for all $f \in L^2(\mathcal{X}, \mu)$,

$$(5.9) \quad \begin{aligned} \int_{B(x, r')} |f - f_{B(x, r')}|^2 d\mu &= \min_{a \in \mathbb{R}} \int_{B(x, r')} |f - a|^2 d\mu \\ &\geq \min_{a \in \mathbb{R}} \int_{B_\Psi(x, r)} |f - a|^2 d\mu = \int_{B_\Psi(x, r)} |f - f_{B_\Psi(x, r)}|^2 d\mu. \end{aligned}$$

The Poincaré inequality $\text{PI}(\beta)$ for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ follows from (5.8) and (5.9). The converse is similar.

(b) Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfy $\text{CS}(\Psi)$ with constants $C_1, C_2, A \geq 1$. Let $x \in X, r > 0$ be arbitrary, and let $r' > 0, A' > 1$ be as given in (5.7). By $\text{CS}(\Psi)$, there exists a cutoff function φ for $B(x, r') \subset B(x, Ar')$ such that for

any $u \in \mathcal{F}^\mu \cap L^\infty$,

$$(5.10) \quad \int_{B(x,Ar') \setminus B(x,r')} u^2 d\Gamma(\varphi, \varphi) \leq C_1 \int_{B(x,Ar') \setminus B(x,r')} d\Gamma(u, u) + \frac{C_2}{\Psi(x, r')} \int_{B(x,Ar') \setminus B(x,r')} u^2 d\mu.$$

Clearly by (5.7), φ is also a cutoff function for $B_\Psi(x, r) \subset B_\Psi(A'r)$. Since $\text{supp } \Gamma(\varphi, \varphi) \subset B(x, Ar') \setminus B(x, r')$, by (5.7), we have

$$(5.11) \quad \int_{B_\Psi(x,A'r) \setminus B_\Psi(x,r')} u^2 d\Gamma(\varphi, \varphi) = \int_{B(x,Ar') \setminus B(x,r')} u^2 d\Gamma(\varphi, \varphi).$$

Combining (5.10), (5.11), (5.7), and Proposition 5.7(c), we obtain the cutoff energy inequality $\text{CS}(\beta)$ for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$. The converse is again similar. \square

We will extend $\text{CS}(\Psi)$ to an inequality for cutoff functions for $B(x, R) \subset B(x, R + r)$. We will use the following elementary inequality involving energy measures.

LEMMA 5.10. *Let $(\mathcal{E}, \mathcal{F}^\mu)$ be a regular Dirichlet form on $L^2(\mathcal{X}, \mu)$ with energy measure $\Gamma(\cdot, \cdot)$. Then for any quasi-continuous functions $f, \varphi_1, \varphi_2 \in \mathcal{F}^\mu \cap L^\infty$, we have*

$$\int_{\mathcal{X}} f^2 d\Gamma(\varphi_1 \vee \varphi_2, \varphi_1 \vee \varphi_2) \leq \int_{\mathcal{X}} f^2 d\Gamma(\varphi_1, \varphi_1) + \int_{\mathcal{X}} f^2 d\Gamma(\varphi_2, \varphi_2).$$

Proof. Let $\varphi_0 = \varphi_1 \vee \varphi_2$. By [FOT94, Th. 1.4.2(i), (ii)], we have $\varphi_0 \in \mathcal{F}^\mu, f^2 \in \mathcal{F}^\mu$. By [FOT94, last equation on p. 206], for each j , we have

$$\int_{\mathcal{X}} f^2 d\Gamma(\varphi_j, \varphi_j) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{f^2, \mu} \left((\varphi_j(Y_t) - \varphi_j(Y_0))^2 \right).$$

Here $\mathbb{E}_{f^2, \mu}$ denotes the expectation where Y_0 has the distribution $f^2 d\mu$. Combining this with the elementary estimate,

$$(\varphi_0(Y_t) - \varphi_0(Y_0))^2 \leq \max_{i=1,2} (\varphi_i(Y_t) - \varphi_i(Y_0))^2 \leq \sum_{i=1}^2 (\varphi_i(Y_t) - \varphi_i(Y_0))^2,$$

we obtain the desired inequality. \square

The cutoff energy inequality $\text{CS}(\Psi)$ has the following self improving property.

PROPOSITION 5.11 (cutoff energy inequality for all annuli). *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfy (MD) and $\text{CS}(\Psi)$ for some regular scale function Ψ . There exist $C_E, \gamma > 0$ such that the following holds. For all $R > 0, r > 0, x_0 \in \mathcal{X}$ with*

$B_1 = B(x_0, R)$, $B_2 = B(x_0, R + r)$ and $U = B_2 \setminus B_1$, there exists a cutoff function φ for $B_1 \subset B_2$ such that for any $f \in \mathcal{F}^\mu \cap L^\infty$,

$$(5.12) \quad \int_U f^2 d\Gamma(\varphi, \varphi) \leq \frac{1}{8} \int_U d\Gamma(f, f) + C_E \left(\frac{R+r}{r}\right)^\gamma \frac{1}{\Psi(x_0, r)} \int_U f^2 d\mu.$$

Proof. Let $f \in \mathcal{F}^\mu \cap L^\infty$. Let $A > 1$, C_1, C_2 be the constants in **CS**(Ψ). Replacing A by $\lceil A \rceil$ if necessary, we can assume that $A \in \mathbb{N}$. Let $n \geq 8(A+8)$, and cover $B(x_0, R+r)$ by balls $B_i = B(z_i, r/n)$, $i \in I$ such that $z_i \in B(x_0, R+r)$ and the balls $B(z_i, r/2n)$ are disjoint. Then using (MD) there exists a constant N (which does not depend on n) such that any $y \in U$ is in at most N of the balls $B_i^* = B(z_i, Ar/n)$. Let $U_i = B_i^* \setminus B_i$.

By **CS**(Ψ), there exists a cutoff function φ_i for $B_i \subset B_i^*$ such that

$$(5.13) \quad \int_{U_i} f^2 d\Gamma(\varphi_i, \varphi_i) \leq C_1 \int_{U_i} d\Gamma(f, f) + \frac{C_2}{\Psi(z_i, r/n)} \int_{U_i} f^2 d\mu.$$

Now let $2 \leq j \leq n - A - 1$, $j \in \mathbb{N}$, and let $I_j = \{i \in I : z_i \in B(x_0, R + jr/n)\}$. Set

$$\psi_j = \max_{i \in I_j} \varphi_i.$$

Then $\psi_j \equiv 1$ on $B(x_0, R + (j - 1)r/n)$, and it is zero outside $B(x_0, R + (j + A)r/n)$. Thus ψ_j is a cutoff function for $B(x_0, R + (j - 2)r/n) \subset B(x_0, R + (j + A + 1)r/n)$. We have $d(z_i, x_0) \leq R + r$ for all $i \in I$, so using (5.3),

$$(5.14) \quad \frac{\Psi(x_0, r)}{\Psi(z_i, r/n)} \leq C \left(\frac{R+r}{r}\right)^{\beta_2 - \beta_1} n^{\beta_2}.$$

Let $V_j = B(x_0, R + (j + A + 1)r/n) \setminus B(x_0, R + (j - 2)r/n)$, so that $\text{supp}(\Gamma(\psi_j, \psi_j)) \subset V_j$.

Let h_j be a cutoff function for $\text{supp}(\Gamma(\psi_j, \psi_j)) \subset V_j$. By **Lemma 5.10**,

$$(5.15) \quad \begin{aligned} \int_{\mathcal{X}} f^2 d\Gamma(\psi_j, \psi_j) &= \int_{\mathcal{X}} f^2 h_j d\Gamma(\psi_j, \psi_j) \\ &\leq \sum_{i \in I_j} \int_{\mathcal{X}} f^2 h_j d\Gamma(\varphi_i, \varphi_i) \leq \sum_{i \in I_j} \int_{V_j} f^2 d\Gamma(\varphi_i, \varphi_i). \end{aligned}$$

Now let

$$\varphi = \frac{1}{n - 2A - 4} \sum_{j=A+3}^{n-A-2} \psi_j.$$

Then φ is a cutoff function for $B(x_0, R) \subset B(x_0, R + r)$. Since every point in $B(x_0, R + r)$ is in the support of at most $A + 4$ of the energy measures $\Gamma(\psi_j, \psi_j)$, by Cauchy-Schwarz inequality we have

$$(5.16) \quad \int_{\mathcal{X}} f^2 d\Gamma(\varphi, \varphi) \leq (A + 4)(n - 2A - 4)^{-2} \sum_{j=A+3}^{n-A-2} \int_{\mathcal{X}} f^2 d\Gamma(\psi_j, \psi_j).$$

Combining (5.15) and (5.16),

$$\int_{\mathcal{X}} f^2 d\Gamma(\varphi, \varphi) \leq (A + 4)(n - 2A - 4)^{-2} \sum_{j=A+3}^{n-A-2} \sum_{i \in I} \int_{V_j} f^2 d\Gamma(\varphi_i, \varphi_i).$$

Set $\tilde{I} = \{i \in I : \text{supp}(\Gamma(\varphi_i, \varphi_i)) \subset B(x_0, R + r) \setminus B(x_0, R)\}$. If $A + 2 \leq j \leq n - A - 1$ and $\text{supp}(\Gamma(\varphi_i, \varphi_i)) \cap V_j \neq \emptyset$, by the triangle inequality $\text{supp}(\Gamma(\varphi_i, \varphi_i)) \subset B(x_0, R + r) \setminus B(x_0, R)$. Therefore it suffices to consider only the indices $i \in \tilde{I}$ in (5.15). Since for each i , $\text{supp}(\Gamma(\varphi_i, \varphi_i))$ intersects at most $4(A + 4)$ different V_j 's, we have,

$$(5.17) \quad \int_{\mathcal{X}} f^2 d\Gamma(\varphi, \varphi) \leq 4(A + 4)^2(n - 2A - 4)^{-2} \sum_{i \in \tilde{I}} \int_{B(x_0, R+r) \setminus B(x_0, R)} f^2 d\Gamma(\varphi_i, \varphi_i).$$

Combining (5.17), (5.13), and (5.14), and using that every point is in at most N different B_i^* , we obtain

$$\begin{aligned} & \int_{\mathcal{X}} f^2 d\Gamma(\varphi, \varphi) \\ & \leq \frac{4(A + 4)^2}{(n - 2A - 4)^2} \left(C_1 \sum_{i \in \tilde{I}} \int_{U_i} d\Gamma(f, f) + \frac{C_2 C n^{\beta_2}}{\Psi(x_0, r)} \left(\frac{R + r}{r} \right)^{\beta_2 - \beta_1} \sum_{i \in \tilde{I}} \int_{U_i} f^2 d\mu \right) \\ & \leq \frac{4N(A + 4)^2}{(n - 2A - 4)^2} \left(C_1 \int_U d\Gamma(f, f) + \frac{C_2 C n^{\beta_2}}{\Psi(x_0, r)} \left(\frac{R + r}{r} \right)^{\beta_2 - \beta_1} \int_U f^2 d\mu \right). \end{aligned}$$

Finally, we choose n large enough so that $4N(A + 4)^2(n - 2A - 4)^{-2}C_1 \leq 1/8$. □

Remark 5.12. Note that this quite general argument enables us to deduce a cutoff energy inequality on arbitrary annuli from $\text{CS}(\Psi)$; see [MSC17, Lemma 2.1]. Further, if $\Psi(x, r) = \Psi(y, r)$ for all $x, y \in X$ and $r > 0$, we can modify the proof by using (5.3) with $x = y$ instead of using (5.14), so that $\gamma = 0$ in (5.12).

Definition 5.13. Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a MMD space, and let Ψ be a regular scale function. We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the capacity estimate $(\text{cap})_\Psi$ if there exist $\kappa \in (0, 1)$ and $C > 1$ such that for any ball $x \in \mathcal{X}, r > 0$,

$$(\text{cap})_\Psi \quad C^{-1} \frac{\mu(B(x, r))}{\Psi(x, r)} \leq \text{Cap}_{B(x, r)}(B(x, \kappa r)) \leq C \frac{\mu(B(x, r))}{\Psi(x, r)}.$$

If $\Psi(x, r) = r^\beta$ for all $x \in \mathcal{X}, r > 0$, we denote the property $(\text{cap})_\Psi$ by $(\text{cap})_\beta$.

We will now apply these results in the context of a change of measure on an MMD space. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ be a MMD space that satisfies the EHI and Assumptions 2.3 and 2.5. Let $(\mathcal{E}, \mathcal{F}_e)$ denote the corresponding extended Dirichlet space (cf. [FOT94, Lemma 1.5.4]), and let μ be the measure

constructed in [Theorem 4.2](#). By construction, μ is a positive Radon measure charging no set of capacity zero and possessing full support. Let $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ denote the time changed Dirichlet space with respect to μ . We have that $\mathcal{F}^m = \mathcal{F}_e \cap L^2(\mathcal{X}, m)$, $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X}, \mu)$ and $\mathcal{E}^\mu(f, f) = \mathcal{E}(f, f)$ for all $f \in \mathcal{F}^\mu$; cf. [[FOT94](#), p. 275]. Moreover, the domain of the extended Dirichlet space is the same for both the Dirichlet forms $(\mathcal{E}, \mathcal{F}^m, L^2(\mathcal{X}, m))$ and $(\mathcal{E}, \mathcal{F}^\mu, L^2(\mathcal{X}, \mu))$.

THEOREM 5.14. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ be a length MMD space that satisfies the EHI and [Assumptions 2.3](#) and [2.5](#). Let μ be the measure constructed in [Theorem 4.2](#). Then the function Ψ defined by $\Psi(x, 0) = 0$ and*

$$(5.18) \quad \Psi(x, r) = \frac{\mu(B(x, r))}{\text{Cap}_{B(x, r)}(B(x, r/8))}, \quad r > 0,$$

is a regular scale function. Furthermore, the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality [PI\(\$\Psi\$ \)](#) and the cutoff energy inequality [CS\(\$\Psi\$ \)](#).

Proof. By volume doubling and [Corollary 3.13](#), there exists $C_2 > 0$ such that for all $r > 0$ and for all $x, y \in \mathcal{X}$ with $d(x, y) \leq r$, we have

$$(5.19) \quad C_2^{-1}\Psi(x, r) \leq \Psi(y, r) \leq C_2\Psi(x, r).$$

If $R \leq r$, the inequalities are immediate from property (b) in [Theorem 4.2](#) and [\(5.19\)](#). If $s < r < R$, then writing

$$\frac{\Psi(x, r)}{\Psi(y, s)} = \frac{\Psi(x, r)}{\Psi(x, R)} \cdot \frac{\Psi(y, R)}{\Psi(y, s)} \cdot \frac{\Psi(x, R)}{\Psi(y, R)}$$

and bounding each of the three terms on the right using [Theorem 4.2](#) and [\(5.19\)](#) gives [\(5.3\)](#). Thus Ψ is a regular scale function.

Let d_Ψ and $\beta > 0$ be as given by [Proposition 5.7](#). Write $B_\Psi(\cdot, \cdot)$ for balls in the d_Ψ metric. We now show that $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies [\(cap\) \$_\beta\$](#) . By [Lemma 5.2](#), there exist $A > 8, \kappa \in (0, 1)$ such that for all $x \in X, r > 0$,

$$B(x, s_1) \subset B_\Psi(x, \kappa r) \subset B(x, s_2) \subset B(x, 8s_2) \subset B_\Psi(x, r) \subset B(x, As_1)$$

for some $s_1, s_2 > 0$. By domain monotonicity of capacity, we have

$$(5.20) \quad \text{Cap}_{B(x, As_1)}(B(x, s_1)) \leq \text{Cap}_{B_\Psi(x, r)}(B_\Psi(x, \kappa r)) \leq \text{Cap}_{B(x, 8s_2)}(B(x, s_2)).$$

By [Proposition 5.7\(c\)](#) and the regularity of Ψ , s_1 and s_2 are both comparable with $\Psi(x, s_1) \asymp \Psi(x, s_2) \asymp r^\beta$. Therefore by (VD), [Lemmas 3.5](#) and [3.12](#), [\(3.13\)](#), and [\(5.20\)](#), we have

$$(5.21) \quad \text{Cap}_{B(x, As_1)}(B(x, s_1)) \asymp \text{Cap}_{B(x, 8s_2)}(B(x, s_2)) \asymp \frac{\mu(B(x, s_2))}{\Psi(x, s_2)} \asymp \frac{\mu(B_\Psi(x, r))}{r^\beta}.$$

Combining [\(5.20\)](#) and [\(5.21\)](#), we have that $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies [\(cap\) \$_\beta\$](#) . By [Lemma 5.3](#) and [Proposition 5.7\(b\)](#), $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI.

By [Remark 5.6](#) the space (\mathcal{X}, d_Ψ) is uniformly perfect, and the measure μ on (\mathcal{X}, d_Ψ) satisfies (RVD). Thus by [[GHL15](#), Th. 1.2], since $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI and $(\text{cap})_\beta$, it satisfies $\text{PI}(\beta)$ and $\text{CS}(\beta)$. We now conclude using [Lemma 5.9](#). \square

THEOREM 5.15. *Let (\mathcal{X}, d) be a complete, locally compact, length metric space with a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F}^m)$ on $L^2(\mathcal{X}, m)$ that satisfies [Assumptions 2.3](#) and [2.5](#). The following are equivalent:*

- (a) $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ satisfies the EHI.
- (b) There exist a doubling Radon measure μ on (\mathcal{X}, d) that is mutually absolutely continuous with respect to m and a regular scale function Ψ , such that the time-changed MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality $\text{PI}(\Psi)$ and the cutoff energy inequality $\text{CS}(\Psi)$.
- (c) There exist a doubling Radon measure μ on (\mathcal{X}, d) that is mutually absolutely continuous with respect to m , a metric d_Ψ on \mathcal{X} that is quasisymmetric to d , and $\beta > 0$, such that the time-changed MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality $\text{PI}(\beta)$ and the cutoff energy inequality $\text{CS}(\beta)$ for some $\beta > 0$.

Proof. (a) \Rightarrow (b) This follows from [Theorem 5.14](#).

(b) \Rightarrow (c) Let d_Ψ and $\beta > 0$ be as given by [Proposition 5.7](#). Quasisymmetry of d_Ψ follows from [Proposition 5.7\(b\)](#). Then $\text{PI}(\beta)$ and $\text{CS}(\beta)$ for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ follow from [Lemma 5.9](#).

(c) \Rightarrow (a) By [Remark 5.6](#), (\mathcal{X}, d) and therefore (\mathcal{X}, d_Ψ) are uniformly perfect. Thus μ satisfies (RVD). By [Proposition 5.11](#) and [Remark 5.12](#), we obtain the condition (CSA) in [[GHL15](#)]. Then by [[GHL15](#), Th. 1.2], we obtain EHI for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$. Since d_Ψ and d are quasisymmetric, the desired EHI follows from [Lemma 5.3](#). \square

Proof of [Theorem 1.3](#). The relation $\mathcal{E} \asymp \mathcal{E}'$ implies that the energy measure $d\Gamma'(f, f)$ for \mathcal{E}' satisfies

$$(5.22) \quad C^{-1}d\Gamma(f, f) \leq d\Gamma'(f, f) \leq Cd\Gamma(f, f) \text{ for all } f \in \mathcal{F};$$

see [[LJ78](#), Prop. 1.5.5(b)]. This implies stability of Poincaré and cutoff energy inequalities under such perturbations. Therefore, the desired EHI follows from stability of property (b) in [Theorem 5.15](#) (or alternatively (c)). \square

We remark that the cutoff energy inequality in [Theorems 5.15](#) and [5.14](#) could be replaced by the slightly weaker generalized capacity estimate given in [[GHL15](#)].

Remark 5.16. (1) The approach using quasisymmetry given in this section implicitly contains an alternate proof to the main results in [[Bas13](#)].

(2) [Theorem 5.15](#) shows that, after suitable transformations of measure and metric, the stability of EHI follows from the stability of the $\text{PHI}(\beta)$; see [[BBK06](#), p. 485 and Def. 2.1(d)] for the definition of $\text{PHI}(\beta)$. It is known that the index satisfies $\beta \geq 2$; see [[Hin02](#), p. 252]. One might ask if we can improve [Theorem 5.15\(c\)](#) to obtain $\text{PHI}(2)$. A paper in preparation [[KM](#)] shows that this is not possible in general, but on the other hand the Sierpinski gasket provides a non-trivial example where this is possible; see [[Kig08](#)]. See [[Kaj13](#), §9] for further discussion on this problem.

(3) The constant $\beta > 0$ in [Theorem 5.15](#) can be made arbitrarily large by a “snowflake transform” of the metric $d_\Psi \mapsto d_\Psi^\varepsilon$, where $\varepsilon \in (0, 1)$. We can ask how small β can be. Recall that a *conformal gauge* on a set \mathcal{X} is a maximal collection of metrics on \mathcal{X} such that each pair of metrics from the collection are quasisymmetric. By analogy with conformal Hausdorff dimension (see [[MT10](#), Def. 2.2.1] or [[Hei01](#), p. 121]), we can define the *conformal walk dimension* of a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ as the infimum of all β such that there exist a quasisymmetric metric d_Ψ and a Revuz measure μ with full support such that the time changed MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\text{PHI}(\beta)$. The conformal walk dimension is always at least 2, and by [Theorem 5.15](#) it is finite if and only if the space satisfies EHI. This raises the following questions: Can the conformal walk dimension be finite and strictly greater than 2? Is the infimum in the definition of conformal walk dimension always attained?

(4) By [[GHL15](#), Th. 1.2] the modified space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies heat kernel upper and lower bounds; see [[GHL15](#)] for details.

(5) The classical parabolic Harnack inequality $\text{PHI}(2)$ implies that vector space of harmonic functions with fixed polynomial growth is finite dimensional [[CM97](#), Th. 0.7]. This result of Colding and Minicozzi settled a conjecture of Yau on manifolds with non-negative Ricci curvature. This result was extended by P. Li [[Li97](#), Th. 1] to spaces satisfying a mean value inequality for harmonic functions with respect to a doubling measure. This theorem of Li along with our doubling measure μ in [Theorem 4.2](#) implies that the vector space of harmonic functions with fixed polynomial growth is finite dimensional on any space satisfying the EHI. Note that one cannot directly use [[Li97](#), Th. 1] to obtain the above result because there are manifolds that satisfy EHI but whose Riemannian measure is not doubling.

6. Examples: Weighted Riemannian manifolds and graphs

In this section we return to our two main examples, and give sufficient conditions for these spaces to satisfy the local regularity hypotheses [2.3](#) and [2.5](#).

We first recall some standard definitions in Riemannian geometry. Let (\mathcal{X}, g) be a Riemannian manifold, and let ν and ∇ denote the Riemannian measure and the Riemannian gradient respectively. In local coordinates

(x_1, x_2, \dots, x_n) , we have

$$\nabla f = \sum_{i,j=1}^n g^{i,j} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}, \quad d\nu = \sqrt{\det g(x)} dx,$$

where $\det g$ denotes the determinant of the metric tensor $(g_{i,j})$ and $(g^{i,j}) = (g_{i,j})^{-1}$ is the co-metric tensor. For a function $f \in C^\infty(\mathcal{X})$, we denote the length of the gradient by $|\nabla f| = (g(\nabla f, \nabla f))^{1/2}$. The Laplace-Beltrami operator Δ is given in local coordinates by

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{i,j} \sqrt{\det g} \frac{\partial}{\partial x_j} \right).$$

A *weighted manifold* (\mathcal{X}, g, μ) is a Riemannian manifold (\mathcal{X}, g) endowed with a measure μ that has a smooth (strictly) positive density w with respect to ν . Let w be the smooth function such that

$$d\mu = w d\nu.$$

On the weighted manifold (M, g, μ) , one associates a *weighted Laplace operator* Δ_μ given by

$$\Delta_\mu f = \Delta f + g(\nabla(\ln w), \nabla f) \text{ for all } f \in C^\infty(\mathcal{X}).$$

We say that the weighted manifold (M, g, μ) has *controlled weights* if the function w defined above satisfies

$$\sup_{x,y \in \mathcal{X}: d(x,y) \leq 1} \frac{w(x)}{w(y)} < \infty,$$

where d is the Riemannian distance function. The corresponding Dirichlet form on $L^2(\mathcal{X}, \mu)$ is given by

$$\mathcal{E}(f_1, f_2) = \int_{\mathcal{X}} g(\nabla f_1, \nabla f_2) d\mu, \quad f_1, f_2 \in \mathcal{F},$$

where \mathcal{F} is the weighted Sobolev space of functions in $L^2(\mathcal{X}, \mu)$ whose distributional gradient is also in $L^2(\mathcal{X}, \mu)$. We refer the reader to Grigor'yan's survey [Gri06] for details of the construction of the heat kernel, Markov semigroup and Brownian motion on weighted manifolds for motivation, as well as applications.

Our second example is weighted graphs. Let $\mathbb{G} = (\mathbb{V}, E)$ be an infinite graph, such that each vertex x has finite degree. For $x \in \mathbb{V}$, we write $x \sim y$ if $\{x, y\} \in E$. For $D \subset \mathbb{V}$, define

$$\partial D = \{y \in D^c : y \sim x \text{ for some } x \in D\}.$$

We define a metric on \mathbb{V} by taking $d(x, y)$ to be the length of the shortest path connecting x and y . We define balls by

$$B_d(x, r) = \{y \in \mathbb{V} : d(x, y) < r\}.$$

Let $w : E \rightarrow (0, \infty)$ be a function that assigns weight w_e to the edge e . We write w_{xy} for $w_{\{x,y\}}$ and define

$$(6.1) \quad w_x = \sum_{y \sim x} w_{xy}.$$

We extend w to a measure on \mathbb{V} by setting $w(A) = \sum_{x \in A} w_x$. We call (\mathbb{V}, E, w) a *weighted graph*. An *unweighted graph* has $w_e \equiv 1$.

The Dirichlet form associated with this weighted graph is given by taking

$$\mathcal{E}_{\mathbb{G}}(f, f) = \frac{1}{2} \sum_x \sum_{y \sim x} w_{xy} (f(y) - f(x))^2,$$

with domain $\mathcal{F} = \{f \in L^2(\mathbb{V}, w) : \mathcal{E}_{\mathbb{G}}(f, f) < \infty\}$. We define the Laplacian on \mathbb{G} by setting

$$\Delta_{\mathbb{G}} f(x) = \frac{1}{w_x} \sum_{y \sim x} w_{xy} (f(y) - f(x)).$$

We say that a function h is *harmonic* on a set $D \subset \mathbb{V}$ if $\Delta_{\mathbb{G}} h(x) = 0$ for all $x \in D$. (Note that for $\Delta_{\mathbb{G}} h(x)$ to be defined for $x \in D$, we need h to be defined on the set $D \cup \partial D$.)

The statement of the elliptic Harnack inequality for a weighted graph is analogous to the EHI for a MMD space. We say $\mathbb{G} = (V, E, w)$ satisfies the EHI if there exists $C_H < \infty$ such that if $x_0 \in \mathbb{V}$, $R \geq 1$, and $h : B(x_0, 2R+1) \rightarrow \mathbb{R}_+$ is harmonic in $B(x_0, 2R)$, then

$$\sup_{B_d(x_0, R)} h \leq C_H \inf_{B_d(x_0, R)} h.$$

The *cable system of a weighted graph* gives a natural embedding of a graph in a connected metric length space. Choose a direction for each edge $e \in E$, let $(I_e, e \in E)$ be a collection of copies of the open unit interval, and set

$$\mathcal{X} = \mathbb{V} \cup (\cup_{e \in E} I_e).$$

(Following [Var85] we call the sets I_e *cables*.) We define a metric d_c on \mathcal{X} by using Euclidean distance on each cable. If $x \in \mathbb{V}$ and $e = (x, y)$ is an oriented edge, we set $d_c(x, t) = 1 - d_c(y, t) = t$ for $t \in I_e$. We then extend d_c to a metric on \mathcal{X} ; note that this agrees with the graph metric for $x, y \in \mathbb{V}$. We take m to be the measure on \mathcal{X} that assigns zero mass to points in \mathbb{V} , and mass $w_e |s - t|$ to any interval $(s, t) \subset I_e$. For more details on this construction. see [Var85], [BB04].

We say that a function f on \mathcal{X} is *piecewise differentiable* if it is continuous at each vertex $x \in \mathbb{V}$, is differentiable on each cable, and has one-sided derivatives at the endpoints. Let \mathcal{F}_0 be the set of piecewise differentiable functions f with compact support. Given two such functions we set

$$d\Gamma(f, g)(t) = f'(t)g'(t)m(dt).$$

(While the sign of f' and g' depends on the orientation of the cable, this does not affect their product.) We then define

$$\mathcal{E}(f, g) = \int_{\mathcal{X}} d\Gamma(f, g)(t), \quad f, g \in \mathcal{F}_0,$$

and we take \mathcal{F} to be the completion of \mathcal{F}_0 with respect to the norm

$$\|f\|_{\mathcal{E}_1} = \left(\int f^2 dm + \mathcal{E}(f, f) \right)^{1/2}.$$

We extend \mathcal{E} to \mathcal{F} , and it is straightforward to verify that $(\mathcal{E}, \mathcal{F})$ is a closed regular strongly local Dirichlet form. We call $(\mathcal{X}, d_c, m, \mathcal{E}, \mathcal{F})$ the *cable system* of the graph \mathbb{G} . We define harmonic functions for the cable system as in [Section 1](#).

We remark that (up to a constant time change) the associated Hunt process X behaves like a Brownian motion on each cable and like a “Walsh Brownian motion” (see [\[Wal78\]](#)) at each vertex: starting at x it makes excursions along the cable $I_{\{x,y\}}$ at a rate proportional to w_{xy}/w_x .

There is a natural bijection between harmonic functions on the graph \mathbb{G} and the cable system \mathcal{X} . If h is harmonic on a domain $D \subset \mathcal{X}$, then $h|_{\mathbb{V}}$ satisfies $\Delta_{\mathbb{G}}h(x) = 0$ for any $x \in \mathbb{V}$ such that $B(x, 1) \subset D$. Conversely, let $D_0 \subset \mathbb{V}$, and suppose that $h : D_0 \cup \partial D_0 \rightarrow \mathbb{R}$ is \mathbb{G} -harmonic. Let D be the open subset of \mathcal{X} that consists of D_0 and all cables with an endpoint in D_0 . Define \bar{h} by setting $\bar{h}(x) = h(x)$, $x \in D_0 \cup \partial D_0$ and taking \bar{h} to be linear on each cable. Then \bar{h} is harmonic on D .

Definition 6.1. We say that \mathbb{G} has *controlled weights* if there exists $p_0 > 0$ such that

$$(6.2) \quad \frac{w_{xy}}{w_x} \geq p_0 \text{ for all } x \in \mathbb{V}, y \sim x.$$

This is called the p_0 condition in [\[GT02\]](#). Note that it implies that vertices have degree at most $1/p_0$, so that an unweighted graph satisfies controlled weights if and only if the vertex degrees are uniformly bounded.

LEMMA 6.2. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be the cable system of a weighted graph $\mathbb{G} = (\mathbb{V}, E, w)$. If \mathcal{X} satisfies the EHI with constant C_H , then \mathbb{G} has controlled weights.*

Proof. (By looking at a linear (harmonic) function in a single cable we have that $C_H \geq 3$.) Let $x_0 \in \mathbb{V}$, and let x_i , $i = 1, \dots, n$ be the neighbors of x_0 . Let $r < \frac{1}{2}$, and let y_i, z_i be the points on the cable $\gamma(x_0, x_i)$ with $d(x_0, y_i) = r$, $d(x_0, z_i) = 2r$. Set $p_j = w_{x_0, x_j}/w_{x_0}$.

Let $D = B(x_0, 2r)$ and h_j be the harmonic function in $B(x_0, 2r)$ with $h_j(z_i) = \delta_{ij}$. We have $h_j(x_0) = p_j$, $h_j(y_i) = \frac{1}{2}p_j$ if $i \neq j$ and $h_j(y_j) = \frac{1}{2}(1 + p_j)$.

So using the EHI with $i \neq j$,

$$2h(y_j) = 1 + p_j \leq 2C_H h(y_i) = C_H p_j,$$

which leads to the required lower bound on p_j . □

Remark 6.3. See [Bar05] for an example that shows that the EHI for a weighted graph, as opposed to its cable system, does not imply controlled weights.

It is straightforward to verify

LEMMA 6.4. *Let \mathbb{G} have controlled weights. The EHI holds for \mathbb{G} if and only if it holds for the associated cable system.*

We conclude this section by showing that a large class of weighted manifolds and cable systems satisfy our local regularity hypotheses (BG). To this end, we introduce a local parabolic Harnack inequality which turns out to be strong enough to imply (BG).

Definition 6.5. We say a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the local parabolic Harnack inequality $(\text{PHI}(2))_{\text{loc}}$ if there exist $R > 0, C_R > 0$ such that for all $x \in \mathcal{X}, 0 < r \leq R$, any non-negative weak solution u of $(\partial_t + \mathcal{L})u = 0$ on $(0, r^2) \times B(x, r)$ satisfies

$$(\text{PHI}(2))_{\text{loc}} \quad \sup_{(r^2/4, r^2/2) \times B(x, r/2)} u \leq C_R \inf_{(3r^2/4, r^2) \times B(x, r/2)} u;$$

here \mathcal{L} is the generator corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F}, L^2(\mathcal{X}, \mu))$.

LEMMA 6.6.

- (a) *Let (\mathcal{M}, g, w) be a weighted Riemannian manifold with controlled weights such that (\mathcal{M}, g) is quasi-isometric to a manifold with Ricci curvature bounded below. Then (\mathcal{M}, g, w) satisfies $(\text{PHI}(2))_{\text{loc}}$.*
- (b) *Let $\mathbb{G} = (\mathbb{V}, E, w)$ be a weighted graph with controlled weights. Then its cable system satisfies $(\text{PHI}(2))_{\text{loc}}$.*

Proof. (a) If (\mathcal{M}', g') has Ricci curvature bounded below then (\mathcal{M}', g') satisfies $(\text{PHI}(2))_{\text{loc}}$ by the Li-Yau estimates. By [HS, Th. 2.7], the property $(\text{PHI}(2))_{\text{loc}}$ is stable under quasi-isometries and under introducing controlled weights.

(b) By taking $R < 1$ this reduces to looking at either a single cable (i.e., an interval) or a finite union of cables. See [BM] for more details. □

LEMMA 6.7. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies $(\text{PHI}(2))_{\text{loc}}$. Then $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies Assumption 2.3 and (BG).*

Proof. We refer the reader to [BM] for the proof of [Assumption 2.3](#). By [HSC01, Th. 2.7] the heat kernel on this space satisfies a two-sided Gaussian bound at small time scales. These imply the volume doubling property at small scales.

Using the heat kernel upper bounds given in [HSC01, Lemma 3.9], we obtain the following Green's function upper bound. There exist $A > 1$, $a \in (0, 1)$, $C_0, r_0 > 0$ such that for all $x \in \mathcal{X}$, $r \in (0, r_0)$ and for all $y \in B(x, Ar)$ such that $d(x, y) = ar$, we have

$$g_{B(x, Ar)}(x, y) \leq C_0 \frac{r^2}{m(B(x, r))}.$$

A matching lower bound follows from [HSC01, Lemmas 3.7 and 3.8], after adjusting r_0, a if necessary.

Clearly, $((\text{PHI}(2))_{\text{loc}})$ implies a local EHI for small scales. By using the local EHI along with the results in Section 2 (see [Remark 3.16](#)), there exist $r_0, C_1 > 0$ such that

$$\begin{aligned} C_1^{-1} \frac{m(B(x, r))}{r^2} &\leq \text{Cap}_{B(x, 8r)}(B(x, r)) \\ &\leq \frac{m(B(x, r))}{r^2} \quad \text{for all } x \in \mathcal{X} \text{ for all } r \in (0, r_0). \end{aligned}$$

This implies (2.8) with $\gamma_2 = 2$. Hence (BG) follows. \square

Proof of Theorem 1.4. [Assumption 2.5](#) follows from [Lemmas 6.6](#) and [6.7](#). [Assumption 2.3](#) follows from [BM]. The conclusions now follow from [Theorem 1.3](#). \square

7. Stability under rough isometries

As well as stability of the EHI under bounded perturbation of weights, our results also imply stability under rough isometries.

Definition 7.1. For each $i = 1, 2$, let $(\mathcal{Y}_i, d_i, \mu_i)$ be either a metric measure space or a weighted graph. A map $\varphi : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ is a *rough isometry* if there exist constants $C_1 > 0$ and $C_2, C_3 > 1$ such that

$$(7.1) \quad X_2 = \bigcup_{x \in X_1} B_{d_2}(\varphi(x), C_1),$$

$$(7.2) \quad C_2^{-1}(d_1(x, y) - c_1) \leq d_2(\varphi(x), \varphi(y)) \leq C_2(d_1(x, y) + c_1) \text{ for } x \in \mathcal{Y}_1,$$

$$(7.3) \quad \begin{aligned} C_3^{-1} \mu_1(B_{d_1}(x, C_1)) &\leq \mu_2(B_{d_2}(\varphi(x), c_1)) \\ &\leq C_3 \mu_1(B_{d_1}(x, C_1)) \text{ for } x, y \in \mathcal{Y}_1. \end{aligned}$$

If there exists a rough isometry between two spaces, they are said to be *roughly isometric*. (One can check this is an equivalence relation.)

This concept was introduced by Gromov [Gro81] (under the name quasi-isometry) in the context of groups and by Kanai [Kan85] (under the name rough isometry) for metric spaces; in both cases they just required conditions (7.1) and (7.2). Condition (7.3) is a natural extension when one treats measure spaces; see [CSC95] and [BBK06].

If two spaces are roughly isometric then they have similar large scale structure. However, as the EHI implies some local regularity, we need to impose some local regularity on the spaces in the class we consider.

Definition 7.2. We say a MMD space satisfies a local EHI (denoted EHI_{loc}) if there exist $r_0 \in (0, \infty)$ and $C_L < \infty$ such that whenever $2r < r_0$, $x \in \mathcal{X}$ and h is a non-negative harmonic function on $B(x, 2r)$, then

$$\text{ess sup}_{B(x,r)} h \leq C_L \text{ess inf}_{B(x,r)} h.$$

Remark 7.3. An easy chaining argument shows that if \mathcal{X} satisfies EHI_{loc} with constants r_0 and C_L , then for any $r_1 > r_0$, there exists $C'_L = C_L(r_1)$ such that \mathcal{X} satisfies EHI_{loc} with constants r_1 and C'_L .

Definition 7.4. Let $\mathcal{X} = (\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space. We say \mathcal{X} satisfies *local regularity* (LR) if there exists $r_0 \in (0, 1)$, $C_L < \infty$ such that the following conditions hold:

- (B1) \mathcal{X} satisfies (BG);
- (B2) the Green's function and operator satisfies Assumption 2.3;
- (B3) \mathcal{X} satisfies EHI_{loc} with constants r_0 and C_L ;
- (B4) there exists $C_0 > 0$ such that for all $x_0 \in \mathcal{X}$ and for all $r \in (0, r_0)$, there exists a cutoff function φ for $B(x_0, r/2) \subset B(x_0, r)$ such that

$$\int_{B(x_0,r)} d\Gamma(\varphi, \varphi) \leq C_0 m(B(x_0, r)).$$

The final condition (B4) links m with the energy measure $d\Gamma(\cdot, \cdot)$ at small length scales.

LEMMA 7.5.

- (a) Let (\mathcal{M}, g, w) be a weighted Riemannian manifold with controlled weights such that (\mathcal{M}, g) is quasi-isometric to a manifold with Ricci curvature bounded below. Then (\mathcal{M}, g, w) satisfies (LR).
- (b) Let $\mathbb{G} = (\mathbb{V}, E, w)$ be a weighted graph with controlled weights. Then its cable system satisfies (LR).

Proof. Properties (B1)–(B3) all follow from Lemmas 6.6 and 6.7. For (B4), it is sufficient to look at a cutoff function $\varphi(x)$ that is piecewise linear in $d(x, x_0)$. □

Our main theorem concerning stability under rough isometries is the following.

THEOREM 7.6 (Stability under rough isometries). *Let $\mathcal{X}_i = (\mathcal{X}_i, d_i, m_i, \mathcal{E}_i, \mathcal{F}_i)$, $i = 1, 2$ be MMD spaces that satisfy (LR). Suppose that \mathcal{X}_1 satisfies the EHI, and \mathcal{X}_2 is roughly isometric to \mathcal{X}_1 . Then \mathcal{X}_2 satisfies the EHI.*

Sketch of the proof. The basic approach goes back to the seminal works of Kanai [Kan85], [Kan86b], [Kan86a]; see [CSC95], [HK04], [BBK06] for further developments.

We use the characterization of EHI in [Theorem 5.15](#) and transfer functional inequalities and volume estimates from one space to the other. A key step of this transfer is carried out by a discretization procedure using weighted graphs.

We can approximate an MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ by a weighted graph as follows. For a small enough ε , we choose an ε -net \mathbb{V} of the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ that is a maximal ε -separated subset of X . The set \mathbb{V} forms the vertices of a graph whose edges E are given by $u \sim v$ if and only if $d(u, v) \leq 3\varepsilon$. Define weights by $w_{uv} = m(B(u, \varepsilon)) + m(B(v, \varepsilon))$ if $\{u, v\} \in E$. (Many other choices are possible.) We then define w_x as in [\(6.1\)](#) and hence obtain a measure w on \mathbb{V} . It is easy to verify that the metric measure spaces (\mathcal{X}, d, m) and (\mathbb{V}, E, w) are roughly isometric.

The next step is to transfer functions between MMD space and its net. This transfer of functions has the property that the norms and energy measures are comparable on balls (up to constants and linear scaling of balls), which in turn implies that functional inequalities such as the Poincaré inequality and cutoff energy inequality can be transferred between a MMD space and its net. Using the notation of [\[SC04\]](#), we denote by **rst** a “restriction map” that takes a function $f : \mathcal{X} \rightarrow \mathbb{R}$ on the MMD space to a function $\mathbf{rst}(f) : \mathbb{V} \rightarrow \mathbb{R}$ on the graph defined by

$$\mathbf{rst}(f)(v) = \frac{1}{m(B(v, \varepsilon))} \int_{B(v, \varepsilon)} f(y) m(dy) \text{ for } v \in \mathbb{V}.$$

Similarly, we denote by **ext** an “extension map” that takes a function $f : \mathbb{V} \rightarrow \mathbb{R}$ on the net to a function $\mathbf{ext}(f) : \mathcal{X} \rightarrow \mathbb{R}$ on the MMD space defined by

$$\mathbf{ext}(f)(x) = \sum_{v \in \mathbb{V}} f(v) \chi_v(x),$$

where $(\chi_v)_{v \in \mathbb{V}}$ is a “nice” partition of unity on \mathcal{X} indexed by the vertices of the net V satisfying the following properties:

- (i) $\sum_{v \in \mathbb{V}} \chi_v = 1$;
- (ii) there exists $c \in (0, 1)$ such that $\chi_v \geq c$ on $B(x, \varepsilon/2)$ for all $v \in \mathbb{V}$;
- (iii) $\chi_v \equiv 0$ on $B(v, 2\varepsilon)^c$ for all $v \in \mathbb{V}$;

- (iv) there exists $C > 0$ such that $\chi_v \in \mathcal{F}^m$ and $\mathcal{E}(\chi_v, \chi_v) \leq Cm(B(x, \varepsilon))$ for all $v \in V$.

The maps **rst** and **ext** are (roughly) inverses of each other, and they preserve norms and energy measures on balls. Therefore volume doubling, the Poincaré inequality, and the cutoff energy inequality can be transferred between a MMD space and its net.

A difficulty that is not present in the previous settings in [CSC95], [HK04], [BBK06] arises from the change of measure in the characterization of the EHI in Theorem 5.15. This change of symmetric measure does not affect the energy measures in the cutoff energy and Poincaré inequalities. However the integrals on the left side of the Poincaré inequality and the final integral in the cutoff energy inequality involve the measure μ constructed in Theorem 4.2. Let g be such that $d\mu = gdm$. The integrals for the cutoff energy and Poincaré inequalities on the net then are taken with respect to the measure **rst**(g) $d\mu$. It is easy to verify using (4.3) that the metric measure spaces (\mathcal{X}, d, μ) and the net equipped with the measure **rst**(g) $d\mu$ are roughly isometric, and therefore integrals with respect to the measures $g dm$ and **rst**(g) $d\mu$ are comparable on balls.

Thus, starting with the space \mathcal{X}_1 we take $g_1 = d\mu_1/dm_1$, where μ_1 is the measure given by Theorem 4.2. Write \mathbb{V}_i for the nets for \mathcal{X}_i , $i = 1, 2$. We take $\tilde{g}_1 = \mathbf{rst}(g_1)$ and then transfer \tilde{g}_1 to a function \tilde{g}_2 on \mathbb{V}_2 using the rough isometry between \mathbb{V}_1 and \mathbb{V}_2 . The function $g_2 = \mathbf{ext}(\tilde{g}_2)$ then gives a measure $d\mu_2 = g_2 dm_2$ on \mathcal{X}_2 . As in [CSC95], [HK04], [BBK06] we can then transfer the cutoff energy and Poincaré inequalities across this chain of spaces, and deduce that the space $(\mathcal{X}_2, d_2, \mu_2, \mathcal{E}_2, \mathcal{F}_2)$ satisfies the conditions in Theorem 5.15(b), and therefore satisfies the EHI. \square

Proof of Theorem 1.5. This is a direct consequence of Lemma 7.5 and Theorem 7.6. \square

We conclude this paper by suggesting a characterization of the EHI in terms of capacity, or equivalently effective conductance. Let D be a bounded domain in \mathcal{X} . As in [CF12] we can define a reflected Dirichlet space $\tilde{\mathcal{F}}_D$; the associated diffusion \tilde{X} is the process X reflected on (a) boundary of D . (For the case of manifolds or graphs, this reflected process can be constructed in a straightforward fashion). For disjoint subsets A_1, A_2 of D , define

$$C_{\text{eff}}(A_1, A_2; D) = \inf\{\mathcal{E}_D(f, f) : f|_{A_1} = 1, f|_{A_2} = 0, f \in \tilde{\mathcal{F}}_D\}.$$

Let $\mathcal{D}(x_0, R) = \{(x, y) \in B(x_0, R) : x, y \in B(x_0, R/2), d(x, y) \geq R/3\}$. As in [Bar05] we say that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *dumbbell condition* if there

exists C_D such that for all $x_0 \in \mathcal{X}$, $R > 0$, writing $D = B(x_0, R)$ we have

$$\begin{aligned} & \sup_{(x,y) \in \mathcal{D}(x_0,R)} C_{\text{eff}}(B(x, R/8), B(y, R/8); D) \\ & \leq C_D \inf_{(x,y) \in \mathcal{D}(x_0,R)} C_{\text{eff}}(B(x, R/8), B(y, R/8); D). \end{aligned}$$

[Bar05] asked if the dumbbell condition characterizes EHI. However G. Kozma [Koz05] remarked that a class of spherically symmetric trees satisfy the dumbbell condition, but fail to satisfy EHI. These trees also fail to satisfy (MD). We can therefore modify the question in [Bar05] as follows.

Problem 7.7. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ satisfy (LR), the dumbbell condition and metric doubling. Does this space satisfy the EHI?

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References

- [AB15] S. ANDRES and M. T. BARLOW, Energy inequalities for cutoff functions and some applications, *J. Reine Angew. Math.* **699** (2015), 183–215. [MR 3305925](#). [Zbl 1347.31003](#). <https://doi.org/10.1515/crelle-2013-0009>.
- [Bar05] M. T. BARLOW, Some remarks on the elliptic Harnack inequality, *Bull. London Math. Soc.* **37** no. 2 (2005), 200–208. [MR 2119019](#). [Zbl 1067.31002](#). <https://doi.org/10.1112/S0024609304003893>.
- [BB99] M. T. BARLOW and R. F. BASS, Brownian motion and harmonic analysis on Sierpinski carpets, *Canad. J. Math.* **51** no. 4 (1999), 673–744. [MR 1701339](#). [Zbl 0945.60071](#). <https://doi.org/10.4153/CJM-1999-031-4>.
- [BB04] M. T. BARLOW and R. F. BASS, Stability of parabolic Harnack inequalities, *Trans. Amer. Math. Soc.* **356** no. 4 (2004), 1501–1533. [MR 2034316](#). [Zbl 1034.60070](#). <https://doi.org/10.1090/S0002-9947-03-03414-7>.
- [BBK06] M. T. BARLOW, R. F. BASS, and T. KUMAGAI, Stability of parabolic Harnack inequalities on metric measure spaces, *J. Math. Soc. Japan* **58** no. 2 (2006), 485–519. [MR 2228569](#). [Zbl 1102.60064](#). <https://doi.org/10.2969/jmsj/1149166785>.
- [BM] M. T. BARLOW and M. MURUGAN, Boundary Harnack principle and elliptic Harnack inequality, *J. Math. Soc. Japan*, to appear, 2018. [arXiv 1701.01782](#).
- [Bas13] R. F. BASS, A stability theorem for elliptic Harnack inequalities, *J. Eur. Math. Soc. (JEMS)* **15** no. 3 (2013), 857–876. [MR 3085094](#). [Zbl 1266.31005](#). <https://doi.org/10.4171/JEMS/379>.
- [BG72] E. BOMBIERI and E. GIUSTI, Harnack’s inequality for elliptic differential equations on minimal surfaces, *Invent. Math.* **15** (1972), 24–46. [MR 0308945](#). [Zbl 0227.35021](#). <https://doi.org/10.1007/BF01418640>.

- [CF12] Z.-Q. CHEN and M. FUKUSHIMA, *Symmetric Markov Processes, Time Change, and Boundary Theory*, *London Math. Soc. Monogr. Ser.* **35**, Princeton University Press, Princeton, NJ, 2012. MR 2849840. Zbl 1253.60002.
- [CY75] S. Y. CHENG and S. T. YAU, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** no. 3 (1975), 333–354. MR 0385749. Zbl 0312.53031. <https://doi.org/10.1002/cpa.3160280303>.
- [CM97] T. H. COLDING and W. P. MINICOZZI, II, Harmonic functions on manifolds, *Ann. of Math. (2)* **146** no. 3 (1997), 725–747. MR 1491451. Zbl 0928.53030. <https://doi.org/10.2307/2952459>.
- [CSC95] T. COULHON and L. SALOFF-COSTE, Variétés riemanniennes isométriques à l’infini, *Rev. Mat. Iberoamericana* **11** no. 3 (1995), 687–726. MR 1363211. Zbl 0845.58054. <https://doi.org/10.4171/RMI/190>.
- [Del99] T. DELMOTTE, Parabolic Harnack inequality and estimates of Markov chains on graphs, *Rev. Mat. Iberoamericana* **15** no. 1 (1999), 181–232. MR 1681641. Zbl 0922.60060. <https://doi.org/10.4171/RMI/254>.
- [Del02] T. DELMOTTE, Graphs between the elliptic and parabolic Harnack inequalities, *Potential Anal.* **16** no. 2 (2002), 151–168. MR 1881595. Zbl 1081.39012. <https://doi.org/10.1023/A:1012632229879>.
- [FOT94] M. FUKUSHIMA, Y. ŌSHIMA, and M. TAKEDA, *Dirichlet Forms and Symmetric Markov Processes*, *De Gruyter Stud. Math.* **19**, Walter de Gruyter & Co., Berlin, 1994. MR 1303354. Zbl 0838.31001. <https://doi.org/10.1515/9783110889741>.
- [Gri91] A. A. GRIGOR’YAN, The heat equation on noncompact Riemannian manifolds, *Mat. Sb.* **182** no. 1 (1991), 55–87. MR 1098839. Zbl 0743.58031. <https://doi.org/10.1070/SM1992v072n01ABEH001410>.
- [Gri95] A. GRIGOR’YAN, Heat kernel of a noncompact Riemannian manifold, in *Stochastic Analysis* (Ithaca, NY, 1993), *Proc. Sympos. Pure Math.* **57**, Amer. Math. Soc., Providence, RI, 1995, pp. 239–263. MR 1335475. Zbl 0829.58041. <https://doi.org/10.1090/pspum/057/1335475>.
- [Gri06] A. GRIGOR’YAN, Heat kernels on weighted manifolds and applications, in *The Ubiquitous Heat Kernel*, *Contemp. Math.* **398**, Amer. Math. Soc., Providence, RI, 2006, pp. 93–191. MR 2218016. Zbl 1106.58016. <https://doi.org/10.1090/conm/398/07486>.
- [GH14] A. GRIGOR’YAN and J. HU, Heat kernels and Green functions on metric measure spaces, *Canad. J. Math.* **66** no. 3 (2014), 641–699. MR 3194164. Zbl 1293.35128. <https://doi.org/10.4153/CJM-2012-061-5>.
- [GHL14] A. GRIGOR’YAN, J. HU, and K.-S. LAU, Heat kernels on metric measure spaces, in *Geometry and Analysis of Fractals*, *Springer Proc. Math. Stat.* **88**, Springer, Heidelberg, 2014, pp. 147–207. MR 3276002. Zbl 1323.58018. https://doi.org/10.1007/978-3-662-43920-3_6.
- [GHL15] A. GRIGOR’YAN, J. HU, and K.-S. LAU, Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces,

- J. Math. Soc. Japan* **67** no. 4 (2015), 1485–1549. MR 3417504. Zbl 1331.35152. <https://doi.org/10.2969/jmsj/06741485>.
- [GSC05] A. GRIGOR'YAN and L. SALOFF-COSTE, Stability results for Harnack inequalities, *Ann. Inst. Fourier (Grenoble)* **55** no. 3 (2005), 825–890. MR 2149405. Zbl 1115.58024. Available at http://aif.cedram.org/item?id=AIF_2005_55_3_825_0.
- [GT02] A. GRIGOR'YAN and A. TELCS, Harnack inequalities and sub-Gaussian estimates for random walks, *Math. Ann.* **324** no. 3 (2002), 521–556. MR 1938457. Zbl 1011.60021. <https://doi.org/10.1007/s00208-002-0351-3>.
- [Gro81] M. GROMOV, Hyperbolic manifolds, groups and actions, in *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference* (State Univ. New York, Stony Brook, N.Y., 1978), *Ann. of Math. Stud.* **97**, Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213. MR 0624814. Zbl 0467.53035.
- [HK04] B. M. HAMBLY and T. KUMAGAI, Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries, in *Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Part 2, Proc. Sympos. Pure Math.* **72**, Amer. Math. Soc., Providence, RI, 2004, pp. 233–259. MR 2112125. Zbl 1065.60041.
- [HSC01] W. HEBISCH and L. SALOFF-COSTE, On the relation between elliptic and parabolic Harnack inequalities, *Ann. Inst. Fourier (Grenoble)* **51** no. 5 (2001), 1437–1481. MR 1860672. Zbl 0988.58007. Available at http://aif.cedram.org/item?id=AIF_2001_51_5_1437_0.
- [Hei01] J. HEINONEN, *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001. MR 1800917. Zbl 0985.46008. <https://doi.org/10.1007/978-1-4613-0131-8>.
- [Hin02] M. HINO, On short time asymptotic behavior of some symmetric diffusions on general state spaces, *Potential Anal.* **16** no. 3 (2002), 249–264. MR 1885762. Zbl 1015.47029. <https://doi.org/10.1023/A:1014033208581>.
- [KRS12] A. KÄENMÄKI, T. RAJALA, and V. SUOMALA, Existence of doubling measures via generalised nested cubes, *Proc. Amer. Math. Soc.* **140** no. 9 (2012), 3275–3281. MR 2917099. Zbl 1277.28017. <https://doi.org/10.1090/S0002-9939-2012-11161-X>.
- [Kaj13] N. KAJINO, Analysis and geometry of the measurable Riemannian structure on the Sierpiński gasket, in *Fractal Geometry and Dynamical Systems in Pure and Applied Mathematics. I. Fractals in Pure Mathematics, Contemp. Math.* **600**, Amer. Math. Soc., Providence, RI, 2013, pp. 91–133. MR 3203400. Zbl 1325.28001. <https://doi.org/10.1090/conm/600/11932>.
- [KM] N. KAJINO and M. MURUGAN, in preparation.
- [Kan85] M. KANAI, Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds, *J. Math. Soc. Japan* **37** no. 3 (1985), 391–413. MR 0792983. Zbl 0554.53030. <https://doi.org/10.2969/jmsj/03730391>.

- [Kan86a] M. KANAI, Analytic inequalities, and rough isometries between noncompact Riemannian manifolds, in *Curvature and Topology of Riemannian Manifolds* (Katata, 1985), *Lecture Notes in Math.* **1201**, Springer, Berlin, 1986, pp. 122–137. MR 0859579. Zbl 0593.53026. <https://doi.org/10.1007/BFb0075650>.
- [Kan86b] M. KANAI, Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** no. 2 (1986), 227–238. MR 0833199. Zbl 0577.53031. <https://doi.org/10.2969/jmsj/03820227>.
- [Kas07] M. KASSMANN, Harnack Inequalities: An Introduction, *Bound. Value Probl.* (2007), Art. ID 81415, 21. MR 2291922. Zbl 1144.35002. <https://doi.org/10.1155/2007/81415>.
- [Kig08] J. KIGAMI, Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate, *Math. Ann.* **340** no. 4 (2008), 781–804. MR 2372738. Zbl 1143.28004. <https://doi.org/10.1007/s00208-007-0169-0>.
- [Kig12] J. KIGAMI, *Resistance Forms, Quasisymmetric Maps and Heat Kernel Estimates*, **216** Mem. Amer. Math. Soc. no. 1015, 2012. MR 2919892. Zbl 1246.60099. <https://doi.org/10.1090/S0065-9266-2011-00632-5>.
- [Koz05] G. KOZMA, personal communication, 2005.
- [Kum04] T. KUMAGAI, Heat kernel estimates and parabolic Harnack inequalities on graphs and resistance forms, *Publ. Res. Inst. Math. Sci.* **40** no. 3 (2004), 793–818. MR 2074701. Zbl 1067.60070. Available at <http://projecteuclid.org/euclid.prim/1145475493>.
- [LJ78] Y. LE JAN, Mesures associées à une forme de Dirichlet. Applications, *Bull. Soc. Math. France* **106** no. 1 (1978), 61–112. MR 0508949. Zbl 0393.31008. <https://doi.org/10.24033/bsmf.1864>.
- [Li97] P. LI, Harmonic sections of polynomial growth, *Math. Res. Lett.* **4** no. 1 (1997), 35–44. MR 1432808. Zbl 0880.53039. <https://doi.org/10.4310/MRL.1997.v4.n1.a4>.
- [LY86] P. LI and S.-T. YAU, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** no. 3-4 (1986), 153–201. MR 0834612. Zbl 0611.58045. <https://doi.org/10.1007/BF02399203>.
- [LS98] J. LUUKKAINEN and E. SAKSMAN, *Every complete doubling metric space carries a doubling measure*, **126**, 1998. MR 1443161. Zbl 0897.28007. <https://doi.org/10.1090/S0002-9939-98-04201-4>.
- [Lyo87] T. LYONS, Instability of the Liouville property for quasi-isometric Riemannian manifolds and reversible Markov chains, *J. Differential Geom.* **26** no. 1 (1987), 33–66. MR 0892030. Zbl 0599.60011. <https://doi.org/10.4310/jdg/1214441175>.
- [MT10] J. M. MACKAY and J. T. TYSON, *Conformal Dimension, Univ. Lecture Ser.* **54**, Amer. Math. Soc., Providence, RI, 2010, Theory and application. MR 2662522. Zbl 1201.30002. <https://doi.org/10.1090/ulect/054>.

- [Mos71] J. MOSER, On a pointwise estimate for parabolic differential equations, *Comm. Pure Appl. Math.* **24** (1971), 727–740. MR 0288405. Zbl 0227.35016. <https://doi.org/10.1002/cpa.3160240507>.
- [Mos61] J. MOSER, On Harnack’s theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577–591. MR 0159138. Zbl 0111.09302. <https://doi.org/10.1002/cpa.3160140329>.
- [Mos64] J. MOSER, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.* **17** (1964), 101–134. MR 0159139. Zbl 0149.06902. <https://doi.org/10.1002/cpa.3160170106>.
- [MSC15] M. MURUGAN and L. SALOFF-COSTE, *Harnack Inequalities and Gaussian Estimates for Random Walks on Metric Measure Spaces*, 2015. arXiv 1506.07539.
- [MSC17] M. MURUGAN and L. SALOFF-COSTE, Davies’ method for anomalous diffusions, *Proc. Amer. Math. Soc.* **145** no. 4 (2017), 1793–1804. MR 3601569. Zbl 1360.60152. <https://doi.org/10.1090/proc/13324>.
- [SC92a] L. SALOFF-COSTE, A note on Poincaré, Sobolev, and Harnack inequalities, *Internat. Math. Res. Notices* no. 2 (1992), 27–38. MR 1150597. Zbl 0769.58054. <https://doi.org/10.1155/S1073792892000047>.
- [SC92b] L. SALOFF-COSTE, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.* **36** no. 2 (1992), 417–450. MR 1180389. Zbl 0735.58032. <https://doi.org/10.4310/jdg/1214448748>.
- [SC95] L. SALOFF-COSTE, Parabolic Harnack inequality for divergence-form second-order differential operators, *Potential Anal.* **4** no. 4 (1995), 429–467, Potential theory and degenerate partial differential operators (Parma). MR 1354894. Zbl 0840.31006. <https://doi.org/10.1007/BF01053457>.
- [SC04] L. SALOFF-COSTE, Analysis on Riemannian co-compact covers, in *Surveys in Differential Geometry. Vol. IX, Surv. Differ. Geom.* **9**, Int. Press, Somerville, MA, 2004, pp. 351–384. MR 2195413. Zbl 1082.31006. <https://doi.org/10.4310/SDG.2004.v9.n1.a10>.
- [Stu96] K. T. STURM, Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality, *J. Math. Pures Appl.* (9) **75** no. 3 (1996), 273–297. MR 1387522. Zbl 0854.35016.
- [Var85] N. T. VAROPOULOS, Long range estimates for Markov chains, *Bull. Sci. Math.* (2) **109** no. 3 (1985), 225–252. MR 0822826. Zbl 0583.60063.
- [VK87] A. L. VOL’BERG and S. V. KONYAGIN, On measures with the doubling condition, *Izv. Akad. Nauk SSSR Ser. Mat.* **51** no. 3 (1987), 666–675. MR 0903629. Zbl 0649.42010. <https://doi.org/10.1070/IM1988v030n03ABEH001034>.
- [Wal78] J. B. WALSH, A diffusion with a discontinuous local time, in *Temps Locaux, Astérisque* **52–53**, Math. Soc. France, Paris, 1978, pp. 37–45. MR 0509476. Zbl 0385.60063.
- [Wu98] J.-M. WU, Hausdorff dimension and doubling measures on metric spaces, *Proc. Amer. Math. Soc.* **126** no. 5 (1998), 1453–1459. MR 1443418. Zbl 0897.28008. <https://doi.org/10.1090/S0002-9939-98-04317-2>.

- [Yau75] S. T. YAU, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201–228. [MR 0431040](#). [Zbl 0291.31002](#). <https://doi.org/10.1002/cpa.3160280203>.

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