Boundary Harnack principle and elliptic Harnack inequality

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Abstract

We prove a scale-invariant boundary Harnack principle for inner uniform domains over a large family of Dirichlet spaces. A novel feature of our work is that we do not assume volume doubling property for the symmetric measure.

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1 Introduction

Let $(\mathcal{X}, d)$ be a metric space, and assume that associated with this space is a structure which gives a family of harmonic functions on domains $D \subset \mathcal{X}$. (For example, $\mathbb{R}^d$ with the usual definition of harmonic functions.) The *elliptic Harnack inequality* (EHI) holds if there exists a constant C_H such that, whenever h is non-negative and harmonic in a ball $B(x, r)$, then, writing $\frac{1}{2}B = B(x, r/2)$,

$$\operatorname*{ess\,sup}_{\frac{1}{2}B} h \le C_H \operatorname*{ess\,inf}_{\frac{1}{2}B} h. \tag{1.1}$$

Thus the EHI controls harmonic functions in a domain D away from the boundary ∂D. On the other hand, the *boundary Harnack principle* (BHP) controls the ratio of two positive harmonic functions near the boundary of a domain. The BHP given in [Anc] states that if $D \subset \mathbb{R}^d$ is a Lipschitz domain, $\xi \in \partial D$, $r > 0$ is small enough, then for any pair u, v of non-negative harmonic functions in D which vanish on $\partial D \cap B(\xi, 2r)$,

$$\frac{u(x)}{v(x)} \le C \frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap B(\xi, r). \tag{1.2}$$

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The BHP is a key component in understanding the behaviour of harmonic functions near the boundary. It will in general lead to a characterisation of the Martin boundary, and there is a close connection between BHP and a Carleson estimate – see [ALM, Aik08]. (See also [Aik08] for a discussion of different kinds of BHP.)

The results in [Anc] have been extended in several ways. The first direction has been to weaken the smoothness hypotheses on the domain D; for example [Aik01] proves a BHP for uniform domains in Euclidean space. A second direction is to consider functions which are harmonic with respect to more general operators. The standard Laplacian is the (infinitesimal) generator of the semigroup for Brownian motion, and it is natural to ask about the BHP for more general Markov processes, with values in a metric space $(\mathcal{X}, d)$. In [GyS] the authors prove a BHP for inner uniform domains in a measure metric space $(\mathcal{X}, d, m)$ with a Dirichlet form which satisfies the standard parabolic Harnack inequality (PHI). These results are extended in [L] to spaces which satisfy a parabolic Harnack inequality with anomalous space-time scaling. In most cases the BHP has been proved for Markov processes which are symmetric, but see [LS] for the BHP for some more general processes. All the papers cited above study the harmonic functions associated with continuous Markov processes: see [Bog, BKK] for a BHP for a class of jump processes.

The starting point for this paper is the observation that the BHP is a purely elliptic result, and one might expect that the proof would only use elliptic data. However, the generalizations of the BHP beyond the Euclidean case in [GyS, LS, L] all use parabolic data, or more precisely bounds on the heat kernel of the process.

The main result of this paper is as follows. See Sections 2, 3 and 4 for unexplained definitions and notation.

Theorem 1.1. *Let $(\mathcal{X}, d)$ be a complete, separable, locally compact, length space, and let μ be a non atomic Radon measure on $(\mathcal{X}, d)$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular strongly local Dirichlet form on $L^2(\mathcal{X}, \mu)$. Assume that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the elliptic Harnack inequality, and has Green functions which satisfy the regularity hypothesis Assumption 4.9. Let $U \subsetneq \mathcal{X}$ be an inner uniform domain. Then there exist $A_0, C_1 \in (1, \infty)$, $R(U) \in (0, \infty]$ such that for all $\xi \in \partial_{\widetilde{U}} U$, for all $0 < r < R(U)$ and any two non-negative functions u, v that are harmonic on $B_U(\xi, A_0 r)$ with Dirichlet boundary conditions along $\partial_{\widetilde{U}} U \cap B_{\widetilde{U}}(\xi, 2A_0 r)$, we have*

$$\operatorname*{ess\,sup}_{x \in B_U(\xi,r)} \frac{u(x)}{v(x)} \le C_1 \operatorname*{ess\,inf}_{x \in B_U(\xi,r)} \frac{u(x)}{v(x)}.$$

The constant $R(U)$ depends only on the inner uniformity constants of U and diameter(U)*, and can be chosen to be $+\infty$ if U is unbounded.*

Remark 1.2. (1) The constant A_0 above depends only on the inner uniformity constants for the domain U, and C_1 depends only on these constants and the constants in the EHI. (2) Since the EHI is weaker than the PHI, our result extends the BHP to a wider class of spaces; also our approach has the advantage that we can dispense with heat kernel bounds. Our main result provides new examples of differential operators that satisfy the BHP even in $\mathbb{R}^n$ – see [GS, (2.1) and Example 6.14].

(3) By the standard oscillation lemma (see [GT2, Lemma 5.2]), any locally bounded harmonic function admits a continuous version. The elliptic Harnack inequality implies that any non-negative harmonic function is locally bounded. Therefore, under our assumptions, every non-negative harmonic function admits a continuous version.
(4) Let μ' be a measure which is mutually absolutely continuous with respect to the measure μ in the Theorem above, and suppose that $d\mu'/d\mu$ is bounded away from 0 and infinity on compact subsets of $\mathcal{X}$. Then (see Remark 4.13) this change of measure does not change the family of harmonic functions, or the Green functions, and the hypotheses of Theorem 1.1 hold for $(\mathcal{X}, d, \mu', \mathcal{E}, \mathcal{F})$. On the other hand, heat kernel bounds and parabolic Harnack inequality are not in general preserved by such a change of measure because $d\mu'/d\mu$ need not be bounded away from 0 or infinity on $\mathcal{X}$.

The contents of this paper are as follows. In Section 2 we give the definition and basic properties of inner uniform domains in length spaces. Section 3 reviews the properties of Dirichlet forms and the associated Hunt processes. In Section 4 we give the definition of harmonic function in our context, state Assumption 4.9, and give some consequences. In particular, we prove the essential technical result that Green functions are locally in the domain of the Dirichlet form – see Lemma 4.10. The key comparisons of Green functions, which follow from the EHI, and were proved in [BM], are given in Proposition 4.11. In the second part of this section we give some sufficient conditions for Assumption 4.9 to hold, in terms of local ultracontracivity. We conclude Section 4 with two examples: weighted manifolds and cable systems of graphs.

After these rather lengthy preliminaries, Section 5 gives the proof of Theorem 1.1. We follow Aikawa's approach in [Aik01], which proved the BHP for uniform domains in $\mathbb{R}^n$. This method has been adapted to more general settings [ALM, GyS, LS, L]. The papers [GyS, LS, L] all consider domains in more general metric spaces, and use heat kernel estimates to obtain two sided estimates for the Green function in a domain; these estimates are then used in the proof of the BHP. For example [L, Lemma 4.5] gives upper and lower bounds on $g_D(x, y)$ when D is a domain of diameter R, and the points x, y are separated from ∂D and each other by a distance greater than δR. These bounds are of the form $\Psi(R)/\mu(B(x, R))$; here $\Psi : [0, \infty) \to [0, \infty)$ is a global space time scaling function. (See [L] for the precise statement.) In our argument we use instead the comparison of Green functions given by Proposition 4.11.

We use c, c', C, C' for strictly positive constants, which may change value from line to line. Constants with numerical subscripts will keep the same value in each argument, while those with letter subscripts will be regarded as constant throughout the paper. The notation $C_0 = C_0(a)$ means that the constant C_0 depends only on the constant a.

2 Inner uniform domains

In this section we introduce the geometric assumptions on the underlying metric space, and the corresponding domains.

Definition 2.1 (Length space)**.** Let $(\mathcal{X}, d)$ be a metric space. The length $L(\gamma) \in [0, \infty]$

of a continuous curve $\gamma : [0,1] \to \mathcal{X}$ is given by

$$L(\gamma) = \sup \sum_i d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1]$. It is clear that $L(\gamma) \geq d(\gamma(0), \gamma(1))$. A metric space is a *length space* if $d(x,y)$ is equal to the infimum of the lengths of continuous curves joining x and y.

For the rest of this paper we will assume that $(\mathcal{X}, d)$ is a complete, separable, locally compact, length space. We write $\overline{A}$ and ∂A for the closure and boundary respectively of a subset A in $\mathcal{X}$. By the Hopf–Rinow–Cohn-Vossen theorem (cf. [BBI, Theorem 2.5.28]) every closed metric ball in $(\mathcal{X}, d)$ is compact. It also follows that there exists a geodesic path $\gamma(x,y)$ (not necessarily unique) between any two points $x, y \in \mathcal{X}$. We write $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$ for open balls in $(\mathcal{X}, d)$.

Next, we introduce the intrinsic distance d_U induced by an open set $U \subset \mathcal{X}$.

Definition 2.2 (Intrinsic distance)**.** Let $U \subset \mathcal{X}$ be a connected open subset. We define the *intrinsic distance* d_U by

$$d_U(x,y) = \inf \{L(\gamma) : \gamma : [0,1] \to U \text{ continuous}, \gamma(0) = x, \gamma(1) = y\}.$$

It is well-known that (U, d_U) is a length space (cf. [BBI, Exercise 2.4.15]). We now consider its completion.

Definition 2.3 (Balls in intrinsic metric)**.** Let $U \subset \mathcal{X}$ be connected and open. Let $\widetilde{U}$ denote the completion of (U, d_U), equipped with the natural extension of d_U to $\widetilde{U} \times \widetilde{U}$. For $x \in \widetilde{U}$ we define

$$B_{\widetilde{U}}(x,r) = \left\{ y \in \widetilde{U} : d_U(x,y) < r \right\}.$$

Set

$$B_U(x,r) = U \cap B_{\widetilde{U}}(x,r).$$

If $x \in U$, then $B_U(x,r)$ simply corresponds to the open ball in (U, d_U). However, the definition of $B_U(x,r)$ also makes sense for $x \in \widetilde{U} \setminus U$.

Definition 2.4 (Boundary and distance to the boundary)**.** We denote the boundary of U with respect to the inner metric by

$$\partial_{\widetilde{U}} U = \widetilde{U} \setminus U,$$

and the distance to the boundary by

$$\delta_U(x) = \inf_{y \in \partial_{\widetilde{U}} U} d_U(x,y) = \inf_{y \in \mathcal{X} \setminus U} d(x,y).$$

For any open set $V \subset U$, let $\overline{V}^{d_U}$ denote the completion of V with respect to the metric d_U. We denote the boundary of V with respect to $\widetilde{U}$ by

$$\partial_{\widetilde{U}} V = \overline{V}^{d_U} \setminus V.$$

Definition 2.5 (Inner uniform domain)**.** Let U be a connected, open subset of a length space $(\mathcal{X}, d)$. Let $\gamma : [0,1] \to U$ be a rectifiable, continuous curve in U. Let $c_U, C_U \in (0, \infty)$. We say γ is a (c_U, C_U)-*inner uniform curve* if

$$L(\gamma) \leq C_U d_U(\gamma(0), \gamma(1)),$$

and

$$\delta_U(\gamma(t)) \geq c_U \min\left(d_U(\gamma(0), \gamma(t)), d_U(\gamma(1), \gamma(t))\right) \text{ for all } t \in [0,1].$$

The domain U is called a (c_U, C_U)-*inner uniform domain* if any two points in U can be joined by a (c_U, C_U)-inner uniform curve.

The following lemma extends the existence of inner uniform curves between any two points in U in Definition 2.5 to the existence of inner uniform curves between any two points in $\widetilde{U}$.

Lemma 2.6. *Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length space. Let U be a (c_U, C_U)-inner uniform domain and let $\widetilde{U}$ denote the completion of U with respect to the inner metric d_U. Then for any two points x, y in $(\widetilde{U}, d_U)$, there exists a (c_U, C_U)-uniform curve in the d_U metric.*

Proof. Let $x, y \in \widetilde{U}$. There exist sequences $(x_n), (y_n)$ in U such that $x_n \to y, y_n \to y$ in the d_U metric as $n \to \infty$. Let $\gamma_n : [0,1] \to U, n \in \mathbb{N}$ be a (c_U, C_U)-inner uniform curve in U from x_n to y_n with constant speed parametrization. By [BBI, Theorem 2.5.28], the curves γ_n can be viewed as being in the compact space $\overline{B_U(x, 2C_U d_U(x,y))}^{d_U}$ for all large enough n. By a version of Arzela-Ascoli theorem the desired inner uniform curve γ from x to y can be constructed as a sub-sequential limit of the curves (γ_n) – see [BBI, Theorem 2.5.14]. □

The following geometric property of a metric space $(\mathcal{X}, d)$ will play an important role in the paper.

Definition 2.7 (Metric doubling property)**.** We say that a metric space $(\mathcal{X}, d)$ satisfies the *metric doubling property* if there exists $C_M > 0$ such that any ball $B(x, r)$ can be covered by at most C_M balls of radius $r/2$.

Let $\overline{U} \subset \mathcal{X}$ denote the closure of U in $(\mathcal{X}, d)$. Let $p : (\widetilde{U}, d_U) \to (\overline{U}, d)$ denote the natural projection map, that is p is the unique continuous map such that p restricted to U is the identity map on U. The following lemma allows us to compare balls with respect to the d and d_U metrics.

Lemma 2.8. *Let $(\mathcal{X}, d)$ be a complete, length space satisfying the metric doubling property. Let $U \subset \mathcal{X}$ be a connected, open, (c_U, C_U)-inner uniform domain. Then there exists $\widetilde{C_U} > 1$ such that for all balls $B(p(x), r/\widetilde{C_U})$ with $x \in \widetilde{U}$ and $r > 0$, we have*

$$B_{\widetilde{U}}(x, r/\widetilde{C_U}) \subset D' \subset B_{\widetilde{U}}(x, r),$$

where D' is the connected component of $p^{-1}(B(p(x), r/\widetilde{C_U}) \cap \overline{U})$ containing x.

Proof. See [LS, Lemma 3.7] where this is proved under the hypothesis of volume doubling, and note that the argument only uses metric doubling. (Alternatively, a doubling measure exists by [LuS, Theorem 1], and one can then use [LS]). $\square$

The following lemma shows that every point in an inner uniform domain is close to a point that is sufficiently far away from the boundary.

Lemma 2.9. *([GyS, Lemma 3.20]) Let U be a (c_U, C_U)-inner uniform domain in a length space $(\mathcal{X}, d)$. For every inner ball $B = B_{\widetilde{U}}(x, r)$ with the property that $B \neq B_{\widetilde{U}}(x, 2r)$ there exists a point $x_r \in B$ with*

$$d_U(x, x_r) = r/4 \quad \textit{and} \quad \delta_U(x_r) \geq \frac{c_U r}{4}.$$

Lemma 2.10. *Let U be a (c_U, C_U)-inner uniform domain in a length space $(\mathcal{X}, d)$. If $x, y \in U$, then there exists a (c_U, C_U)-inner uniform curve γ connecting x and y with $\delta_U(z) \geq \frac{1}{2} c_U\, (\delta_U(x) \wedge \delta_U(y))$ for all $z \in \gamma$.*

Proof. Write $t = \delta_U(x) \wedge \delta_U(y)$. Let γ be an inner uniform curve from x to y and let $z \in \gamma$. If $d_U(x, z) \leq \frac{1}{2}t$, then $\delta_U(z) \geq \delta_U(x) - d_U(x, z) \geq \frac{1}{2}t$, and the same bound holds if $d_U(y, z) \leq \frac{1}{2}t$. Finally if $d_U(x, z) \wedge d_U(y, z) \geq \frac{1}{2}t$, then $\delta_U(z) \geq \frac{1}{2}c_U t$. $\square$

3 Dirichlet spaces and Hunt processes

Let $(\mathcal{X}, d)$ be a locally compact, separable, metric space and let μ be a Radon measure with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular, strongly local Dirichlet form on $L^2(\mathcal{X}, \mu)$ – see [FOT]. Recall that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *strongly local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with compact supports, such that f is constant in a neighbourhood of $\mathrm{supp}(g)$. We call $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ a *metric measure Dirichlet* space, or *MMD space* for short.

Let $\mathcal{L}$ be the generator of $(\mathcal{E}, \mathcal{F})$ in $L^2(\mathcal{X}, \mu)$; that is $\mathcal{L}$ is a self-adjoint and non-positive-definite operator in $L^2(\mathcal{X}, \mu)$ with domain $\mathcal{D}(\mathcal{L})$ that is dense in $\mathcal{F}$ such that

$$\mathcal{E}(f, g) = -\langle \mathcal{L}f, g \rangle,$$

for all $f \in \mathcal{D}(\mathcal{L})$ and for all $g \in \mathcal{F}$; here $\langle \cdot, \cdot \rangle$, is the inner product in $L^2(\mathcal{X}, \mu)$. The associated *heat semigroup*

$$P_t = e^{t\mathcal{L}}, t \geq 0,$$

is a family of contractive, strongly continuous, Markovian, self-adjoint operators in $L^2(\mathcal{X}, \mu)$. We set

$$\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \langle f, g \rangle, \quad ||f||_{\mathcal{E}_1} = \mathcal{E}_1(f, f)^{1/2}. \tag{3.1}$$

It is known that corresponding to a regular Dirichlet form, there exists an essentially unique Hunt process $X = (X_t, t \geq 0, \mathbb{P}^x, x \in \mathcal{X})$. The relation between the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$ and the associated Hunt process is given by the identity

$$P_t f(x) = \mathbb{E}^x f(X_t),$$

for all $f \in L^\infty(\mathcal{X},\mu)$, for every $t>0$, and for μ-almost all $x \in \mathcal{X}$. Also associated with the Dirichlet form and $f \in \mathcal{F}$ is the energy measure $d\Gamma(f,f)$. This is defined to be the unique Radon measure such that for all $g \in \mathcal{F} \cap \mathcal{C}_c(\mathcal{X})$, we have

$$\int_{\mathcal{X}} g \, d\Gamma(f,f) = 2\mathcal{E}(f,fg) - \mathcal{E}(f^2,g).$$

We have

$$\mathcal{E}(f,f) = \frac{1}{2}\int_{\mathcal{X}} d\Gamma(f,f).$$

Definition 3.1. *For an open subset of U of $\mathcal{X}$, we define the following function spaces associated with $(\mathcal{E},\mathcal{F})$.*

$$\begin{aligned}
\mathcal{F}_{\mathrm{loc}}(U) &= \left\{u \in L^2_{\mathrm{loc}}(U,\mu) : \forall \text{ relatively compact open } V \subset U, \exists u^\# \in \mathcal{F}, u = u^\#\big|_V \ \mu\text{-a.e.}\right\},\\
\mathcal{F}(U) &= \left\{u \in \mathcal{F}_{\mathrm{loc}}(U) : \int_U |u|^2 \, d\mu + \int_U d\Gamma(u,u) < \infty\right\},\\
\mathcal{F}_c(U) &= \{u \in \mathcal{F}(U) : \text{the essential support of } u \text{ is compact in } U\},\\
\mathcal{F}^0(U) &= \text{ the closure of } \mathcal{F}_c(U) \text{ in } \mathcal{F} \text{ in the norm } ||\cdot||_{\mathcal{E}_1}.
\end{aligned}$$

We define capacities for $(\mathcal{X},d,\mu,\mathcal{E},\mathcal{F})$ as follows. Let U be an open subset of $\mathcal{X}$. By $A \Subset U$, we mean that the closure of A is a compact subset of U. For $A \Subset U$ we set

$$\mathrm{Cap}_U(A) = \inf\{\mathcal{E}(f,f) : f \in \mathcal{F}^0(U) \text{ and } f \geq 1 \text{ in a neighbourhood of } A\}. \tag{3.2}$$

A statement depending on $x \in A$ is said to hold *quasi-everywhere on A* (abbreviated as q.e. on A), if there exists a set $N \subset A$ of zero capacity such that the statement is true for every $x \in A \backslash N$. It is known that every function $f \in \mathcal{F}$ admits a quasi continuous version, which is unique up to a set of zero capacity (cf. [FOT, Theorem 2.1.3]). Throughout this paper we will assume that every $f \in \mathcal{F}$ is represented by its quasi-continuous version.

For an open set U an equivalent definition of $\mathcal{F}^0(U)$ is given by

$$\mathcal{F}^0(U) = \{u \in \mathcal{F} : \widetilde{u} = 0 \text{ q.e. on } \mathcal{X} \setminus U\}, \tag{3.3}$$

where $\widetilde{u}$ is a quasi-continuous version of u – see [FOT, Theorem 4.4.3(i)]. Thus we can identify $\mathcal{F}^0(U)$ as a subset of $L^2(U,\mu)$, where in turn $L^2(U,\mu)$ is identified with the subspace $\{u \in L^2(\mathcal{X},\mu) : u = 0 \ \mu\text{-a.e. on } \mathcal{X} \setminus U\}$.

Definition 3.2. For an open set $U \subset \mathcal{X}$, we define the *part of the Dirichlet form $(\mathcal{E},\mathcal{F})$ on U* by

$$\mathcal{D}(\mathcal{E}_U) = \mathcal{F}^0(U) \text{ and } \mathcal{E}_U(f,g) = \mathcal{E}(f,g) \text{ for } f,g \in \mathcal{F}^0(U).$$

By [CF, Theorem 3.3.9] $(\mathcal{E}_U,\mathcal{F}^0(U))$ is a regular, strongly local Dirichlet form on $L^2(U,\mu)$. We write $(P^U_t, t \geq 0)$ for the associated semigroup, and call (P^U_t) the *semigroup of X killed on exiting U*. The Dirichlet form $(\mathcal{E}_U,\mathcal{F}^0(U))$ is associated with the process X killed upon exiting U – see [CF, Theorem 3.3.8(ii)].

For an open set U we need to consider functions that vanish on a portion of the boundary of U, and we therefore define the following local spaces associated with $(\mathcal{E}_U, \mathcal{F}^0(U))$.

Definition 3.3. Let $V \subset U$ be open subsets of $\mathcal{X}$. Set

$$\begin{aligned}\mathcal{F}^0_{\mathrm{loc}}(U,V) = \{ & f \in L^2_{\mathrm{loc}}(V,\mu) : \forall \text{ open } A \subset V \text{ relatively compact in } \overline{U} \text{ with} \\ & d_U(A, U \setminus V) > 0, \exists f^\sharp \in \mathcal{F}^0(U) : f^\sharp = f \ \mu\text{-a.e. on } A\}.\end{aligned}$$

Note that $\mathcal{F}^0_{\mathrm{loc}}(U,V) \subset \mathcal{F}_{\mathrm{loc}}(V)$. Roughly speaking, a function in $\mathcal{F}^0_{\mathrm{loc}}(U,V)$ vanishes along the portion of boundary given by $\partial_{\widetilde{U}} V \cap \partial_{\widetilde{U}} U$.

4 Harmonic functions and Green functions

4.1 Harmonic functions

We begin by defining harmonic functions for a strongly local, regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$.

Definition 4.1. Let $U \subset \mathcal{X}$ be open. A function $u : U \to \mathbb{R}$ is *harmonic* on U if $u \in \mathcal{F}_{\mathrm{loc}}(U)$ and for any function $\phi \in \mathcal{F}_c(U)$ there exists a function $u^\# \in \mathcal{F}$ such that $u^\# = u$ in a neighbourhood of the essential support of ϕ and

$$\mathcal{E}(u^\#, \phi) = 0.$$

Remark 4.2. (a) By the locality of $(\mathcal{E}, \mathcal{F})$, $\mathcal{E}(u^\#, \phi)$ does not depend on the choice of $u^\#$ in Definition 4.1.

(b) If U and V are open subsets of $\mathcal{X}$ with $V \subset U$ and u is harmonic in U, then the restriction $u\big|_V$ is harmonic in V. This follows from the locality of $(\mathcal{E}, \mathcal{F})$.

(c) It is known that $u \in L^\infty_{\mathrm{loc}}(U,\mu)$ is harmonic in U if and only if it satisfies the following property: for every relatively compact open subset V of U, $t \mapsto \widetilde{u}(X_{t\wedge\tau_V})$ is a uniformly integrable $\mathbb{P}^x$-martingale for q.e. $x \in V$. (Here $\widetilde{u}$ is a quasi continuous version of u on V.) This equivalence between the weak solution formulation in Definition 4.1 and the probabilistic formulation using martingales is given in [Che, Theorem 2.11].

Definition 4.3. Let $V \subset U$ be open. We write $\widetilde{V}^{d_U}$ for the closure of V in $(\widetilde{U}, d_U)$. We say that a harmonic function $u : V \to \mathbb{R}$ satisfies *Dirichlet boundary conditions on the boundary* $\partial_{\widetilde{U}} U \cap \widetilde{V}^{d_U}$ if $u \in \mathcal{F}^0_{\mathrm{loc}}(U,V)$.

4.2 Elliptic Harnack inequality

Definition 4.4 (Elliptic Harnack inequality)**.** We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *local elliptic Harnack inequality*, denoted $\mathrm{EHI}_{\mathrm{loc}}$, if there exist constants $C_H < \infty$, $R_0 \in (0, \infty]$ and

$\delta \in (0,1)$ such that, for any ball $B(x,R) \subset \mathcal{X}$ satisfying $R \in (0,R_0)$, and any non-negative function $u \in \mathcal{F}_{\mathrm{loc}}(B(x,R))$ that is harmonic on $B(x,R)$, we have

$$\operatorname*{ess\,sup}_{z \in B(x,\delta R)} u(z) \le C_H \operatorname*{ess\,inf}_{z \in B(x,\delta R)} u(z). \tag{EHI}$$

We say that $(\mathcal{E},\mathcal{F})$ satisfies the *elliptic Harnack inequality*, denoted (EHI), if $\mathrm{EHI}_{\mathrm{loc}}$ holds with $R_0 = \infty$.

An easy chaining argument along geodesics shows that if the EHI holds for some $\delta \in (0,1)$, then it holds for any other $\delta' \in (0,1)$. Further, if the local EHI holds for some R_0, then it holds (with of course a different constant C_H) for any $R_0' \in (0,\infty)$.

We recall the definition of Harnack chain – see [JK, Section 3]. For a ball $B = B(x,r)$, we use the notation $M^{-1}B$ to denote the ball $B(x, M^{-1}r)$.

Definition 4.5 (Harnack chain)**.** Let $U \subsetneq \mathcal{X}$ be a connected open set. For $x,y \in U$, an *M-Harnack chain from x to y* in U is a sequence of balls $B_1, B_2, \ldots, B_n$ each contained in U such that $x \in M^{-1}B_1, y \in M^{-1}B_n$, and $M^{-1}B_i \cap M^{-1}B_{i+1} \neq \emptyset$, for $i = 1,2,\ldots,n-1$. The number n of balls in a Harnack chain is called the *length* of the Harnack chain. For a domain U write $N_U(x,y;M)$ for the length of the shortest M-Harnack chain in U from x to y.

Remark 4.6. Suppose that $(\mathcal{E},\mathcal{F})$ satisfies the elliptic Harnack inequality with constants C_H and δ. If u is a positive continuous harmonic function on a domain U, then

$$C_H^{-N_U(x_1,x_2;\delta^{-1})} u(x_1) \le u(x_2) \le C_H^{N_U(x_1,x_2;\delta^{-1})} u(x_1). \tag{4.1}$$

for all x_1, $x_2 \in U$.

Lemma 4.7. *Let $(\mathcal{X},d)$ be a locally compact, separable, length space that satisfies the metric doubling property. Let $U \subsetneq \mathcal{X}$ be a (c_U, C_U)-inner uniform domain in $(\mathcal{X},d)$. Then for each $M > 1$ there exists $C \in (0,\infty)$, depending only on c_U, C_U and M, such that for all $x,y \in U$*

$$C^{-1} \log\left(\frac{d_U(x,y)}{\min(\delta_U(x),\delta_U(y))} + 1\right) \le N_U(x,y;M) \le C \log\left(\frac{d_U(x,y)}{\min(\delta_U(x),\delta_U(y))} + 1\right) + C.$$

Proof. See [GO, Equation (1.2) and Theorem 1.1] or [Aik15, Theorem 3.8 and 3.9] for a similar statement for the quasi-hyperbolic metric on U; the result then follows by a comparison between the quasi-hyperbolic metric and the length of Harnack chains as in [Aik01, p. 127]. $\square$

4.3 Green function

Let $(\mathcal{E},\mathcal{F})$ be a regular, strongly local Dirichlet form and let $\Omega \subsetneq \mathcal{X}$ be open. We define

$$\lambda_{\min}(\Omega) = \inf_{u \in \mathcal{F}^0(\Omega)\setminus\{0\}} \frac{\mathcal{E}_\Omega(u,u)}{\|u\|_2^2}.$$

Writing $\mathcal{L}^\Omega$ for the generator of $(\mathcal{E}_\Omega, \mathcal{F}^0(\Omega))$, we have $\lambda_{\min}(\Omega) = \inf \operatorname{spectrum}(-\mathcal{L}^\Omega)$.

The next Lemma gives the existence and some fundamental properties of the Green operator on a domain $\Omega \subset \mathcal{X}$.

Lemma 4.8. *([GH1, Lemma 5.1]) Let $(\mathcal{E}, \mathcal{F})$ be a regular, Dirichlet form in $L^2(\mathcal{X}, \mu)$ and let $\Omega \subset \mathcal{X}$ be open and satisfy $\lambda_{\min}(\Omega) > 0$. Let $\mathcal{L}^\Omega$ be the generator of $(\mathcal{E}_\Omega, \mathcal{F}^0(\Omega))$, and let $G^\Omega = (-\mathcal{L}^\Omega)^{-1}$ be the inverse of $-\mathcal{L}^\Omega$ on $L^2(\Omega, \mu)$. Then the following statements hold:*

(i) $\left\|G^\Omega\right\| \leq \lambda_{\min}(\Omega)^{-1}$, that is, for any $f \in L^2(\Omega, \mu)$,

$$\left\|G^\Omega f\right\|_{L^2(\Omega)} \leq \lambda_{\min}(\Omega)^{-1} \left\|f\right\|_{L^2(\Omega)};$$

(ii) for any $f \in L^2(\Omega)$, we have that $G^\Omega f \in \mathcal{F}^0(\Omega)$, and

$$\mathcal{E}_\Omega(G^\Omega f, \phi) = \langle f, \phi \rangle \text{ for any } \phi \in \mathcal{F}^0(\Omega);$$

(iii) for any $f \in L^2(\Omega)$,

$$G^\Omega f = \int_0^\infty P_s^\Omega f \, ds;$$

(iv) G^Ω is positivity preserving: $G^\Omega f \geq 0$ if $f \geq 0$.

We now state our fundamental assumption on the Green function.

Assumption 4.9. Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length space and let μ be a non-atomic Radon measure on $(\mathcal{X}, d)$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(\mathcal{X}, \mu)$. Let $\Omega \subset \mathcal{X}$ be a non-empty bounded open set with $\operatorname{diameter}(\Omega, d) \leq \operatorname{diameter}(\mathcal{X}, d)/5$. Assume that $\lambda_{\min}(\Omega) > 0$, and there exists a function $g_\Omega(x, y)$ defined for $(x, y) \in \Omega \times \Omega$ with the following properties:

(i) (Integral kernel) $G^\Omega f(x) = \int_\Omega g_\Omega(x, z) f(x) \, \mu(dz)$ for all $f \in L^2(\Omega)$ and μ-a.e. $x \in \Omega$;

(ii) (Symmetry) $g_\Omega(x, y) = g_\Omega(y, x) \geq 0$ for all $(x, y) \in \Omega \times \Omega \setminus \operatorname{diag}$;

(iii) (Continuity) $g_\Omega(x, y)$ is jointly continuous in $(x, y) \in \Omega \times \Omega \setminus \operatorname{diag}$;

(iv) (Maximum principles) If $x_0 \in U \Subset \Omega$, then

$$\inf_{U \setminus \{x_0\}} g_\Omega(x_0, \cdot) = \inf_{\partial U} g_\Omega(x_0, \cdot),$$
$$\sup_{\Omega \setminus U} g_\Omega(x_0, \cdot) = \sup_{\partial U} g_\Omega(x_0, \cdot).$$

We now give some consequences of this assumption; in the next subsection we will give some sufficient conditions for it to hold.

We begin by showing that the Green function $g_\Omega(x,\cdot)$ is harmonic in $\Omega\setminus\{x\}$ and vanishes along the boundary of Ω. Since we are using Definition 4.1, we need first to prove that this function is locally in the domain of the Dirichlet form, that is that $g_\Omega(x,\cdot)\in\mathcal{F}_{\mathrm{loc}}(\Omega\setminus\{x\})$. For this it is enough that $g_\Omega(x,\cdot)\in\mathcal{F}^0_{\mathrm{loc}}(\Omega,\Omega\setminus\{x\})$. This result was shown under more restrictive hypothesis (Gaussian or sub-Gaussian heat kernel estimates) in [GyS, Lemma 4.7] and by similar methods in [L, Lemma 4.3]. Our proof is based on a different approach (see [GyS, Theorem 4.16]), using Lemma 4.8.

Lemma 4.10. *Let $(\mathcal{X},d,\mu,\mathcal{E},\mathcal{F})$, Ω be as in Assuption 4.9. For any fixed $x\in\Omega$, the function $y\mapsto g_\Omega(x,y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(\Omega,\Omega\setminus\{x\})$, and is harmonic in $\Omega\setminus\{x\}$.*

Proof. Fix $x\in\Omega$. Let $V\subset\Omega$ be an open set such that $\overline{V}\subset\overline{\Omega}\setminus\{x\}$. Let Ω_1,Ω_2 be precompact open sets such that $\overline{\Omega}\subset\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$. Let $r>0$ be such that $B(x,4r)\subset\Omega\cap V^c$. Let $\phi\in\mathcal{F}$ be a continuous function such that $0\le\phi\le 1$ and

$$\phi=\begin{cases}1 & \text{on } B(x,3r)^c\cap\Omega_1,\\ 0 & \text{on } B(x,2r)\cup\left(\overline{\Omega_2}\right)^c.\end{cases}$$

Since $\varphi\equiv 1$ on V, to prove that $g_\Omega(x,\cdot)\in\mathcal{F}^0_{\mathrm{loc}}(\Omega,\Omega\setminus\{x\})$ it is sufficient to prove that $\varphi g_\Omega(x,\cdot)\in\mathcal{F}^0(\Omega)$.

For $k\ge 1$ set $B_k=B(x,r/k)$. Consider the sequence of functions defined by

$$h_k(y)=\frac{1}{\mu(B_k)}\int_{B_k}g_\Omega(z,y)\mu(dz),\quad y\in\Omega,\ k\in\mathbb{N}. \tag{4.2}$$

By Lemma 4.8(ii) we have $h_k\in\mathcal{F}^0(\Omega)$ for all $k\ge 1$.

By the maximum principle, we have

$$M:=\sup_{z\in\overline{B(x,r)},y\in B(x,2r)^c}g_\Omega(z,y)=\sup_{z\in\overline{B(x,r)},y\in\partial B(x,2r)}g_\Omega(z,y)<\infty,$$

since the image of the compact set $\overline{B(x,r)}\times\partial B(x,2r)$ under the continuous map of g_Ω is bounded. Thus the functions ϕh_k are bounded uniformly by $M\mathbb{1}_{\Omega\setminus B(x,2r)}$. By the continuity of $g_\Omega(\cdot,\cdot)$ on $\Omega\times\Omega\setminus\mathrm{diag}$, the functions ϕh_k converge pointwise to $\phi g_\Omega(x,\cdot)$ on Ω, and using dominated convergence this convergence also holds in $L^2(\Omega)$.

For the remainder of the proof we identify $L^2(\Omega)$ with the subspace

$$\left\{f\in L^2(\mathcal{X}):\ f=0,\quad \mu-\text{a.e. on }\Omega^c\right\}.$$

Similarly, we view $\mathcal{F}^0(\Omega)$ as a subspace of $\mathcal{F}(\Omega)$ –see [CF, (3.2.2) and Theorem 3.3.9]. In particular, we can view the functions ϕh_i as functions over $\mathcal{X}$. By [FOT, Theorem 1.4.2(ii),(iii)], $\phi h_i=\phi(h_i\wedge M)\in\mathcal{F}^0(\Omega)$, and $\phi^2 h_i=\phi^2(h_i\wedge M)\in\mathcal{F}^0(\Omega)$.

We now show that ϕh_i is Cauchy in the seminorm induced by $\mathcal{E}(\cdot,\cdot)$. By the Leibniz rule (cf. [FOT, Lemma 3.2.5]) we have

$$\mathcal{E}(\phi(h_i - h_j), \phi(h_i - h_j)) = \int_{\mathcal{X}} (h_i - h_j)^2 \, d\Gamma(\phi,\phi) + \mathcal{E}(h_i - h_j, \phi^2(h_i - h_j)). \tag{4.3}$$

Since $\mathbb{1}_{B_i} - \mathbb{1}_{B_j}$ and $\varphi^2(h_i - h_j)$ have disjoint support the second term is zero by Lemma 4.8(ii).

For the first term in (4.3), we use the fact that the function h_i vanishes on Ω^c together with strong locality to obtain

$$\int_{\mathcal{X}} (h_i - h_j)^2 \, d\Gamma(\phi,\phi) = \int_{\overline{B(x,3r)}\setminus B(x,2r)} (h_i - h_j)^2 \, d\Gamma(\phi,\phi). \tag{4.4}$$

Let $F = \overline{B(x,3r)} \setminus B(x,2r)$. Since $\overline{B(x,r)} \times F$ is compact, by Assumption 4.9(iii) the function $g_\Omega(\cdot,\cdot)$ is uniformly continuous on $\overline{B(x,r)} \times F$. This in turn implies that h_i converges uniformly to $g_\Omega(x,\cdot)$ as $i \to \infty$ on F, and so by (4.4) we have that ϕh_i is Cauchy in the $(\mathcal{E}_\Omega(\cdot,\cdot))^{1/2}$-seminorm. Since ϕh_i converges pointwise and in $L^2(\Omega)$ to $\phi g_\Omega(x,\cdot)$, and $(\mathcal{E}_\Omega, \mathcal{F}^0(\Omega))$ is a closed form, this implies that $\phi g_\Omega(x,\cdot) \in \mathcal{F}^0(\Omega)$.

Finally, we show that $g_\Omega(x,\cdot)$ is harmonic on $\Omega \setminus \{x\}$. Let $\psi \in \mathcal{F}_c(\Omega \setminus \{x\})$, and let $V \Subset \Omega$ be a precompact open set containing $\operatorname{supp}(\psi)$ such that $d(x,V) > 0$. Choose $r > 0$ such that $B(x,4r) \cap V = \emptyset$, and let φ and h_k be as defined above. Then as $\varphi \equiv 1$ on V, using strong locality,

$$\mathcal{E}(\varphi h_k, \psi) = \mathcal{E}(h_k, \psi) = \mu(B_k)^{-1} \langle \mathbb{1}_{B_k}, \psi \rangle = 0.$$

As φh_k converge to $\phi g_\Omega(x,\cdot)$ in the $\mathcal{E}_1(\cdot,\cdot)^{1/2}$ norm, it follows that $\mathcal{E}(\phi g_\Omega(x,\cdot), \psi) = 0$. This allows us to conclude that $g_\Omega(x,\cdot)$ is harmonic on $\Omega \setminus \{x\}$. $\square$

The elliptic Harnack inequality enables us to relate capacity and Green functions, and also to control their fluctuations on bounded regions of $\mathcal{X}$.

Proposition 4.11. *(See [BM, Section 3]). Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet space satisfying the EHI and Assumption 4.9. Then the following hold:*

(a) For all $A_1, A_2 \in (1,\infty)$, there exists $C_0 = C_0(A_1, A_2, C_H) > 1$ such that for all bounded open sets D and for all $x_0 \in \mathcal{X}, r > 0$ that satisfy $B(x_0, A_1 r) \subset D$, we have

$$g_D(x_1,y_1) \le C_0 g_D(x_2,y_2) \quad \forall x_1,y_1,x_2,y_2 \in B(x_0,r),$$

satisfying $d(x_i,y_i) \ge r/A_2$, for $i = 1,2$.

(b) For all $A \in (1,\infty)$, there exists $C_1 = C_1(A, C_H) > 1$ such that for all bounded open sets D and for all $x_0 \in \mathcal{X}, r > 0$ that satisfy $B(x_0, Ar) \subset D$, we have

$$\inf_{y \in \partial B(x_0,r)} g_D(x_0,y) \le \operatorname{Cap}_D\left(\overline{B(x_0,r)}\right)^{-1} \le \operatorname{Cap}_D\left(B(x_0,r)\right)^{-1} \le C_1 \inf_{y \in \partial B(x_0,r)} g_D(x_0,y).$$

(c) *For all* $1 \le A_1 \le A_2 < \infty$ *and* $a \in (0,1]$ *there exists* $C_2 = C_2(a, A_1, A_2, C_H) > 1$ *such that for* $x \in \mathcal{X}$*, and* $r > 0$ *with* $r \le \operatorname{diameter}(\mathcal{X})/5A$*,*

$$\operatorname{Cap}_{B(x_0,A_2r)}(B(x_0,ar)) \le \operatorname{Cap}_{B(x_0,A_1r)}(B(x_0,r)) \le C_2 \operatorname{Cap}_{B(x_0,A_2r)}(B(x_0,ar)).$$

(d) *For all* $A_2 > A_1 \ge 2$ *there exists* $C_3 = C_3(A_1, A_2, C_H) > 1$ *such that for all* $x, y \in \mathcal{X}$*, with* $d(x,y) = r > 0$ *and such that* $r \le \operatorname{diameter}(\mathcal{X})/5A_2$*,*

$$g_{B(x,A_1r)}(x,y) \le g_{B(x,A_2r)}(x,y) \le C_3 g_{B(x,A_1r)}(x,y).$$

(e) $(\mathcal{X}, d)$ *satisfies metric doubling.*

The statements given above are slightly stronger than those in [BM, Section 3], but Proposition 4.11 easily follows from the results there using additional chaining arguments.

We will need the following maximum principle for Green functions.

Lemma 4.12. *Suppose that* $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ *and* Ω *satisfy Assumption 4.9. Let* $c_0 \in (0,1)$*,* $y, y^* \in \Omega$*, such that* $B(y^*, r) \Subset \Omega$ *and*

$$g_\Omega(y,x) \ge c_0 g_\Omega(y^*,x) \quad \text{for all } x \in \partial B(y^*,r).$$

Then

$$g_\Omega(y,x) \ge c_0 g_\Omega(y^*,x) \quad \text{for all } x \in \Omega \setminus (\{y\} \cup B(y^*,r)).$$

Proof. If $y \in B(y^*,r)$, then the function $g_\Omega(y,\cdot) - c_0 g_\Omega(y^*,\cdot)$ is bounded and harmonic on $\Omega \setminus B(y^*,r)$, so the result follows by the maximum principle in [GH1, Lemma 4.1(ii)].

Now suppose that $y \notin B(y^*,r)$. Choose $r' > 0$ small enough so that $B(y,4r') \subset \Omega \setminus B(y^*,r)$, and as in (4.2) set

$$h_n(x) = \mu(B(y,r'/n))^{-1} \int_{B(y,r'/n)} g_\Omega(z,x)\mu(dz).$$

Then $h_n \in \mathcal{F}^0(\Omega)$ and $h_n \to g_\Omega(y,\cdot)$ pointwise on $\Omega \setminus \{y\}$. Let $M = 2\sup_{z \in \partial B(y^*,r)} g_\Omega(z,y^*)$. By [FOT, Corollary 2.2.2 and Lemma 2.2.10] the functions $M \wedge h_n$ are bounded and superharmonic on Ω.

Set $f_n = M \wedge h_n - c_1 g_\Omega(y^*,\cdot)$, where $c_1 \in (0,c_0)$. Then f_n is a bounded, superharmonic function in $\Omega \setminus B(y^*,r)$ that is non-negative on the boundary of $\Omega \setminus B(y^*,r)$ for all sufficiently large n. By the maximum principle in [GH1, Lemma 4.1(ii)] we obtain that $f_n \ge 0$ in $\Omega \setminus B(y^*,r)$ for all sufficiently large n. Since $c_1 \in (0,c_0)$ was arbitrary, we obtain the desired conclusion by letting $n \to \infty$. □

Remark 4.13. Suppose that Assumption 4.9 holds for $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$. Let μ' be a measure which is mutually absolutely continuous with respect to the measure μ, and suppose that $d\mu'/d\mu$ is uniformly bounded away from 0 and infinity on bounded sets. It is straightforward to verify that if Ω is a bounded domain and the operator G'_Ω is defined by

$$G'_\Omega f(x) = \int_\Omega g_\Omega(x,y) f(y) \mu'(dy), \text{ for } f \in L^2(\Omega, \mu'),$$

then G'_Ω is the Green operator on the domain Ω for the Dirichlet form $(\mathcal{E}, \mathcal{F}')$ on $L^2(\mathcal{X}, \mu')$, where $\mathcal{F}'$ is the domain of the time-changed Dirichlet space (Cf. [FOT, p. 275]). It follows that $g_\Omega(x,y)$ is the Green function for both $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ and $(\mathcal{X}, d, \mu', \mathcal{E}, \mathcal{F}')$, and thus that Assumption 4.9 holds for $(\mathcal{X}, d, \mu', \mathcal{E}, \mathcal{F}')$.

4.4 Sufficient conditions for Assumption 4.9

We begin by recalling the definition of an ultracontractive semigroup, a notion introduced in [DS].

Definition 4.14. Let $(\mathcal{X}, d, \mu)$ be a metric measure space. Let $(P_t)_{t\geq 0}$ be the semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$. We say that the semigroup $(P_t)_{t\geq 0}$ is *ultracontractive* if P_t is a bounded operator from $L^2(\mathcal{X}, \mu)$ to $L^\infty(\mathcal{X}, \mu)$ for all $t > 0$. We say that $(\mathcal{E}, \mathcal{F})$ is *ultracontractive* if the associated heat semigroup is ultracontractive.

We will use the weaker notion of local ultracontractivity introduced in [GT2, Definition 2.11].

Definition 4.15. We say that a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, F)$ is *locally ultracontractive* if for all open balls B, the killed heat semigroup (P_t^B) given by Definition 3.2 is ultracontractive.

It is well-known that ultracontractivity of a semigroup is equivalent to the existence of an essentially bounded heat kernel at all strictly positive times.

Lemma 4.16. *([Dav, Lemma 2.1.2]) Let U be a bounded open subset of $\mathcal{X}$.*
(a) Suppose that (P_t^U) is ultracontractive. Then for each $t > 0$ the operator P_t^U has an integral kernel $p^U(t, \cdot, \cdot)$ which is jointly measurable in $U \times U$ and satisfies

$$0 \leq p^U(t,x,y) \leq \left\| P_{t/2}^U \right\|^2_{L^2(\mu) \to L^\infty(\mu)} \text{ for } \mu \times \mu\text{-a.e. } (x,y) \in U \times U.$$

(b) If P_t^U has an integral kernel $p^U(t,x,y)$ satisfying

$$0 \leq p^U(t,x,y) \leq a_t < \infty$$

for all $t > 0$ and for $\mu \times \mu$-a.e. $(x,y) \in U \times U$, then $(P_t^U)_{t\geq 0}$ is ultracontractive with

$$\left\| P_t^U \right\|_{L^2(\mu) \to L^\infty(\mu)} \leq a_t^{1/2} \text{ for all } t > 0.$$

The issue of joint measurability is clarified in [GT2, p. 1227].

We now introduce a second assumption on the Dirichlet form $(\mathcal{E}, \mathcal{F})$, and will prove below that it implies Assumption 4.9.

Assumption 4.17. Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length space and let μ be a non-atomic Radon measure on $(\mathcal{X}, d)$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(\mathcal{X}, \mu)$. We assume that the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ is locally ultracontractive and that $\lambda_{\min}(\Omega) > 0$ for any non-empty bounded open set $\Omega \subset \mathcal{X}$ with diameter$(\Omega, d) \leq$ diameter$(\mathcal{X}, d)/5$.

This assumption gives the existence of a Green function satisfying Assumption 4.9.

Lemma 4.18. *(See [GH1, Lemma 5.2 and 5.3])). Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet space which satisfies the EHI$_{\mathrm{loc}}$ and Assumption 4.17. Then Assumption 4.9 holds.*

Remark 4.19. A similar result is stated in [GH1, Lemma 5.2]. Unfortunately, the proof in [GH1] of Assumption 4.9(i) and (iv) have gaps, which we do not know how to fix without the extra hypothesis of local ultracontractivity. For (i) the problem occurs in the proof of [GH1, (5.8)] from [GH1, (5.7)]. In particular, while one has in the notation of [GH1] that $G^\Omega f_k \to G^\Omega f$ in $L^2(\Omega)$, this does not imply pointwise convergence. On the other hand the proof of [GH1, (5.8)] does require pointwise convergence at the specific point x.

The following example helps to illustrate this gap. Consider the Dirichlet form $\mathcal{E}(f, f) = \|f\|_2^2$ on $\mathbb{R}^n$; this satisfies [GH1, (5.7)] for any bounded open domain Ω with $g_x^\Omega \equiv 0$ but it fails to satisfy [GH1, (5.8)]. This Dirichlet form does not satisfy the hypothesis of [GH1, Lemma 5.2] since it is local rather than strongly local, but it still illustrates the problem, since strong locality was not used in the proof of [GH1, (5.8)] from [GH1, (5.7)].

Proof of Lemma 4.18. Let Ω be a non-empty bounded open set with diameter$(\Omega, d) \leq$ diameter$(\mathcal{X}, d)/5$; we need to verify properties (i)–(iv) of Assumption 4.9.

We use the construction in [GH1]. We denote by $g_\Omega(\cdot, \cdot)$ the function constructed in [GH1, Lemma 5.2] off the diagonal, and extend it to $\Omega \times \Omega$ by taking g_Ω equal to 0 on the diagonal. By [GH1, Lemma 5.2] the function $g_\Omega(\cdot, \cdot)$ satisfies (ii) and (iii). (The proofs of (ii) and (iii) do not use [GH1, (5.8)].)

Next, we show (i), using the additional hypothesis of local ultracontractivity. Define the operators

$$S_t = P_t^\Omega \circ G^\Omega = G^\Omega \circ P_t^\Omega, \quad t \geq 0.$$

Formally we have $S_t = \int_t^\infty P_s ds$. Since P_t^Ω is a contraction on all $L^p(\Omega)$, we have by Lemma 4.8(i)

$$\|S_t\|_{L^2 \mapsto L^2} \leq \left\|G^\Omega\right\|_{L^2 \mapsto L^2} \left\|P_t^\Omega\right\|_{L^2 \mapsto L^2} < \infty. \tag{4.5}$$

Therefore by [FOT, Lemma 1.4.1], for each $t \geq 0$ there exists a positive symmetric Radon measure σ_t on $\Omega \times \Omega$ such that for all functions $f_1, f_2 \in L^2(\Omega)$, we have

$$\langle f_1, S_t f_2 \rangle = \int_{\Omega \times \Omega} f_1(x) f_2(y) \sigma_t(dx, dy). \tag{4.6}$$

By [FOT, Lemma 1.4.1] $S_{t+r} - S_t$ is a positive symmetric operator on $L^2(\Omega)$. Thus $(\sigma_t, t \in \mathbb{R}_+)$ is a family of symmetric positive measures on $\Omega \times \Omega$ with $\sigma_s \geq \sigma_t$ for all $0 \leq s \leq t$. Note that the measures σ_t are finite, since for any $t \geq 0$ by using Lemma 4.8(i), we have

$$\sigma_t(\Omega \times \Omega) \leq \sigma_0(\Omega \times \Omega) = \langle \mathbb{1}_\Omega, G^\Omega \mathbb{1}_\Omega \rangle \leq ||\mathbb{1}_\Omega||_2^2 \lambda_{\min}(\Omega)^{-1} = \mu(\Omega)\lambda_{\min}(\Omega)^{-1}. \tag{4.7}$$

Let A, B be measurable subsets of Ω. Since P_t^Ω is a strongly continuous semigroup, we have

$$\sigma_0(A \times B) = \langle G^\Omega \mathbb{1}_A, \mathbb{1}_B \rangle = \lim_{t \downarrow 0} \langle S_t \mathbb{1}_A, \mathbb{1}_B \rangle = \lim_{t \downarrow 0} \sigma_t(A \times B).$$

The above equation implies that, for all measurable subsets $F \subset \Omega \times \Omega$, we have

$$\lim_{n \to \infty} \sigma_{1/n}(F) = \sigma_0(F). \tag{4.8}$$

For each $t > 0$ by local ultracontractivity, we have

$$\|S_t\|_{L^2 \to L^\infty} \leq \left\|G^\Omega\right\|_{L^2 \to L^2} \left\|P_t^\Omega\right\|_{L^2 \to L^\infty} < \infty. \tag{4.9}$$

By [GH2, Lemma 3.3], there exists a jointly measurable function $s_t(\cdot, \cdot)$ on $\Omega \times \Omega$ such that

$$\langle f_1, S_t f_2 \rangle_{L^2(\Omega)} = \int_\Omega \int_\Omega f_1(x) f_2(y) s_t(x,y) \mu(dx)\mu(dy), \tag{4.10}$$

for all $f_1, f_2 \in L^2(\Omega)$. By (4.6) and (4.10), we have

$$\sigma_{1/n}(dx, dy) = s_{1/n}(x,y)\mu(dx)\mu(dy) \tag{4.11}$$

for all $n \in \mathbb{N}$. Therefore by (4.11), (4.7), (4.8) and Vitali-Hahn-Saks theorem (cf. [Yos, p. 70]), the measure σ_0 is absolutely continuous with respect to the product measure $\mu \times \mu$ on $\Omega \times \Omega$. Let $s(\cdot, \cdot)$ be the Radon-Nikoym derivative of σ_0 with respect to $\mu \times \mu$. By (4.6) and Fubini's theorem, for all $f \in L^2(\Omega)$ and for almost all $x \in \Omega$,

$$G^\Omega f(x) = \int_\Omega s(x,y) f(y)\, \mu(dy). \tag{4.12}$$

If B and B' are disjoint open balls in Ω, then for all $f_1 \in L^2(B), f_2 \in L^2(B')$, we have

$$\begin{aligned} \langle G^\Omega f_1, f_2 \rangle &= \int_B \int_{B'} f_1(x) f_2(y) s(x,y) \mu(dy)\mu(dx) \\ &= \int_B \int_{B'} f_1(x) f_2(y) g_\Omega(x,y) \mu(dy)\mu(dx). \end{aligned} \tag{4.13}$$

We used [GH1, (5.7)], along with (4.10), to obtain the above equation. By the same argument as in [GH2, Lemma 3.6(a)], (4.13) implies that

$$s(x,y) = g_\Omega(x,y) \text{ for } \mu \times \mu\text{-almost every } (x,y) \in B \times B'. \tag{4.14}$$

By an easy covering argument $\Omega \times \Omega \setminus \mathrm{diag}$ can be covered by countably many sets of the form $B_i \times B_i'$, $i \in \mathbb{N}$, such that B_i and B_i' are disjoint balls contained in Ω. Therefore by (4.14), we have

$$s(x,y) = g_\Omega(x,y) \tag{4.15}$$

for almost every $(x,y) \in \Omega \times \Omega$. In the last line we used that since μ is non-atomic the diagonal has measure zero. Combining (4.15) and (4.12) gives property (i).

The maximum principles in (iv) are proved in [GH1, Lemma 5.3] by showing the corresponding maximum principles for the approximations of Green functions given by (4.2). This maximum principle for the Green functions follows from [GH1, Lemma 4.1], provided that the functions h_k in (4.2) satisfy the three properties that $h_k \in \mathcal{F}^0(\Omega)$, h_k is superharmonic, and $h_k \in L^\infty(\Omega)$. As in [GH1], the first two conditions can be checked by using Lemma 4.8(i) and (ii). To verify that $h_k \in L^\infty(\Omega)$ it is sufficient to prove that $G^\Omega \mathbb{1}_\Omega \in L^\infty(\Omega)$. By Lemma 4.8(iii) and the semigroup property, for any $t > 0$,

$$G^\Omega \mathbb{1}_\Omega = \int_0^t P_s^\Omega \mathbb{1}_\Omega \, ds + P_t^\Omega \circ G^\Omega \mathbb{1}_\Omega.$$

For the first term, we use $\left\| P_s^\Omega \right\|_{L^\infty \to L^\infty} \le 1$, and for the second term we use local ultracontractivity and Lemma 4.8(i). $\square$

We now introduce some conditions which imply local ultracontractivity. We say that a function $u = u(x,t)$ is *caloric* in a region $Q \subset \mathcal{X} \times (0,\infty)$ if u is a weak solution of $(\partial_t + \mathcal{L})u = 0$ in Q; here $\mathcal{L}$ is the generator corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F}, L^2(\mathcal{X},\mu))$. Let $\Psi : [0,\infty) \to [0,\infty)$ have the property that there exist constants $1 < \beta_1 \le \beta_2 < \infty$ and $C > 0$ such that

$$C^{-1}(R/r)^{\beta_1} \le \Psi(R)/\Psi(r) \le C(R/r)^{\beta_2}. \tag{4.16}$$

Definition 4.20. *We say that a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the local volume doubling property* $(\mathrm{VD})_{\mathrm{loc}}$*, if there exist $R \in (0,\infty]$, $C_{VD} > 0$ such that*

$$V(x,2r) \le C_{VD} V(x,r) \text{ for all } x \in \mathcal{X} \text{ and for all } 0 < r \le R. \tag{$(\mathrm{VD})_{\mathrm{loc}}$}$$

We say a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the local Poincaré inequality $(\mathrm{PI}(\Psi))_{\mathrm{loc}}$*, if there exist $R \in (0,\infty]$, $C_{PI} > 0$, and $A \ge 1$ such that*

$$\int_{B(x,r)} \left| f - f_{B(x,r)} \right|^2 d\mu \le C_{PI} \Psi(r) \int_{B(x,Ar)} d\Gamma(f,f) \tag{$(\mathrm{PI}(\Psi))_{\mathrm{loc}}$}$$

for all $x \in \mathcal{X}$ and for all $0 < r \le R$, where $\Gamma(f,f)$ denotes the energy measure, and $f_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu$.

We say that a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the local parabolic Harnack inequality $(\mathrm{PHI}(\Psi))_{\mathrm{loc}}$*, if there exist $R \in (0, \infty]$, $C_{PHI} > 0$ such that for all $x \in \mathcal{X}$, for all $0 < r \leq R$, any non-negative caloric function u on $(0, r^2) \times B(x, r)$ satisfies*

$$\sup_{(\Psi(r)/4, \Psi(r)/2) \times B(x, r/2)} u \leq C_{PHI} \inf_{(3\Psi(r)/4, \Psi(r)) \times B(x, r/2)} u, \qquad (\mathrm{PHI}(\Psi))_{\mathrm{loc}}$$

We will write $(\mathrm{PI}(\beta))_{\mathrm{loc}}$ *and* $(\mathrm{PHI}(\beta))_{\mathrm{loc}}$ *for the conditions* $(\mathrm{PI}(\Psi))_{\mathrm{loc}}$ *and* $(\mathrm{PHI}(\Psi))_{\mathrm{loc}}$ *if* $\Psi(r) = r^\beta$*.*

Lemma 4.21. *Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length space with* $\mathrm{diameter}(\mathcal{X}) = \infty$*, let μ be a Radon measure on $(\mathcal{X}, d)$ with full support and let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(\mathcal{X}, \mu)$. If $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the properties* $(\mathrm{VD})_{\mathrm{loc}}$ *and* $(\mathrm{PI}(2))_{\mathrm{loc}}$*, then $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies Assumption 4.9, and the property* $\mathrm{EHI}_{\mathrm{loc}}$*.*

Proof. First, the property $(\mathrm{PHI}(2))_{\mathrm{loc}}$ is satisfied; this is immediate from [CS, Theorem 8.1] and [HS, Theorem 2.7], which prove that the property $(\mathrm{PHI}(2))_{\mathrm{loc}}$ is equivalent to the conjunction of the properties $(\mathrm{PI}(2))_{\mathrm{loc}}$ and $(\mathrm{VD})_{\mathrm{loc}}$.

To prove Assumption 4.9 it is sufficient to verify the property $\mathrm{EHI}_{\mathrm{loc}}$ and Assumption 4.17. Of these, the property $\mathrm{EHI}_{\mathrm{loc}}$ follows immediately from the local PHI.

The heat kernel corresponding to $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies Gaussian upper bounds for small times by [HS, Theorem 2.7]. Since the heat kernel of the killed semigroup is dominated by the heat kernel of $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$, local ultracontractivity follows using the property $(\mathrm{VD})_{\mathrm{loc}}$. The fact that μ is non-atomic follows from the property $(\mathrm{VD})_{\mathrm{loc}}$ due to a reverse volume doubling property – see [HS, (2.5)].

By domain monotonicity, it suffices to verify that $\lambda_{\min}(B(x,r)) > 0$ for all balls $B = B(x,r)$ with $0 < r < \mathrm{diameter}(\mathcal{X})/4$. Consider a ball $B(z,r)$ such that $B(x,r) \cap B(z,r) = \emptyset$ and $d(x,z) \leq 3r$. By the Gaussian lower bound for small times [HS, Theorem 2.7] and the property $(\mathrm{VD})_{\mathrm{loc}}$, there exists $t_0 > 0, \delta \in (0,1)$ such that

$$\mathbb{P}^y(X_{t_0} \in B(z,r)) \geq \delta, \quad \forall y \in B(x,r),$$

where $(X_t)_{t \geq 0}$ is the diffusion corresponding to the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$. This implies that $P^B_{t_0} \mathbb{1}_B \leq (1-\delta) \mathbb{1}_B$, which in turn implies that

$$P^B_t \mathbb{1}_B \leq (1-\delta)^{\lfloor (t/t_0) \rfloor} \mathbb{1}_B, \quad \forall t \geq 0.$$

It follows that $\left\| G^B \mathbb{1}_B \right\|_{L^\infty} < \infty$. By Riesz–Thorin interpolation we have $\left\| G^B \right\|_{L^2 \to L^2} \leq \left\| G^B \right\|^{1/2}_{L^1 \to L^1} \left\| G^B \right\|^{1/2}_{L^\infty \to L^\infty}$, while by duality $\left\| G^B \right\|_{L^\infty \to L^\infty} = \left\| G^B \right\|_{L^1 \to L^1}$. Thus

$$\lambda_{\min}(B)^{-1} = \left\| G^B \right\|_{L^2 \to L^2} \leq \left\| G^B \right\|_{L^\infty \to L^\infty} = \left\| G^B \mathbb{1}_B \right\|_{L^\infty} < \infty.$$

$\square$

4.5 Examples

In this section, we give some examples of MMD spaces which satisfy Assumption 4.9: weighted Riemannian manifolds and cable systems of weighted graphs. We also briefly describe some classes of regular fractals which satisfy Assumption 4.9 – see Remark 4.23.

Example 1. (Weighted Riemannian manifolds.) Let $(\mathcal{M}, g)$ be a Riemannian manifold, and ν and ∇ denote the Riemannian measure and the Riemannian gradient respectively. Write $d = d_g$ for the Riemannian distance function. A *weighted manifold* $(\mathcal{M}, g, \mu)$ is a Riemmanian manifold $(\mathcal{M}, g)$ endowed with a measure μ that has a smooth (strictly) positive density w with respect to the Riemannian measure ν. The *weighted Laplace operator* Δ_μ on $(\mathcal{M}, g, \mu)$ is given by

$$\Delta_\mu f = \Delta f + g\left(\nabla\left(\ln w\right), \nabla f\right), \quad f \in \mathcal{C}^\infty(\mathcal{M}).$$

We say that the weighted manifold $(\mathcal{M}, g, \mu)$ has *controlled weights* if w satisfies

$$\sup_{x,y\in\mathcal{M}:d(x,y)\le 1} \frac{w(x)}{w(y)} < \infty.$$

The construction of heat kernel, Markov semigroup and Brownian motion for a weighted Riemannian manifold $(\mathcal{M}, g, \mu)$ is outlined in [Gri, Sections 3 and 8]. The corresponding Dirichlet form on $L^2(\mathcal{M}, \mu)$ given by

$$\mathcal{E}_w(f_1, f_2) = \int_{\mathcal{X}} g(\nabla f_1, \nabla f_2)\, d\mu, \quad f_1, f_2 \in \mathcal{F},$$

where $\mathcal{F}$ is the weighted Sobolev space of functions in $L^2(\mathcal{M}, \mu)$ whose distributional gradient is also in $L^2(\mathcal{M}, \mu)$. See [Gri] and [CF, pp. 75–76] for more details.

Example 2. (Cable systems of weighted graphs.) Let $\mathbb{G} = (\mathbb{V}, E)$ be an infinite graph, such that each vertex x has finite degree. For $x \in \mathbb{V}$ we write $x \sim y$ if $\{x, y\} \in E$. Let $w : E \to (0, \infty)$ be a function which assigns weight w_e to the edge e. We write w_{xy} for $w_{\{x,y\}}$, and define

$$w_x = \sum_{y\sim x} w_{xy}.$$

We call $(\mathbb{V}, E, w)$ a *weighted graph*. An *unweighted graph* has $w_e = 1$ for all $e \in E$. We say that $\mathbb{G}$ has *controlled weights* if there exists $p_0 > 0$ such that

$$\frac{w_{xy}}{w_x} \ge p_0 \text{ for all } x \in \mathbb{V}, y \sim x. \tag{4.17}$$

The *cable system of a weighted graph* gives a natural embedding of a graph in a connected metric length space. Choose a direction for each edge $e \in E$, let $(I_e, e \in E)$ be a collection of copies of the open unit interval, and set

$$\mathcal{X} = \mathbb{V} \cup \bigcup_{e\in E} I_e.$$

(We call the sets I_e *cables*). We define a metric d_c on $\mathcal{X}$ by using Euclidean distance on each cable, and then extending it to a metric on $\mathcal{X}$; note that this agrees with the graph metric for $x, y \in \mathbb{V}$. Let m be the measure on $\mathcal{X}$ which assigns zero mass to points in $\mathbb{V}$, and mass $w_e|s-t|$ to any interval $(s,t) \subset I_e$. It is straightforward to check that $(\mathcal{X}, d_c, m)$ is a MMD space. For more details on this construction see [V, BB3].

We say that a function f on $\mathcal{X}$ is piecewise differentiable if it is continuous at each vertex $x \in \mathbb{V}$, is differentiable on each cable, and has one sided derivatives at the endpoints. Let $\mathcal{F}_d$ be the set of piecewise differentiable functions f with compact support. Given two such functions we set

$$d\Gamma(f,g)(t) = f'(t)g'(t)m(dt), \quad \mathcal{E}(f,g) = \int_{\mathcal{X}} d\Gamma(f,g)(t), \quad f, g \in \mathcal{F}_d,$$

and let $\mathcal{F}$ be the completion of $\mathcal{F}_d$ with respect to the $\mathcal{E}_1^{1/2}$norm. We extend $\mathcal{E}$ to $\mathcal{F}$; it is straightforward to verify that $(\mathcal{E}, \mathcal{F})$ is a closed regular strongly local Dirichlet form. We call $(\mathcal{X}, d_c, m, \mathcal{E}, \mathcal{F})$ the *cable system* of the graph $\mathbb{G}$.

We will now show that both these examples satisfy the conditions $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$, and therefore Assumption 4.9.

Lemma 4.22. *(a) Let $(\mathcal{M}, g, \mu)$ be a weighted Riemmanian manifold with controlled weights w which is quasi isometric to a Riemannian manifold $(\mathcal{M}', g')$ with Ricci curvature bounded below. Then the MMD space $(\mathcal{M}, d_g, \mu, \mathcal{E}_w)$ satisfies the conditions* $(\mathrm{VD})_{\mathrm{loc}}$ *and* $(\mathrm{PI}(2))_{\mathrm{loc}}$.
(b) Let $\mathbb{G}$ be a weighted graph with controlled weights. Then the corresponding cable system satisfies the conditions $(\mathrm{VD})_{\mathrm{loc}}$ *and* $(\mathrm{PI}(2))_{\mathrm{loc}}$.

Proof. (a) The properties $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$ for $(\mathcal{M}', g')$ follow from the Bishop-Gromov volume comparison theorem [Cha, Theorem III.4.5] and Buser's Poincaré inequality (see [Sal02, Lemma 5.3.2]) respectively. Since quasi isometry only changes distances and volumes by at most a constant factor, we have that $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$ also hold for $(\mathcal{M}, g)$. The controlled weights condition on w implies that these two conditions also hold for $(\mathcal{M}, g, \mu)$.
(b) Using the controlled weights condition and the uniform bound on vertex degree, one can easily obtain the two properties $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$. □

Remark 4.23. The paper [L] proves a BHP on MMD spaces which are length spaces and satisfy a weak heat kernel estimate associated with a space time scaling function Ψ, where Ψ satisfies the condition (4.16). By [BGK, Theorem 3.2] these spaces satisfy $(\mathrm{PHI}(\Psi))_{\mathrm{loc}}$ with $R = \infty$. The same argument as in Lemma 4.21 then proves that these spaces satisfy Assumption 4.9.

Examples of spaces of this type are the Sierpinski gasket, nested fractals, and generalized Sierpinski carpets – see [BP, Kum1, BB].

5 Proof of Boundary Harnack Principle

In this section we prove Theorem 1.1. For the remainder of the section, we assume the hypotheses of Theorem 1.1, and will fix a (c_U, C_U)-inner uniform domain U. We can assume that $c_U \le \frac{1}{2} \le 2 \le C_U$, and will also assume that the EHI holds with constants $\delta = \frac{1}{2}$ and C_H. We will use A_i to denote constants which just depend on the constants c_U and C_U; other constants will depend on c_U, C_U and C_H.

Since by Proposition 4.11(e), $(\mathcal{X}, d)$ has the metric doubling property, we can use Lemma 2.8. In addition we will assume that

$$\text{diameter}(U) = \infty,$$

so that $R(U) = \infty$. The proof of the general case is the same except that we need to ensure that the balls $B_U(\xi, s)$ considered in the argument are all small enough so that they do not equal U.

Definition 5.1 (Capacitary width)**.** For an open set $V \subset \mathcal{X}$ and $\eta \in (0,1)$, define the capacitary width $w_\eta(V)$ by

$$w_\eta(V) = \inf \left\{ r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}\left(\overline{B(x,r)} \setminus V\right)}{\mathrm{Cap}_{B(x,2r)}\left(\overline{B(x,r)}\right)} \ge \eta \ \forall x \in V \right\}. \tag{5.1}$$

Note that $w_\eta(V)$ is an increasing function of $\eta \in (0,1)$ and is also an increasing function of the set V.

Lemma 5.2. *(See [Aik01, (2.1)] and [GyS, Lemma 4.12]) There exists $\eta = \eta(c_U, C_U, C_H) \in (0,1)$ and $A_1 > 0$ such that*

$$w_\eta\left(\{x \in U : \delta_U(x) < r\}\right) \le A_1 r.$$

Proof. Set $V_r = \{x \in U : \delta_U(x) < r\}$. By Lemma 2.9 there is a constant $A_1 > 1$ such that for any point $x \in V_r$, there is a point $z \in U \cap B(x, A_1 r)$ with the property that $\delta_U(z) > 2r$. By domain monotonicity of capacity, we have

$$\mathrm{Cap}_{B(x,2A_1r)}\left(\overline{B(x,A_1r)} \setminus V_r\right) \ge \mathrm{Cap}_{B(x,2A_1r)}\left(\overline{B(z,r)}\right) \ge \mathrm{Cap}_{B(z,3A_1r)}\left(\overline{B(z,r)}\right).$$

The capacities $\mathrm{Cap}_{B(z,3A_1r)}\left(\overline{B(z,r)}\right)$ and $\mathrm{Cap}_{B(x,2A_1r)}\left(\overline{B(x,A_1r)}\right)$ are comparable by Proposition 4.11(a)-(c), and so the condition in (5.1) holds for some $\eta > 0$, with r replaced by $A_1 r$. $\square$

We now fix $\eta \in (0,1)$ once and for all, small enough such that the conclusion of Lemma 5.2 applies. In what follows, we write $f \asymp g$, if there exists a constant $C_1 = C_1(c_U, C_U, C_H)$ such that $C_1^{-1} g \le f \le C_1 g$.

Definition 5.3. Let $(X_t, t \geq 0, \mathbb{P}^x, x \in \mathcal{X})$ be the Hunt process associated with the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$. For a Borel subset $U \subset \mathcal{X}$ set

$$\tau_U := \inf \{t > 0 : X_t \notin U\}. \tag{5.2}$$

Let $\Omega \subset \mathcal{X}$ be open and relatively compact in $\mathcal{X}$. Since the process (X_t) is continuous, $X_{\tau_\Omega} \in \partial\Omega$ a.s. We define the *harmonic measure* $\omega(x, \cdot, \Omega)$ on $\partial\Omega$ by setting

$$\omega(x, F, \Omega) := \mathbb{P}^x(X_{\tau_\Omega} \in F) \text{ for } F \subset \partial\Omega.$$

The following lemma provides an useful estimate of the harmonic measure in terms of the capacitary width.

Lemma 5.4. *(See [Aik01, Lemma 1], and [GyS, Lemma 4.13]) There exists $a_1 \in (0,1)$ such that for any non-empty open set $V \subset \mathcal{X}$ and for all $x \in \mathcal{X}$, $r > 0$,*

$$\omega\left(x, V \cap \partial B(x,r), V \cap B(x,r)\right) \leq \exp\left(2 - \frac{a_1 r}{w_\eta(V)}\right).$$

Proof. The proof is same as [GyS, Lemma 4.13] except that we use Proposition 4.11(a),(b) instead of [GyS, Lemma 4.8]. □

In the following lemma, we provide an upper bound of the harmonic measure in terms of the Green function. It is an analogue of [Aik01, Lemma 2].

Lemma 5.5. *(Cf. [GyS, Lemma 4.14], [LS, Lemma 4.9] and [L, Lemma 5.3]) There exists $A_2, C_4 \in (0,\infty)$ such that for all $r > 0$, $\xi \in \partial_{\widetilde{U}} U$, there exist $\xi_r, \xi_r' \in U$ that satisfy $d_U(\xi, \xi_r) = 4r$, $\delta_U(\xi_r) \geq 2c_U r$, $d(\xi_r, \xi_r') = c_U r$ and*

$$\omega\left(x, U \cap \partial_{\widetilde{U}} B_U(\xi, 2r), B_U(\xi, 2r)\right) \leq C_4 \frac{g_{B_U(\xi, A_2 r)}(x, \xi_r)}{g_{B_U(\xi, A_2 r)}(\xi_r', \xi_r)}, \quad \forall x \in B_U(\xi, r).$$

Proof. Let $\xi \in \partial_{\widetilde{U}} U$ and $r > 0$. Fix $A_2 \geq 2(12 + C_U)$ so that all (c_U, C_U)-inner uniform curves that connect two points in $B_U(\xi, 12r)$ stay inside $B_U(\xi, A_2 r/2)$. Fix $\xi_r, \xi_r' \in U$ satisfying the given hypothesis: these points exist by Lemma 2.9. Define

$$g'(z) = g_{B_U(\xi, A_2 r)}(z, \xi_r), \quad \text{for } z \in B_U(\xi, A_2 r).$$

Set $s = \min(c_U r, 5r/C_U)$. Note that $B_U(\xi_r, s) = B(\xi_r, s) \subset U$. Since $B(\xi_r, s) \subset B_U(\xi, A_2 r) \setminus B_U(\xi, 2r)$, using the maximum principle given by Assumption 4.9(iv) we have

$$g'(y) \leq \sup_{z \in \partial B(\xi_r, s)} g'(z) \text{ for all } y \in B_U(\xi, 2r).$$

By Proposition 4.11(a), we have

$$\sup_{z \in \partial B(\xi_r, s)} g'(z) \asymp g'(\xi_r'),$$

and hence there exists $\varepsilon_1 > 0$ such that

$$\varepsilon_1 \frac{g'(y)}{g'(\xi_r')} \le \exp(-1) \quad \forall y \in B_U(\xi, 2r).$$

For all non-negative integers j, define

$$U_j := \left\{ x \in B_U(\xi, A_2 r) : \, \exp\left(-2^{j+1}\right) \le \varepsilon_1 \frac{g'(x)}{g'(\xi_r')} < \exp\left(-2^j\right) \right\},$$

so that $B_U(\xi, 2r) = \bigcup_{j \ge 0} U_j \cap B_U(\xi, 2r)$. Set $V_j = \bigcup_{k \ge j} U_k$. We claim that there exist $c_1, \sigma \in (0, \infty)$ such that for all $j \ge 0$

$$w_\eta \left(V_j \cap B_U(\xi, 2r)\right) \le c_1 r \exp\left(-2^j/\sigma\right). \tag{5.3}$$

Let x be an arbitrary point in $V_j \cap B_U(\xi, 2r)$. Let z be the first point in the inner uniform curve from x to ξ_r which is on $\partial_U B_U(\xi_r, c_U r)$. Then by Lemma 4.7 there exists a Harnack chain of balls in $B_U(\xi, A_2 r) \setminus \{\xi_r\}$ connecting x to z of length at most $c_2 \log\left(1 + c_3 r/\delta_U(x)\right)$ for some constants $c_2, c_3 \in (0, \infty)$. Hence, there are constants $\varepsilon_2, \varepsilon_3, \sigma > 0$ such that

$$\exp(-2^j) > \varepsilon_1 \frac{g'(x)}{g'(\xi_r')} \ge \varepsilon_2 \frac{g'(x)}{g'(z)} \ge \varepsilon_3 \left(\frac{\delta_U(x)}{r}\right)^\sigma.$$

The first inequality above follows from definition of V_j, the second follows from Proposition 4.11(a) and the last one follows from Harnack chaining. Therefore, we have

$$V_j \cap B_U(\xi, 2r) \subset \left\{ x \in U : \, \delta_U(x) \le \varepsilon_3^{-1/\sigma} r \exp\left(-2^j/\sigma\right) \right\},$$

which by Lemma 5.2 immediately implies (5.3).

Set $R_0 = 2r$ and for $j \ge 1$,

$$R_j = \left(2 - \frac{6}{\pi^2} \sum_{k=1}^{j} \frac{1}{k^2}\right) r.$$

Then $R_j \downarrow r$ and as in [GyS]

$$\sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{a_1(R_{j-1} - R_j)}{c_2 r \exp(-2^j/\sigma)}\right) < C < \infty; \tag{5.4}$$

here C depends only on σ, c_2 and the constant a_1 in Lemma 5.4.

Let $\omega_0(\cdot) = \omega\left(\cdot, U \cap \partial_{\widetilde{U}} B_U(\xi, 2r), B_U(\xi, 2r)\right)$ and set

$$d_j = \begin{cases} \sup_{x \in U_j \cap B_U(\xi, R_j)} \frac{g'(\xi_r')\omega_0(x)}{g'(x)}, & \text{if } U_j \cap B_U(\xi, R_j) \ne \emptyset, \\ 0, & \text{if } U_j \cap B_U(\xi, R_j) = \emptyset. \end{cases}$$

It suffices to show that $\sup_{j\geq 0} d_j \leq C_1 < \infty$, and this is proved by iteration exactly as in [LS, Lemma 4.9] or [L, Lemma 5.3]. The only difference is that we replace $r^2/V(\xi,r)$ in [LS] (or $\Psi(r)/V(\xi,r)$ in [L]) by $g'(\xi'_r)$. $\square$

By using a balayage formula (cf. [L, Proposition 4.3]) and a standard argument (cf. [GyS, pp. 75-76], [L, Theorem 5.2]), the proof of Theorem 1.1 reduces to the following estimate on the Green function.

Theorem 5.6. *(See [Aik01, Lemma 3].) There exist $C_1, A_4, A_3 \in (1,\infty)$ such that for all $\xi \in \partial_{\widetilde{U}} U$ and for all $r > 0$, we have, writing $D = B_U(\xi, A_4 r)$,*

$$\frac{g_D(x_1,y_1)}{g_D(x_2,y_1)} \leq C_1 \frac{g_D(x_1,y_2)}{g_D(x_2,y_2)} \text{ for all } x_1,x_2 \in B_U(\xi,r),\ y_1,y_2 \in U \cap \partial_{\widetilde{U}} B_U(\xi, A_3 r).$$

Our proof follows Aikawa's approach, replacing the use of bounds on the Green function with Proposition 4.11. However, as we are working on domains in a metric space rather than $\mathbb{R}^d$, we need to be careful with Harnack chaining. On a general metric space one cannot control the length of a Harnack chain in a punctured domain $D \setminus \{z\}$ by the length of a Harnack chain in D, as is done in [ALM, (2.15)]. For a general inner uniform domain D on a metric space, the domain $D \setminus \{z\}$ need not even be connected. Since this is the key argument in this paper, we provide the full proof, and as the proof is long we split it into several Lemmas.

We define

$$A_3 = \max(2 + 2c_U^{-1}, 7). \tag{5.5}$$

Lemma 5.7. *Let $\xi \in \partial_{\widetilde{U}} U$, $r > 0$, and $y_1, y_2 \in U \cap \partial_{\widetilde{U}} B_U(\xi, A_3 r)$. If γ is a (c_U, C_U)-inner uniform curve from y_1 to y_2 in U, then $\gamma \cap B_U(\xi, 2r) = \emptyset$ and $\gamma \subset \overline{B_U(\xi, A_3(C_U+1)r)}$.*

Proof. Let $z \in \gamma$. If $d_U(y_1,z) \wedge d_U(y_2,z) \leq (A_3 - 2)r$, then by the triangle inequality $d_U(z,\xi) \geq 2r$. If $d_U(y_1,z) \wedge d_U(y_2,z) > (A_3-2)r$, then using the inner uniformity of γ,

$$\delta_U(z) \geq c_U\left(d_U(y_1,z) \wedge d_U(y_2,z)\right) > c_U(A_3-2)r \geq 2r,$$

which implies that $z \notin B_U(\xi, 2r)$.

For the second conclusion, note that for all $z \in \gamma$,

$$d_U(\xi,z) \leq A_3 r + \min\left(d_U(y_1,z), d_U(y_2,z)\right) \leq A_3 r + L(\gamma)/2 \leq A_3(C_U+1)r.$$

$\square$

For $\xi \in \partial_{\widetilde{U}} U$ choose $x^*_\xi \in U \cap \partial_{\widetilde{U}} B_U(\xi, r)$ and $y^*_\xi \in U \cap \partial_{\widetilde{U}} B_U(\xi, A_3 r)$ such that $\delta_U(x^*_\xi) \geq c_U r$ and $\delta_U(y^*_\xi) \geq A_3 c_U r$. Note that we have

$$\delta_U(y^*_\xi) \geq A_3 c_U r > 2r, \tag{5.6}$$

so that $B(y^*_\xi, 2r) \subset U$. Let γ_ξ be an inner uniform curve from y^*_ξ to x^*_ξ, and let z^*_ξ be the last point of this curve which is on $\partial B(y^*_\xi, c_U r)$. We will write these points as x^*, y^*, z^* when the choice of the boundary point ξ is clear.

Define

$$A_4 = A_2 + C_U(A_3 + \frac{1}{4}c_U^2 + 8).$$

To prove Theorem 5.6 it is sufficient to prove that, writing $D = B_U(\xi, A_4 r)$, we have for all $x \in B_U(\xi, r)$ and for all $y \in U \cap \partial_{\widetilde{U}} B_U(\xi, A_3 r)$

$$g_D(x, y) \asymp \frac{g_D(x^*, y)}{g_D(x^*, y^*)} g_D(x, y^*). \tag{5.7}$$

Lemma 5.8. *Let $\xi \in \partial_{\widetilde{U}} U$, $r > 0$ and let $D = B_U(\xi, A_4 r)$. If $x \in B_U(\xi, r)$ and $y \in U \cap \partial_{\widetilde{U}} B_U(\xi, A_3 r)$ with $\delta_U(y) \geq \frac{1}{4}c_U^2 r$, then (5.7) holds.*

Proof. Fix $x \in B_U(\xi, r)$. Set

$$u_1(y') = g_D(x, y'), \quad v_1(y') = \frac{g_D(x^*, y')}{g_D(x^*, y^*)} g_D(x, y^*).$$

The functions u_1 and v_1 are harmonic in $D \setminus \{x, x^*\}$, vanish quasi-everywhere on the boundary of D, and satisfy $u_1(y^*) = v_1(y^*)$. Let γ be a (c_U, C_U)-inner uniform curve from y to y^*; by Lemma 5.7 this curve is contained in $U \setminus B_U(\xi, 2r)$. So by Lemma 2.10 $\delta_U(z) \geq \frac{1}{2}c_U\,(\delta_U(y) \wedge \delta_U(y^*)) \geq \frac{1}{8}c_U^3 r$ for $z \in \gamma$. Thus we can find a Harnack chain of balls in $U \setminus \{x, x^*\}$ of radius $\frac{1}{8}c_U^3 r$ with length less than $C = C(c_U, C_U, C_H)$ which connects y and y^*. Therefore, (5.7) follows from (4.1). ☐

Lemma 5.9. *Let $\xi \in \partial_{\widetilde{U}} U$, $r > 0$, and let $D = B_U(\xi, A_4 r)$. If $x \in B_U(\xi, r)$ and $y \in U \cap \partial_{\widetilde{U}} B_U(\xi, A_3 r)$ with $\delta_U(y) < \frac{1}{4}c_U^2 r$, then*

$$g_D(x, y) \geq c\frac{g_D(x^*, y)}{g_D(x^*, y^*)} g_D(x, y^*). \tag{5.8}$$

Proof. Fix y and call u (respectively, v) the left-hand (resp. right-hand) side of (5.8), viewed as a function of x. By Assumption 4.9, u is harmonic in $D \setminus \{y\}$ and v is harmonic in $D \setminus \{y^*\}$. Moreover, both u and v vanish quasi-everywhere on the boundary of D, and $u(x^*) = v(x^*)$.

Let γ_ξ and z^* be as defined above. By Lemma 2.10, we have $\delta_U(z) \geq \frac{1}{2}c_U \delta_U(x^*) \geq \frac{1}{2}c_U^2 r$ for all $z \in \gamma_\xi$, and so this curve lies a distance at least $\frac{1}{4}c_U^2 r$ from y. By the choice of z^* the part of the curve from z^* to x^* lies outside $B_U(y^*, \frac{1}{2}c_U r)$. Thus there exists $N_1 = N_1(c_U, C_U)$ such that there is a 2-Harnack chain of balls in $U \setminus \{y, y^*\}$ connecting x^* with z^* of length at most N_1. Using this we deduce that there exists $C < \infty$ such that

$$C^{-1}v(z^*) \leq v(x^*) \leq Cv(z^*), \quad C^{-1}u(z^*) \leq u(x^*) \leq Cu(z^*). \tag{5.9}$$

Since $B(y^*, 2c_U r) \subset U \setminus \{y\}$, we can use the EHI and Proposition 4.11 to deduce that

$$C^{-1}v(z^*) \le v(z) \le Cv(z^*), \quad C^{-1}u(z^*) \le u(z) \le Cu(z^*) \text{ for all } z \in \partial B_U(y^*, c_U r).$$

Thus there exist c_1, c_2 such that

$$c_1 u(z) \ge u(x^*) = v(x^*) \ge c_1 c_2 v(z) \text{ for all } z \in \partial B_U(y^*, c_U r). \tag{5.10}$$

Using Lemma 4.12 it follows that $u \ge c_2 v$ on $U \setminus B_U(y^*, c_U r)$, proving (5.8). □

For $\xi \in \partial_{\widetilde{U}} U$ set

$$F(\xi) = B_U(\xi, (A_3 + 3)r) \setminus B_U(\xi, (A_3 - 3)r).$$

Let $A_5 = A_3 + A_4$.

Lemma 5.10. *Let $\xi \in \partial_{\widetilde{U}} U$, and $D = B_U(\xi, A_4 r)$. Then*

$$g_D(x, z) \le C_1 g_D(x, y^*), \quad \textit{for all } x \in B_U(\xi, 2r),\ z \in F(\xi). \tag{5.11}$$

Proof. We begin by proving that

$$g_D(x, y) \le C_1 g_D(x^*, y^*), \quad \text{for all } x \in B_U(\xi, 2r),\ y \in F(\xi). \tag{5.12}$$

Let $x \in B_U(\xi, 2r)$, $y \in F(\xi)$. Let $\widetilde{C_U}$ be the constant from Lemma 2.8. We have $D \subset B(y^*, A_5 r)$, and therefore by domain montonicity of the Green function and Proposition 4.11 we have for any $z \in D$ with $d(x, z) \ge r/(2\widetilde{C_U})$,

$$g_D(x, z) \le g_{B(y^*, A_5 r)}(x, z) \le g_{B(y^*, A_5 r)}(x^*, y^*) \le C_1 g_D(x^*, y^*). \tag{5.13}$$

If $d(x, y) \ge r/(2\widetilde{C_U})$ this gives (5.12).

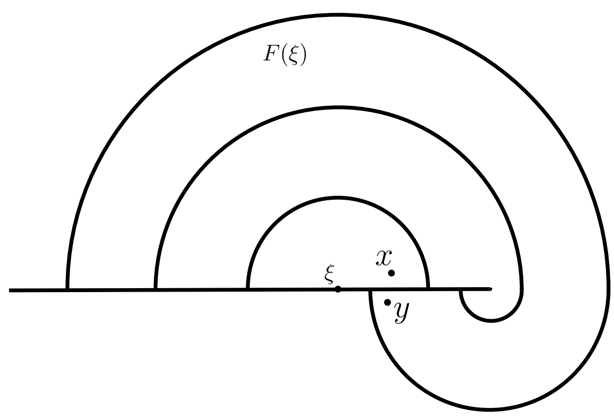

Figure 1: The inner uniform domain $U = \mathbb{R}^2 \setminus ([-1, 0] \times \{0\})$ showing the set $F(\xi)$

Next, we consider the case $d(x, y) < r/(2\widetilde{C_U})$. (See Figure 1 for an example of a slit domain containing such points). Let B_y denote the connected component of

$p^{-1}\left(B(p(y), r/\widetilde{C_U}) \cap \overline{U}\right)$ that contains y. By Lemma 2.8, we have $B_y \subset B_{\widetilde{U}}(y, r)$. As $g_D(x, \cdot)$ is harmonic in $B_y \cap U$, by the maximum principle, we have

$$g_D(x, y) \leq \sup_{z \in U \cap \partial_{\widetilde{U}} B_y} g_D(x, z) \leq \sup_{z \in \partial B(y, r/\widetilde{C_U})} g_D(x, z).$$

By the triangle inequality, we have $d(x, z) \geq r/(2\widetilde{C_U})$ for all $z \in \partial B(y, r/\widetilde{C_U})$, and therefore (5.12) follows from (5.13). This completes the proof of (5.12).

By the continuity of the Green function, we can extend (5.12) as follows:

$$g_D(x, y) \leq C_1 g_D(x^*, y^*), \quad \text{for all } x \in U \cap \overline{B_U(\xi, 2r)}^{d_U},\ y \in F(\xi). \tag{5.14}$$

Now, let $x \in B_U(\xi, 2r)$, $z \in F(\xi)$. Since $g_D(\cdot, z)$ is harmonic in $D \setminus \{z\}$, by the maximum principle we have

$$g_D(x, z) \leq \omega(x, U \cap \partial_{\widetilde{U}} B_U(\xi, 2r), B_U(\xi, 2r)) \sup_{x' \in U \cap \partial_{\widetilde{U}} B_U(\xi, 2r)} g_D(x', z). \tag{5.15}$$

We use Lemma 5.5 to bound the first term, and (5.14) to bound the second, and obtain

$$g_D(x, z) \leq c \frac{g_{B_U(\xi, A_2 r)}(x, \xi_r)}{g_{B_U(\xi, A_2 r)}(\xi_r', \xi_r)} g_D(x^*, y^*). \tag{5.16}$$

We then have by Proposition 4.11(a)-(c), Harnack chaining, and domain monotonicity

$$g_{B_U(\xi, A_2 r)}(\xi_r', \xi_r) \asymp g_D(x^*, y^*), \quad g_{B_U(\xi, A_2 r)}(x, \xi_r) \leq c g_D(x, y^*),$$

and combining these inequalities completes the proof of (5.11). Note that for the second inequality above, one needs to consider two different cases: $\delta_U(x) \leq \frac{1}{2} c_U^2 r$ and $\delta_U(x) > \frac{1}{2} c_U^2 r$. □

Lemma 5.11. *Let $\xi \in \partial_{\widetilde{U}} U$, and $D = B_U(\xi, A_4 r)$. If $x \in B_U(\xi, r)$ and $y \in \partial_U B_U(\xi, A_3 r)$ with $\delta_U(y) < \frac{1}{4} c_U^2 r$, then*

$$g_D(x, y) \leq c \frac{g_D(x^*, y)}{g_D(x^*, y^*)} g_D(x, y^*). \tag{5.17}$$

Proof. Let $\zeta \in \partial_{\widetilde{U}} U$ be a point such that $d_U(y, \zeta) < c_U^2 r/4$, and let ζ_r and ζ_r' be the points given by Lemma 5.5 corresponding to the boundary point ζ. Since $g_D(x, \cdot)$ is harmonic in $B_U(\zeta, 2r)$, we have

$$g_D(x, y) \leq \omega(y, \partial_U B_U(\zeta, 2r), B_U(\zeta, 2r)) \sup_{z \in U \cap \partial_{\widetilde{U}} B_U(\zeta, 2r)} g_D(x, z). \tag{5.18}$$

Since $B_U(\zeta, 2r) \subset F(\xi)$, by Lemma 5.10, the second term in (5.18) is bounded by $c g_D(x, y^*)$. Using Lemma 5.5 to control the first term, we obtain

$$g_D(x, y) \leq c g_D(x, y^*) \frac{g_{B_U(\zeta, A_2 r)}(y, \zeta_r)}{g_{B_U(\zeta, A_2 r)}(\zeta_r', \zeta_r)}. \tag{5.19}$$

Again by Harnack chaining, Proposition 4.11, and domain monotonicity we have

$$g_{B_U(\zeta,A_2r)}(\zeta_r',\zeta_r) \asymp g_D(x^*,y^*),$$

and

$$g_{B_U(\zeta,A_2r)}(y,\zeta_r) \leq c g_D(y,x^*),$$

and combining these estimates completes the proof. $\square$

Proof of Theorem 5.6. The estimate (5.7) follows immediately from Lemmas 5.8, 5.9 and 5.11, and as remarked before, the Theorem follows from (5.7). $\square$

Remark 5.12. One might ask if the converse to Theorem 1.1 holds. That is, suppose $(\mathcal{X},d,\mu,\mathcal{E},\mathcal{F})$ is a MMD space such that for every inner uniform domain the BHP holds. Then does the EHI hold for $(\mathcal{X},d,\mu,\mathcal{E},\mathcal{F})$?

The following example shows this is not the case. Consider the measures μ_α on $\mathbb{R}$ given by $\mu_\alpha(dx) = (1+|x|^2)^{\alpha/2}\,\lambda(dx)$, where λ denotes the Lebesgue measure. (See [GS]). The Dirichlet forms

$$\mathcal{E}_\alpha(f,f) = \int_{\mathbb{R}} |f'(x)|^2 \mu_\alpha(dx)$$

do not satisfy the Liouville property if $\alpha > 1$. This is because the two ends at $\pm\infty$ are transient, so the probability that the diffusion eventually ends up in $(0,\infty)$ is a non-constant positive harmonic function. Since the Liouville property fails, so does the EHI.

On the other hand, the space of inner uniform domains in $\mathbb{R}$ is same as the space of (proper) intervals in $\mathbb{R}$. The space of harmonic functions in a bounded interval vanishing at a boundary point is one dimensional, and hence the BHP holds. We can take $R(U)$ in Theorem 1.1 as $\mathrm{diam}(U)/4$. In view of this example, the following question remains open: *Which diffusions admit the scale invariant BHP for all inner uniform domains?* Theorem 1.1 shows that the EHI provides a sufficient condition for the scale invariant BHP, but the example above shows that the EHI is not necessary.

We now give two examples to which Theorem 1.1 applies but earlier results do not.

Example 5.13. (1) (See [GS, Example 6.14]) Let $n \geq 2$. Consider the measure $\mu_\alpha(dx) = \left(1+|x|^2\right)^{-\alpha/2}\,\lambda(dx)$, where λ is Lebesgue measure on $\mathbb{R}^n$. The second order 'weighted Laplace' operator L_α on $\mathbb{R}^n$ associated with the measure μ_α is given by

$$L_\alpha = \left(1+|x|^2\right)^{-\alpha/2} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left(1+|x|^2\right)^{\alpha/2}\frac{\partial}{\partial x_i}\right) = \Delta + \alpha\frac{x.\nabla}{1+|x|^2}.$$

The operator L_α is the generator of the Dirichlet form

$$\mathcal{E}_\alpha(f,f) = \int_{\mathbb{R}^n} \|\nabla f\|^2\, d\mu_\alpha,$$

on $L^2(\mathbb{R}^n, \mu_\alpha)$. Grigor'yan and Saloff-Coste [GS] show that L_α satisfies the PHI if and only if $\alpha > -n$ but satisfies the EHI for all $\alpha \in \mathbb{R}$. If $\alpha \le n$, the measure μ_α does *not* satisfy the volume doubling property. Assumption 4.9 for this example follows from Lemmas 4.22(a) and 4.21.

(2) The first example of a space that satisfies the EHI but fails to satisfy the volume doubling property was given by Delmotte [Del], in the graph context. A general class of examples similar to [Del] is given in [Bar, Lemma 5.1]. The associated cable systems of these graphs do satisfy the EHI, but do not satisfy a global parabolic Harnack inequality of the kind given in Definition 4.20, i.e. $(\mathrm{PHI}(\Psi))_{\mathrm{loc}}$ with $R = \infty$.

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