

Asymptotics of Entropy Rate in Special Families of Hidden Markov Chains

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Abstract

We generalize a result in [8] and derive an asymptotic formula for entropy rate of a hidden Markov chain around a “weak Black Hole”. We also discuss applications of the asymptotic formula to the asymptotic behaviors of certain channels.

Index Terms—entropy, entropy rate, hidden Markov chain, hidden Markov model, hidden Markov process

1 Introduction

Consider a discrete finite-valued stationary stochastic process $Y = Y_{-\infty}^{\infty} := \{Y_n : n \in \mathbb{Z}\}$. The entropy rate of Y is defined to be

$$H(Y) = \lim_{n \rightarrow \infty} H(Y_{-n}^0)/(n+1);$$

here, $H(Y_{-n}^0)$ denotes the joint entropy of $Y_{-n}^0 := \{Y_{-n}, Y_{-n+1}, \dots, Y_0\}$, and \log is taken to mean the natural logarithm.

If Y is a Markov chain with alphabet $\{1, 2, \dots, B\}$ and transition probability matrix Δ , it is well known that $H(Y)$ can be explicitly expressed with the stationary vector of Y and Δ . A function $Z = Z_{-\infty}^{\infty}$ of the Markov chain Y with the form $Z = \Phi(Y)$ is called a *hidden Markov chain*; here Φ is a function defined on $\{1, 2, \dots, B\}$, taking values in $\mathcal{A} := \{1, 2, \dots, A\}$ (alternatively a hidden Markov chain is defined as a Markov chain observed in noise). For a hidden Markov chain, $H(Z)$ turns out (see Equation (1)) to be the integral of a certain function defined on a simplex with respect to a measure due to Blackwell [4]. However Blackwell’s measure is somewhat complicated and the integral formula appears to be difficult to evaluate in most cases. In general it is very difficult to compute $H(Z)$; so far there is no simple and explicit formula for $H(Z)$.

Recently, the problem of computing the entropy rate of a hidden Markov chain Z has drawn much interest, and many approaches have been adopted to tackle this problem. For

instance, Blackwell’s measure has been used to bound the entropy rate [15] and a variation on the Birch bound [3] was introduced in [5]. An efficient Monte Carlo method for computing the entropy rate of a hidden Markov chain was proposed independently by Arnold and Loeliger [1], Pfister et. al. [17], and Sharma and Singh [19]. The connection between the entropy rate of a hidden Markov chain and the top Lyapunov exponent of a random matrix product has been observed [10, 11, 12, 6]. In [7], it is shown that under mild positivity assumptions the entropy rate of a hidden Markov chain varies analytically as a function of the underlying Markov chain parameters.

Another recent approach is based on computing the coefficients of an asymptotic expansion of the entropy rate around certain values of the Markov and channel parameters. The first result along these lines was presented in [12], where for a binary symmetric channel with crossover probability ε (denoted by BSC(ε)), the Taylor expansion of $H(Z)$ around $\varepsilon = 0$ is studied for a binary hidden Markov chain of order one. In particular, the first derivative of $H(Z)$ at $\varepsilon = 0$ is expressed very compactly as a Kullback-Liebler divergence between two distributions on binary triplets, derived from the marginal of the input process X . Further improvements and new methods for the asymptotic expansion approach were obtained in [16], [20], [21] and [8]. In [16] the authors express the entropy rate for a binary hidden Markov chain where one of the transition probabilities is equal to zero as an asymptotic expansion including a $O(\varepsilon \log \varepsilon)$ term.

This paper is organized as follows. In Section 2 we give an asymptotic formula (Theorem 2.8) for the entropy rate of a hidden Markov chain around a weak Black Hole. The coefficients in the formula can be computed in principle (although explicit computations may be quite complicated in general). The formula can be viewed as a generalization of the Black Hole condition considered in [8]. The *weak* Black Hole case is important for hidden Markov chains obtained as output processes of noisy channels, corresponding to input processes, *for which certain sequences have probability zero*. Examples are given in Section 3. Example 3.1 was already treated in [9] for only the first few coefficients; but in this case, these coefficients were computed quite explicitly.

2 Asymptotic Formula for Entropy Rate

Let W be the simplex, comprising the vectors

$$\{w = (w_1, w_2, \dots, w_B) \in \mathbb{R}^B : w_i \geq 0, \sum_i w_i = 1\},$$

and let W_a be all $w \in W$ with $w_i = 0$ for $\Phi(i) \neq a$. For $a \in \mathcal{A}$, let Δ_a denote the $B \times B$ matrix such that $\Delta_a(i, j) = \Delta(i, j)$ for j with $\Phi(j) = a$, and $\Delta_a(i, j) = 0$ otherwise. For $a \in \mathcal{A}$, define the scalar-valued and vector-valued functions r_a and f_a on W by

$$r_a(w) = w\Delta_a\mathbf{1},$$

and

$$f_a(w) = w\Delta_a/r_a(w).$$

Note that f_a defines the action of the matrix Δ_a on the simplex W .

If Y is irreducible, it turns out that

$$H(Z) = - \int \sum_a r_a(w) \log r_a(w) dQ(w), \quad (1)$$

where Q is *Blackwell's measure* [4] on W . This measure, which satisfies an integral equation dependent on the parameters of the process, is however very hard to extract from the equation in any explicit way.

Definition 2.1. (see [8]) Suppose that for every $a \in \mathcal{A}$, Δ_a is a rank one matrix, and every column of Δ_a is either strictly positive or all zeros. We call this the *Black Hole* case.

It was shown [8] that $H(Z)$ is analytic around a Black Hole and the derivatives of $H(Z)$ can be exactly computed around a Black Hole. In this sequel, we consider weakened assumptions and prove an asymptotic formula for entropy rate of a hidden Markov chain around a “weak Black Hole”, generalizing the corresponding result in [8].

Definition 2.2. Suppose that for every $a \in \mathcal{A}$, Δ_a is either an all zero matrix or a rank one matrix. We call this the *weak Black Hole* case.

We use the standard notation: by $\alpha = \Theta(\beta)$, we mean there exist positive constants C_1, C_2 such that $C_1|\beta| \leq |\alpha| \leq C_2|\beta|$, while by $\alpha = O(\beta)$, we mean there exists a positive constant C such that $|\alpha| \leq C|\beta|$. For a given analytic function $f(\varepsilon)$ around $\varepsilon = 0$, let $\text{ord}(f(\varepsilon))$ denote its order, i.e., the degree of the first non-zero term of its Taylor series expansion around $\varepsilon = 0$. Note that for an analytic function $f(\varepsilon)$ around $\varepsilon = 0$,

$$f(\varepsilon) = \Theta(\varepsilon^k) \iff \text{ord}(f(\varepsilon)) = k.$$

We say $\Delta(\varepsilon)$ is *normally parameterized* by ε ($\varepsilon \geq 0$) if

1. each entry of $\Delta(\varepsilon)$ is an analytic function at $\varepsilon = 0$,
2. when $\varepsilon > 0$, $\Delta(\varepsilon)$ is (non-negative and) irreducible,
3. $\Delta(0)$ is a weak black hole.

In the following, expressions like $p_X(x)$ will be used to mean $P(X = x)$ and we drop the subscripts if the context is clear: $p(x), p(z)$ mean $P(X = x), P(Z = z)$, respectively, and further $p(y|x), p(z_0|z_{-n}^{-1})$ mean $P(Y = y|X = x), P(Z_0 = z_0|Z_{-n}^{-1} = z_{-n}^{-1})$, respectively.

Proposition 2.3. *Suppose that $\Delta(\varepsilon)$ is analytically parameterized by $\varepsilon \geq 0$ and when $\varepsilon > 0$, $\Delta(\varepsilon)$ is non-negative and irreducible. Then for any fixed hidden Markov sequence $z_{-n}^0 \in \mathcal{A}^{n+1}$,*

1. $p(z_{-n}^{-1})$ is analytic around $\varepsilon = 0$;
2. $p(y_i = \cdot | z_{-n}^i) := (p(y_i = b | z_{-n}^i) : b = 1, 2, \dots, B)$ is analytic around $\varepsilon = 0$, where \cdot denotes B possible states of Markov chain Y ,
3. $p(z_0 | z_{-n}^{-1})$ is analytic around $\varepsilon = 0$.

Proof. 1. When $\varepsilon > 0$, $\Delta(\varepsilon)$ is non-negative and irreducible. By Perron-Frobenius theory [18], $\Delta(\varepsilon)$ has a unique positive stationary vector, say $\pi(\varepsilon)$. Since

$$\text{adj}(I - \Delta(\varepsilon))(I - \Delta(\varepsilon)) = \det(I - \Delta(\varepsilon))I = 0$$

(here $\text{adj}(\cdot)$ denotes the adjugate operator on matrices), one can choose $\pi(\varepsilon)$ to be any normalized row vector of $\text{adj}(I - \Delta(\varepsilon))$. So $\pi(\varepsilon)$ can be written as

$$\frac{(\pi_1(\varepsilon), \pi_2(\varepsilon), \dots, \pi_B(\varepsilon))}{\pi_1(\varepsilon) + \pi_2(\varepsilon) + \dots + \pi_B(\varepsilon)},$$

where $\pi_i(\varepsilon)$'s are non-negative analytic functions of ε and the first non-zero term of every $\pi_i(\varepsilon)$'s Taylor series expansion has a positive coefficient. Then we conclude that for each i

$$\text{ord}(\pi_i(\varepsilon)) \geq \text{ord}(\pi_1(\varepsilon) + \dots + \pi_B(\varepsilon)),$$

and thus $\pi(\varepsilon)$, which is uniquely defined on $\varepsilon > 0$, can be continuously extended to $\varepsilon = 0$ via setting $\pi(0) = \lim_{\varepsilon \rightarrow 0} \pi(\varepsilon)$.

Now

$$p(z_{-n}^{-1}) = \pi(\varepsilon)\Delta_{z_{-n}} \cdots \Delta_{z_{-1}} \mathbf{1} = \frac{(\pi_1(\varepsilon), \pi_2(\varepsilon), \dots, \pi_B(\varepsilon))\Delta_{z_{-n}} \cdots \Delta_{z_{-1}} \mathbf{1}}{\pi_1(\varepsilon) + \pi_2(\varepsilon) + \dots + \pi_B(\varepsilon)} =: \frac{f(\varepsilon)}{g(\varepsilon)}, \quad (2)$$

here $\text{ord}(f(\varepsilon)) \geq \text{ord}(g(\varepsilon))$. It then follows that $p(z_{-n}^{-1})$ is analytic around $\varepsilon = 0$.

2. Let $x_{i,-n} = x_{i,-n}(z_{-n}^i)$ denote $p(y_i = \cdot | z_{-n}^i)$. Then one checks that $x_{i,-n}$ satisfies the following iteration:

$$x_{i,-n} = \frac{x_{i-1,-n}\Delta_{z_i}}{x_{i-1,-n}\Delta_{z_i} \mathbf{1}}, \quad -n \leq i \leq -1, \quad (3)$$

starting with $x_{-n-1,-n} = p(y_{-n-1} = \cdot)$. Because Δ is analytically parameterized by ε ($\varepsilon \geq 0$) and $\Delta(\varepsilon)$ is non-negative and irreducible when $\varepsilon > 0$, inductively we can prove (the proof is similar to the proof of 1.) that for any i , $x_{i,-n}$ can be written as follows:

$$x_{i,-n} = \frac{(f_1(\varepsilon), f_2(\varepsilon), \dots, f_B(\varepsilon))}{f_1(\varepsilon) + f_2(\varepsilon) + \dots + f_B(\varepsilon)},$$

where $f_i(\varepsilon)$'s are analytic functions around $\varepsilon = 0$. Note that for each i

$$\text{ord}(f_i(\varepsilon)) \geq \text{ord}(f_1(\varepsilon) + f_2(\varepsilon) + \dots + f_B(\varepsilon)).$$

The existence of the Taylor series expansion of $x_{i,-n}$ around $\varepsilon = 0$ (for any i) then follows.

3. One checks that

$$p(z_0 | z_{-n}^{-1}) = x_{-1,-n}\Delta_{z_0} \mathbf{1}. \quad (4)$$

Analyticity of $p(z_0 | z_{-n}^{-1})$ immediately follows from (4) and analyticity of $x_{-1,-n}$ around $\varepsilon = 0$, which has been shown in 2..

□

Lemma 2.4. Consider two formal series expansion $f(x), g(x) \in \mathbb{R}[[x]]$ such that $f(x) = \sum_{i=0}^{\infty} f_i x^i$ and $g(x) = \sum_{i=0}^{\infty} g_i x^i$, where $g_0 \neq 0$. Let $h(x) \in \mathbb{R}[[x]]$ be the quotient of $f(x)$ and $g(x)$ with $h(x) = \sum_{i=0}^{\infty} h_i x^i$. Then h_i is a function dependent only on f_0, \dots, f_i and g_0, \dots, g_i .

Proof. Comparing the coefficients of all the terms in the following identity:

$$\left(\sum_{i=0}^{\infty} h_i x^i \right) \left(\sum_{i=0}^{\infty} g_i x^i \right) = \sum_{i=0}^{\infty} f_i x^i,$$

we obtain that for any i ,

$$h_0 g_i + h_1 g_{i-1} + \cdots + h_i g_0 = f_i.$$

The lemma then follows from an induction (on i) argument. □

By Proposition 2.3, for any hidden Markov string z_{-m}^0 , the Taylor series expansion of $p(z_0|z_{-m}^{-1})$ around $\varepsilon = 0$ exists. We use $b_j(z_{-m}^0)$ to represent the coefficient of ε^j in the expansion, namely

$$p(z_0|z_{-m}^{-1}) = b_0(z_{-m}^0) + b_1(z_{-m}^0)\varepsilon + b_2(z_{-m}^0)\varepsilon^2 + \cdots. \quad (5)$$

The following lemma shows that under certain conditions, some coefficients $b_j(z_{-m}^0)$ “stabilize”. More precisely, we have:

Lemma 2.5. *Consider a hidden Markov chain Z with normally parameterized $\Delta(\varepsilon)$. For two fixed hidden Markov chain sequences $z_{-m}^0, \hat{z}_{-\hat{m}}^0$ such that*

$$z_{-n}^0 = \hat{z}_{-n}^0, \quad \text{ord}(p(z_{-n}^{-1}|z_{-m}^{-n-1})), \quad \text{ord}(p(\hat{z}_{-n}^{-1}|\hat{z}_{-\hat{m}}^{-n-1})) \leq k$$

for some $n \leq m, \hat{m}$ and some k , we have for j with $0 \leq j \leq n - 4k - 1$,

$$b_j(z_{-m}^0) = b_j(\hat{z}_{-\hat{m}}^0).$$

Proof. Recall that $x_{i,-m} = x_{i,-m}(z_{-m}^i) = p(y_i = \cdot | z_{-m}^i)$ and $\hat{x}_{i,-\hat{m}} = \hat{x}_{i,-\hat{m}}(\hat{z}_{-\hat{m}}^i) = p(y_i = \cdot | \hat{z}_{-\hat{m}}^i)$, where \cdot denotes the possible states of Markov chain Y . Consider the Taylor series expansion of $x_{i,-m}, \hat{x}_{i,-\hat{m}}$ around $\varepsilon = 0$,

$$x_{i,-m} = a_0(z_{-m}^i) + a_1(z_{-m}^i)\varepsilon + a_2(z_{-m}^i)\varepsilon^2 + \cdots \quad (6)$$

$$\hat{x}_{i,-\hat{m}} = a_0(\hat{z}_{-\hat{m}}^i) + a_1(\hat{z}_{-\hat{m}}^i)\varepsilon + a_2(\hat{z}_{-\hat{m}}^i)\varepsilon^2 + \cdots \quad (7)$$

We shall show that $a_j(z_{-m}^i) = a_j(\hat{z}_{-\hat{m}}^i)$ for j with

$$0 \leq j \leq n + i - \sum_{l=-n}^i \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\},$$

where for any hidden Markov sequence z_{-m}^i ,

$$J(z_{-m}^i) = \begin{cases} 1 + \text{ord}(p(z_i|z_{-m}^{i-1})) & \text{if } \text{ord}(p(z_i|z_{-m}^{i-1})) > 0 \\ 0 & \text{if } \text{ord}(p(z_i|z_{-m}^{i-1})) = 0 \end{cases}.$$

Recall that

$$x_{i+1,-m} = \frac{x_{i,-m} \Delta_{z_{i+1}}(\varepsilon)}{x_{i,-m} \Delta_{z_{i+1}}(\varepsilon) \mathbf{1}}. \quad (8)$$

Now with (6) and (7), we have

$$x_{i,-m} \Delta_{z_{i+1}}(\varepsilon) = \sum_{j=0}^{\infty} a_j(z_{-m}^i) \sum_{k=0}^{\infty} \frac{\Delta_{z_{i+1}}^{(k)}(0)}{k!} \varepsilon^k = \sum_{l=0}^{\infty} c_l(z_{-m}^{i+1}) \varepsilon^l, \quad (9)$$

where superscript (k) denotes the k -th order derivative with respect to ε .

We proceed by induction on i (from $-n$ to -1).

First consider the case when $i = -n$. When $\max\{J(z_{-m}^{-n}), J(\hat{z}_{-\hat{m}}^{-n})\} > 0$, the statement is vacuously true; when $J(z_{-m}^{-n}) = J(\hat{z}_{-\hat{m}}^{-n}) = 0$, necessarily $\Delta_{z_{-n}}(0)$ is a rank one matrix, $a_0(z_{-m}^{-n-1}) \Delta_{z_{-n}}(0) \mathbf{1} > 0$ and $a_0(\hat{z}_{-\hat{m}}^{-n-1}) \Delta_{z_{-n}}(0) \mathbf{1} > 0$. Then we have

$$a_0(z_{-m}^{-n}) = \frac{a_0(z_{-m}^{-n-1}) \Delta_{z_{-n}}(0)}{a_0(z_{-m}^{-n-1}) \Delta_{z_{-n}}(0) \mathbf{1}} \stackrel{(*)}{=} \frac{a_0(\hat{z}_{-\hat{m}}^{-n-1}) \Delta_{z_{-n}}(0)}{a_0(\hat{z}_{-\hat{m}}^{-n-1}) \Delta_{z_{-n}}(0) \mathbf{1}} = a_0(\hat{z}_{-\hat{m}}^{-n}),$$

where $(*)$ follows from the fact that $\Delta_{z_{-n}}(0)$ is a rank one matrix.

Now suppose $i \geq -n$ and that $a_j(z_{-m}^i) = a_j(\hat{z}_{-\hat{m}}^i)$ for j with $0 \leq j \leq n + i - \sum_{l=-n}^i \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\}$.

If $\text{ord}(p(z_{i+1}|z_{-m}^i)) > 0$, since the leading coefficient vector of the Taylor series expansion in (9) is non-negative, $c_j(z_{-m}^{i+1}) \equiv \mathbf{0}$ for all j with $0 \leq j \leq J(z_{-m}^{i+1}) - 2$ and $c_{J(z_{-m}^{i+1})-1}(z_{-m}^{i+1}) \neq \mathbf{0}$. So applying Lemma 2.4 to the following expression

$$x_{i+1,-m} = \frac{c_0(z_{-m}^{i+1}) + c_1(z_{-m}^{i+1})\varepsilon + \cdots + c_l(z_{-m}^{i+1})\varepsilon^l + \cdots}{c_0(z_{-m}^{i+1})\mathbf{1} + c_1(z_{-m}^{i+1})\mathbf{1}\varepsilon + \cdots + c_l(z_{-m}^{i+1})\mathbf{1}\varepsilon^l + \cdots} = \frac{\sum_{l=0}^{\infty} c_{l+J(z_{-m}^{i+1})-1}(z_{-m}^{i+1})\varepsilon^l}{\sum_{l=0}^{\infty} c_{l+J(z_{-m}^{i+1})-1}(z_{-m}^{i+1})\mathbf{1}\varepsilon^l}, \quad (10)$$

we conclude that for all j , $a_j(z_{-m}^{i+1})$ depends only on

$$c_l(z_{-m}^{i+1}), \quad J(z_{-m}^{i+1}) - 1 \leq l \leq J(z_{-m}^{i+1}) - 1 + j,$$

implying that $a_j(z_{-m}^{i+1})$ depends only on (or some of)

$$a_l(z_{-m}^i), \quad \Delta_{z_{i+1}}^{(l)}(0), \quad 0 \leq l \leq J(z_{-m}^{i+1}) - 1 + j.$$

A completely parallel argument also applies to the case when $\text{ord}(p(\hat{z}_{i+1}|\hat{z}_{-\hat{m}}^i)) > 0$. More specifically, the statements above for the case $\text{ord}(p(z_{i+1}|z_{-m}^i)) > 0$ are still true if we replace z, x, m with $\hat{z}, \hat{x}, \hat{m}$, which implies that $a_j(\hat{z}_{-\hat{m}}^{i+1})$ depends only on (or some of)

$$a_l(\hat{z}_{-\hat{m}}^i), \quad \Delta_{\hat{z}_{i+1}}^{(l)}(0), \quad 0 \leq l \leq J(\hat{z}_{-\hat{m}}^{i+1}) - 1 + j.$$

Thus when $\max\{J(z_{-m}^{i+1}), J(\hat{z}_{-\hat{m}}^{i+1})\} > 0$, we have $a_j(z_{-m}^{i+1}) = a_j(\hat{z}_{-\hat{m}}^{i+1})$ for j with

$$0 \leq j \leq n + i - \sum_{l=-n}^i \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\} - \max\{J(z_{-m}^{i+1}) - 1, J(\hat{z}_{-\hat{m}}^{i+1}) - 1\}$$

$$= n + (i + 1) - \sum_{l=-n}^{i+1} \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\}.$$

If $\text{ord}(p(z_{i+1}|z_{-m}^i)) = 0$, by (4) necessarily we have

$$a_0(z_{-m}^i)\Delta_{z_{i+1}}(0)\mathbf{1} \neq 0.$$

Again by Lemma 2.4 applied to expression (10), for any j , $a_j(z_{-m}^{i+1})$ depends only on

$$a_l(z_{-m}^i), \quad \Delta_{z_{i+1}}^{(l)}(0), \quad 0 \leq l \leq j,$$

Similarly if $\text{ord}(p(\hat{z}_{i+1}|\hat{z}_{-m}^i)) = 0$, we deduce that for any j , $a_j(\hat{z}_{-m}^{i+1})$ depends only on

$$a_l(\hat{z}_{-m}^i), \quad \Delta_{\hat{z}_{i+1}}^{(l)}(0), \quad 0 \leq l \leq j.$$

Thus if $\max\{J(z_{-m}^{i+1}), J(\hat{z}_{-\hat{m}}^{i+1})\} = 0$, for any j with

$$0 \leq j \leq n + i - \sum_{l=-n}^i \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\} = n + i - \sum_{l=-n}^{i+1} \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\},$$

we have $a_j(z_{-m}^{i+1}) = a_j(\hat{z}_{-m}^{i+1})$.

Now, let $t = n + (i + 1) - \sum_{l=-n}^{i+1} \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\}$. Then one can show that

$$a_t(z_{-m}^{i+1}) = \frac{a_t(z_{-m}^i)\Delta_{z_{i+1}}(0)a_0(z_{-m}^i)\Delta_{z_{i+1}}(0)\mathbf{1} - a_0(z_{-m}^i)\Delta_{z_{i+1}}(0)a_t(z_{-m}^i)\Delta_{z_{i+1}}(0)\mathbf{1}}{(a_0(z_{-m}^i)\Delta_{z_{i+1}}(0)\mathbf{1})^2} + \text{other terms},$$

where the first term in the expression above is equal to $\mathbf{0}$ (since $\Delta_{z_{i+1}}(0)$ is a rank one matrix), and the ‘‘other terms’’ are functions of

$$a_0(z_{-m}^i), \dots, a_{t-1}(z_{-m}^i), \Delta_{z_{i+1}}^{(0)}(0), \dots, \Delta_{z_{i+1}}^{(t)}(0). \quad (11)$$

It follows that $a_j(z_{-m}^{i+1})$ is a function of the same quantities in (11). By a completely parallel argument as above, $a_j(\hat{z}_{-\hat{m}}^{i+1})$ is the same function of of the same quantities in (11). So we have $a_j(z_{-m}^{i+1}) = a_j(\hat{z}_{-\hat{m}}^{i+1})$ for j with

$$0 \leq j \leq n + (i + 1) - \sum_{l=-n}^{i+1} \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\}.$$

Notice that

$$\sum_{l=-n}^{-1} \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\} \leq \sum_{l=-n}^{-1} (J(z_{-m}^l) + J(\hat{z}_{-\hat{m}}^l)) \leq 4k.$$

The lemma then immediately follows from (4) and the proven fact that $a_j(z_{-m}^{-1}) = a_j(\hat{z}_{-\hat{m}}^{-1})$ for j with

$$0 \leq j \leq n - 1 - \sum_{l=-n}^{-1} \max\{J(z_{-m}^l), J(\hat{z}_{-\hat{m}}^l)\}.$$

□

For a mapping $v = v(\varepsilon) : [0, \infty) \rightarrow W$ analytic at $\varepsilon = 0$ and a hidden Markov sequence z_{-n}^0 , define

$$p_v(z_{-n}^{-1}) = v \Delta_{z_{-n}} \cdots \Delta_{z_{-1}} \mathbf{1}, \text{ and } p_v(z_0|z_{-n}^{-1}) = \frac{p_v(z_{-n}^0)}{p_v(z_{-n}^{-1})}.$$

Let $b_{v,j}(z_{-n}^0)$ denote the coefficient of ε^j in the Taylor series expansion of $p_v(z_0|z_{-n}^{-1})$ (note that $b_{v,j}(z_{-n}^0)$ does not depend on ε),

$$p_v(z_0|z_{-n}^{-1}) = \sum_{j=0}^{\infty} b_{v,j}(z_{-n}^0) \varepsilon^j.$$

Using the same inductive approach in Lemma 2.5, we can prove that

Lemma 2.6. *For two mappings $v = v(\varepsilon), \hat{v} = \hat{v}(\varepsilon) : [0, \infty) \rightarrow W$ analytic at $\varepsilon = 0$, if $\text{ord}(p_v(z_{-n}^{-1})), \text{ord}(p_{\hat{v}}(z_{-n}^{-1})) \leq k$, we then have*

$$b_{v,j}(z_{-n}^0) = b_{\hat{v},j}(z_{-n}^0), \quad 0 \leq j \leq n - 4k - 1.$$

Note that for $n \leq m, \hat{m}$, if $v(\varepsilon)$ (or $\hat{v}(\varepsilon)$) is equal to $p(y_{n-1} = \cdot | z_{-m}^{-n-1})$ (or $p(y_{n-1} = \cdot | z_{-\hat{m}}^{-n-1})$), then $p_v(z_{-n}^0)$ (or $p_{\hat{v}}(z_{-n}^0)$) will be equal to $p(z_{-n}^0 | z_{-m}^{-n-1})$ (or $p(z_{-n}^0 | z_{-\hat{m}}^{-n-1})$); and if for a Markov state y , $v(\varepsilon)$ (or $\hat{v}(\varepsilon)$) is equal to $p(y_{n-1} = \cdot | z_{-m}^{-n-1}y)$ (or $p(y_{n-1} = \cdot | z_{-\hat{m}}^{-n-1}y)$), then $p_v(z_{-n}^0)$ (or $p_{\hat{v}}(z_{-n}^0)$) will be equal to $p(z_{-n}^0 | z_{-m}^{-n-1}y)$ (or $p(z_{-n}^0 | z_{-\hat{m}}^{-n-1}y)$). It then immediately follows that

Corollary 2.7. *Given fixed sequences $z_{-m}^0, z_{-\hat{m}}^0, z_{-m}^0 y_{-m-1}, \hat{z}_{-\hat{m}}^0 y_{-\hat{m}-1}$ with $z_{-n}^0 = \hat{z}_{-n}^0$ such that*

$\text{ord}(p(z_{-n}^{-1} | z_{-m}^{-n-1})), \text{ord}(p(\hat{z}_{-n}^{-1} | \hat{z}_{-\hat{m}}^{-n-1})), \text{ord}(p(z_{-n}^{-1} | z_{-m}^{-n-1} y_{-m-1})), \text{ord}(p(\hat{z}_{-n}^{-1} | \hat{z}_{-\hat{m}}^{-n-1} y_{-\hat{m}-1})) \leq k$,
for $n \leq m, \hat{m}$ and some k , we have for j with $0 \leq j \leq n - 4k - 1$,

$$b_j(z_{-m}^0 y_{-m-1}) = b_j(\hat{z}_{-\hat{m}}^0 y_{-\hat{m}-1}) = b_j(z_{-m}^0) = b_j(\hat{z}_{-\hat{m}}^0), \quad (12)$$

where slightly abusing the notation, we define $b_j(z_{-m}^0 y_{-m-1}), b_j(\hat{z}_{-\hat{m}}^0 y_{-\hat{m}-1})$ as the coefficients of the Taylor series expansions of $p(z_0 | z_{-m}^0 y_{-m-1}), p(\hat{z}_0 | \hat{z}_{-\hat{m}}^0 y_{-\hat{m}-1})$, respectively.

Consider expression (5). In the following, we use $p^{<l>}(z_0 | z_{-n}^{-1})$ to denote the truncated (up to the $(l+1)$ -st term) Taylor series expansion of $p(z_0 | z_{-n}^{-1})$, i.e.,

$$p^{<l>}(z_0 | z_{-n}^{-1}) = b_0(z_{-n}^0) + b_1(z_{-n}^0) \varepsilon + b_2(z_{-n}^0) \varepsilon^2 + \cdots + b_l(z_{-n}^0) \varepsilon^l.$$

Theorem 2.8. *For a hidden Markov chain Z with normally parameterized $\Delta(\varepsilon)$, we have for any $k \geq 0$,*

$$H(Z) = H(Z)|_{\varepsilon=0} + \sum_{j=1}^{k+1} f_j \varepsilon^j \log \varepsilon + \sum_{j=1}^k g_j \varepsilon^j + O(\varepsilon^{k+1}), \quad (13)$$

where f_j 's and g_j 's for $j = 1, 2, \dots, k+1$ are functions (more specifically, elementary functions built from log and polynomials) of $\Delta^{(i)}(0)$ for $0 \leq i \leq 6k+6$ and can be computed from $H_{6k+6}(Z(\varepsilon))$.

Proof. First fix n such that $n \geq n_0 = 6k + 6$. Consider the Birch upper bound on $H(Z)$

$$H_n(Z) := H(Z_0|Z_{-n}^{-1}) = - \sum_{z_{-n}^0} p(z_{-n}^0) \log p(z_0|z_{-n}^{-1}).$$

Note that for $j \geq k + 2$,

$$\left| \sum_{\text{ord}(p(z_{-n}^0))=j} p(z_{-n}^0) \log p(z_0|z_{-n}^{-1}) \right| = O(\varepsilon^{k+1}). \quad (14)$$

So, in the following we only consider the sequences z_{-n}^0 with $\text{ord}(p(z_{-n}^0)) \leq k + 1$. For such sequences, since $\text{ord}(p(z_0|z_{-n}^{-1})) \leq \text{ord}(p(z_{-n}^0)) \leq k + 1$, we have

$$|\log p(z_0|z_{-n}^{-1}) - \log p^{<2k+1>}(z_0|z_{-n}^{-1})| = O(\varepsilon^{k+1}); \quad (15)$$

and by Lemma 2.5, we have

$$p^{<2k+1>}(z_0|z_{-n}^{-1}) = p^{<2k+1>}(z_0|z_{-n_0}^{-1}). \quad (16)$$

Now for any fixed $n \geq n_0$,

$$\begin{aligned} H_n(Z) &= \sum_{z_{-n}^0} -p(z_{-n}^0) \log p(z_0|z_{-n}^{-1}) \\ &\stackrel{(a)}{=} \sum_{\text{ord}(p(z_{-n}^0)) \leq k+1} -p(z_{-n}^0) \log p(z_0|z_{-n}^{-1}) + O(\varepsilon^{k+1}) \\ &\stackrel{(b)}{=} \sum_{\text{ord}(p(z_{-n}^0)) \leq k+1} -p(z_{-n}^0) \log p^{<2k+1>}(z_0|z_{-n}^{-1}) + O(\varepsilon^{k+1}) \\ &\stackrel{(c)}{=} \sum_{\text{ord}(p(z_{-n_0}^0)) \leq k+1} -p(z_{-n}^0) \log p^{<2k+1>}(z_0|z_{-n_0}^{-1}) + O(\varepsilon^{k+1}) \\ &= \sum_{\text{ord}(p(z_{-n_0}^0)) \leq k+1} -p(z_{-n_0}^0) \log p^{<2k+1>}(z_0|z_{-n_0}^{-1}) + O(\varepsilon^{k+1}), \end{aligned} \quad (17)$$

where (a) follows from (14); (b) follows from (15); (c) follows from (16), (14) and the fact that

$$\begin{aligned} &\{z_{-n}^0 : \text{ord}(p(z_{-n_0}^0)) \leq k + 1\} \\ &= \{z_{-n}^0 : \text{ord}(p(z_{-n}^0)) \leq k + 1\} \cup \{z_{-n}^0 : \text{ord}(p(z_{-n_0}^0)) \leq k + 1, \text{ord}(p(z_{-n}^0)) \geq k + 2\}. \end{aligned}$$

Expanding (17), we obtain:

$$H_n(Z) = H(Z)|_{\varepsilon=0} + \sum_{j=1}^{k+1} f_j \varepsilon^j \log \varepsilon + \sum_{j=1}^k g_j \varepsilon^j + O(\varepsilon^{k+1}),$$

where f_j 's and g_j 's for $j = 1, 2, \dots, k + 1$ are functions dependent only on $\Delta^{(i)}(0)$ for $0 \leq i \leq n_0$ and can be computed from $H_{n_0}(Z)$ (in fact for fixed j , f_j and g_j are functions

dependent only on $\Delta^{(i)}(0)$ for $0 \leq i \leq 6j + 6$ and can be computed from $H_{6j+6}(Z)$. In particular,

$$\sum_{\text{ord}(p(z_{-n}^0)) \leq k+1} \sum_{\text{ord}(p(z_0|z_{-n_0}^{-1}))=0} -p(z_{-n_0}^0) \log p^{\langle 2k+1 \rangle}(z_0|z_{-n_0}^{-1}) \quad (18)$$

will contribute to $H(Z)|_{\varepsilon=0}$ and the terms ε^j , and

$$\sum_{\text{ord}(p(z_{-n}^0)) \leq k+1} \sum_{\text{ord}(p(z_0|z_{-n_0}^{-1})) > 0} -p(z_{-n_0}^0) \log p^{\langle 2k+1 \rangle}(z_0|z_{-n_0}^{-1}) \quad (19)$$

will contribute to the terms $\varepsilon^j \log \varepsilon$ and the terms ε^j .

Using Corollary 2.7, one can apply similar argument as above to the Birch lower bound

$$\tilde{H}_n(Z) := H(Z_0|Z_{-n}^{-1}Y_{-n-1}) = \sum_{z_{-n}^0, y_{-n-1}} -p(z_{-n}^0 y_{-n-1}) \log p(z_0|z_{-n}^{-1} y_{-n-1}).$$

For the same n_0 , one can show that $\tilde{H}_n(Z)$ takes the same form (17) as $H_n(Z)$, which implies that $H_n(Z)$ and $\tilde{H}_n(Z)$ have exactly the same coefficients of ε^j for $j \leq k$ and of $\varepsilon^j \log \varepsilon$ for $j \leq k + 1$ when $n \geq n_0$. We thus prove the theorem. \square

Remark 2.9. Theorem 2.8 still holds if we assume each entry of $\Delta(\varepsilon)$ is merely a C^{k+1} function of ε in a neighborhood of $\varepsilon = 0$: the proof still works if “analytic” is replaced by “ C^{k+1} ”, and the Taylor series expansions are replaced by Taylor polynomials with remainder. We assumed analyticity of the parametrization only for simplicity.

Remark 2.10. Note that at a Black Hole, we have $\text{ord}(p(z_0|z_{-n}^{-1})) = 0$ for any hidden Markov symbol sequence z_{-n}^0 . Thus, from the discussion surrounding expressions (18) and (19) above, we see that $f_j = 0$ for all j . By the proof of Theorem 2.8, Formula (13) is a Taylor polynomial with remainder; this is consistent with the Taylor series formula for a Black Hole in [8].

Remark 2.11. The proof of Theorem 2.8 shows that for $n \geq n_0$, $H_n(Z), \tilde{H}_n(Z)$ take the same form as in (13) with the same coefficients.

3 Applications to Finite-State Memoryless Channels at High Signal-to-Noise Ratio

Consider a finite-state memoryless channel with stationary input process. Here, $C = \{C_n\}$ is an i.i.d. channel state process over finite alphabet \mathcal{C} with $p_C(c) = q_c$ for $c \in \mathcal{C}$, $X = \{X_n\}$ is a stationary input process, independent of C , over finite alphabet \mathcal{X} and $Z = \{Z_n\}$ is the resulting (stationary) output process over finite alphabet \mathcal{Z} . Let $p(z_n|x_n, c_n) = P(Z_n = z_n|X_n = x_n, C_n = c_n)$ denote the probability that at time n , the channel output symbol is z_n given that the channel state is c_n and the channel input is x_n . The mutual information for such a channel is:

$$I(X, Z) := H(Z) - H(Z|X) \stackrel{(*)}{=} H(Z) - \sum_{x \in \mathcal{X}, z \in \mathcal{Z}} p(x, z) \log p(z|x),$$

where (*) follows from the memoryless property of the channel, and for $x \in \mathcal{X}, z \in \mathcal{Z}$,

$$p(x, z) = \sum_{c \in \mathcal{C}} p(z|x, c)p(x)p(c), \quad p(z|x) = \sum_{c \in \mathcal{C}} p(z|x, c)p(c).$$

Now we introduce an alternative framework, using the concept of channel noise. As above, let C be an i.i.d. channel state process, and let X be a stationary input process, independent of C , over finite alphabets \mathcal{C}, \mathcal{X} . Let \mathcal{E} (resp., \mathcal{Z}) be finite alphabets of abstract error events (resp. output symbols) and let $\Phi : \mathcal{X} \times \mathcal{C} \times \mathcal{E} \rightarrow \mathcal{Z}$ be a function. For each $x \in \mathcal{X}$ and $c \in \mathcal{C}$, let $p(\cdot|x, c)$ be a conditional probability distribution on \mathcal{E} . This defines a jointly distributed stationary process (X, C, E) over $\mathcal{X} \times \mathcal{C} \times \mathcal{E}$. If X is a first order Markov chain with transition probability matrix Π , then (X, C, E) is a Markov chain with transition probability matrix Δ , defined by

$$\Delta_{(x,c,e),(y,d,f)} = \Pi_{xy} \cdot q_d \cdot p(f|y, d)$$

and Φ, Δ define a hidden Markov chain, denoted $Z(\Delta, \Phi)$.

We claim that the output process Z , described in the first paragraph of this section, fits into this alternative framework (when X is a first order Markov chain). To see this, let $\mathcal{E} = \mathcal{X} \times \mathcal{C} \times \mathcal{Z}$, and define $p(e = (x, c, z)|x', c') = p(z|x, c)$ if $x = x'$ and $c = c'$, and 0 otherwise. Define $\Phi(x', c', (x, c, z)) = z$. Then, $Z = Z(\Delta, \Phi)$ is a hidden Markov chain. So, from hereon we adopt the alternative framework.

Now, we assume that X is an irreducible first order Markov chain and that the channel is parameterized by ε such that for each x, c , and e , $p(e|x, c)(\varepsilon)$ are analytic functions of $\varepsilon \geq 0$. For each $\varepsilon \geq 0$, let $\Delta(\varepsilon)$ denote the corresponding transition probability matrix on state set $\mathcal{X} \times \mathcal{C} \times \mathcal{E}$ and $\{Z(\varepsilon)\}$ denote the family of resulting output hidden Markov chains. We also assume that there is a one-to-one function from \mathcal{X} into \mathcal{Z} , $z = z(x)$, such that for all c , $p(z(x)|x, c)(0) = 1$. In other words, ε behaves like a ‘‘composite index’’ indicating how good the channel is, and small ε corresponds to the high signal-to-noise ratio. Then one can verify that $\Delta(0)$ is a weak black hole and $\Delta(\varepsilon)$ is normally parameterized. Thus, by Theorem 2.8, we obtain an asymptotic formula for $H(Z(\varepsilon))$ around $\varepsilon = 0$. We remark that the above naturally generalizes to the case where X is a higher order irreducible Markov chain (through appropriately grouping matrices into blocks).

In the remainder of this section, we give three examples to illustrate the idea.

Example 3.1. [Binary Markov Chains Corrupted by BSC(ε)]

Consider a binary symmetric channel with crossover probability ε . At time n the channel can be characterized by the following equation

$$Z_n = X_n \oplus E_n,$$

where $\{X_n\}$ denotes the input process, \oplus denotes binary addition, $\{E_n\}$ denotes the i.i.d. binary noise with $p_E(0) = 1 - \varepsilon$ and $p_E(1) = \varepsilon$, and $\{Z_n\}$ denotes the corrupted output. Note that this channel only has one channel state, and at $\varepsilon = 0$, $p_{Z|X}(1|1) = 1, p_{Z|X}(0|0) = 1$, so it fits in the alternative framework described in the beginning of Section 3.

Indeed, suppose X is a first order irreducible Markov chain with the transition probability matrix

$$\Pi = \begin{bmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{bmatrix}.$$

Then $Y = \{Y_n\} = \{(X_n, E_n)\}$ is jointly Markov with transition probability matrix (the column and row indices of the following matrix are ordered alphabetically):

$$\Delta = \begin{bmatrix} \pi_{00}(1-\varepsilon) & \pi_{00}\varepsilon & \pi_{01}(1-\varepsilon) & \pi_{01}\varepsilon \\ \pi_{00}(1-\varepsilon) & \pi_{00}\varepsilon & \pi_{01}(1-\varepsilon) & \pi_{01}\varepsilon \\ \pi_{10}(1-\varepsilon) & \pi_{10}\varepsilon & \pi_{11}(1-\varepsilon) & \pi_{11}\varepsilon \\ \pi_{10}(1-\varepsilon) & \pi_{10}\varepsilon & \pi_{11}(1-\varepsilon) & \pi_{11}\varepsilon \end{bmatrix},$$

and $Z = \Phi(Y)$ is a hidden Markov chain with $\Phi(0,0) = \Phi(1,1) = 0$, $\Phi(0,1) = \Phi(1,0) = 1$. When $\varepsilon = 0$,

$$\Delta = \begin{bmatrix} \pi_{00} & 0 & \pi_{01} & 0 \\ \pi_{00} & 0 & \pi_{01} & 0 \\ \pi_{10} & 0 & \pi_{11} & 0 \\ \pi_{10} & 0 & \pi_{11} & 0 \end{bmatrix}, \Delta_0 = \begin{bmatrix} \pi_{00} & 0 & 0 & 0 \\ \pi_{00} & 0 & 0 & 0 \\ \pi_{10} & 0 & 0 & 0 \\ \pi_{10} & 0 & 0 & 0 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & 0 & \pi_{01} & 0 \\ 0 & 0 & \pi_{01} & 0 \\ 0 & 0 & \pi_{11} & 0 \\ 0 & 0 & \pi_{11} & 0 \end{bmatrix},$$

thus both Δ_0 and Δ_1 have rank one. If π_{ij} 's are all positive, then we have a Black Hole case, for which one can derive the Taylor series expansion of $H(Z)$ around $\varepsilon = 0$ [20, 8]; if π_{00} or π_{11} are zero, then this is a weak Black hole case with normal parameterization (of ε), for which Theorem 2.8 can be applied and an asymptotic formula for $H(Z)$ around $\varepsilon = 0$ can be derived.

For a first order Markov chain X with the following transition probability matrix

$$\begin{bmatrix} 1-p & p \\ 1 & 0 \end{bmatrix},$$

where $0 \leq p \leq 1$, it has been shown [16] that

$$H(Z) = H(X) - \frac{p(2-p)}{1+p} \varepsilon \log \varepsilon + O(\varepsilon)$$

as $\varepsilon \rightarrow 0$. This result has been further generalized [9, 13] to the following formula:

$$H(Z) = H(X) + f(X)\varepsilon \log(1/\varepsilon) + g(X)\varepsilon + O(\varepsilon^2 \log \varepsilon), \quad (20)$$

where X is the input Markov chain of any order m with transition probabilities $P(X_t = a_0 | X_{t-m}^{t-1} = a_{-m}^{-1})$, $a_{-m}^0 \in \mathcal{X}^m$, where $\mathcal{X} = \{0, 1\}$, Z is the output process obtained by passing X through a BSC(ε), and $f(X)$ and $g(X)$ can be explicitly computed. Theorem 2.8 can be used to generalize (20) to a formula with higher asymptotic terms. In particular, when $P(X_t = a_0 | X_{t-m}^{t-1} = a_{-m}^{-1}) > 0$ for $a_{-m}^0 \in \mathcal{X}^{m+1}$, we have a Black Hole, in which case, the Taylor series expansions of $H(Z)$ around $\varepsilon = 0$ can be explicitly computed (in principle); when $P(X_t = a_0 | X_{t-m}^{t-1} = a_{-m}^{-1}) = 0$ for some $a_{-m}^0 \in \mathcal{X}^{m+1}$, we have a weak Black Hole, in which case an asymptotic formula of $H(Z)$ around $\varepsilon = 0$ can be obtained.

Example 3.2. [Binary Markov Chains Corrupted by BEC(ε)]

Consider a binary erasure channel with fixed erasure rate ε (denoted by BEC(ε)). At time n the channel can be characterized by the following equation

$$Z_n = \begin{cases} X_n & \text{if } E_n = 0 \\ e & \text{if } E_n = 1 \end{cases},$$

where $\{X_n\}$ denotes the input process, e denotes the erasure, $\{E_n\}$ denotes the i.i.d. binary noise with $p_E(0) = 1 - \varepsilon$ and $p_E(1) = \varepsilon$, and $\{Z_n\}$ denotes the corrupted output. Again this channel only has one channel state, and at $\varepsilon = 0$, $p_{Z|X}(1|1) = 1$, $p_{Z|X}(0|0) = 1$, so it fits in the alternative framework described in the beginning of Section 3.

If the input X is a first order irreducible Markov chain with transition probability matrix

$$\Pi = \begin{bmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{bmatrix},$$

and let Z denote the output process. Then $Y = (X, E)$ is jointly Markov with (the column and row indices of the following matrix are ordered alphabetically)

$$\Delta = \begin{bmatrix} \pi_{00}(1 - \varepsilon) & \pi_{00}\varepsilon & \pi_{01}(1 - \varepsilon) & \pi_{01}\varepsilon \\ \pi_{00}(1 - \varepsilon) & \pi_{00}\varepsilon & \pi_{01}(1 - \varepsilon) & \pi_{01}\varepsilon \\ \pi_{10}(1 - \varepsilon) & \pi_{10}\varepsilon & \pi_{11}(1 - \varepsilon) & \pi_{11}\varepsilon \\ \pi_{10}(1 - \varepsilon) & \pi_{10}\varepsilon & \pi_{11}(1 - \varepsilon) & \pi_{11}\varepsilon \end{bmatrix},$$

and $Z = \Phi(Y)$ is hidden Markov with $\Phi(0, 1) = \Phi(1, 1) = e$, $\Phi(0, 0) = 0$ and $\Phi(1, 0) = 1$.

Now one checks that

$$\Delta_0 = \begin{bmatrix} \pi_{00}(1 - \varepsilon) & 0 & 0 & 0 \\ \pi_{00}(1 - \varepsilon) & 0 & 0 & 0 \\ \pi_{10}(1 - \varepsilon) & 0 & 0 & 0 \\ \pi_{10}(1 - \varepsilon) & 0 & 0 & 0 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & 0 & \pi_{01}(1 - \varepsilon) & 0 \\ 0 & 0 & \pi_{01}(1 - \varepsilon) & 0 \\ 0 & 0 & \pi_{11}(1 - \varepsilon) & 0 \\ 0 & 0 & \pi_{11}(1 - \varepsilon) & 0 \end{bmatrix}, \Delta_e = \begin{bmatrix} 0 & \pi_{00}\varepsilon & 0 & \pi_{01}\varepsilon \\ 0 & \pi_{00}\varepsilon & 0 & \pi_{01}\varepsilon \\ 0 & \pi_{10}\varepsilon & 0 & \pi_{11}\varepsilon \\ 0 & \pi_{10}\varepsilon & 0 & \pi_{11}\varepsilon \end{bmatrix}.$$

One checks that $\Delta(\varepsilon)$ is normally parameterized by ε and thus Theorem 2.8 can be applied. Furthermore, Theorem 2.8 can be applied to the case when the input is an m -th order irreducible Markov chain X to obtain asymptotic formula for $H(Z)$ around $\varepsilon = 0$.

Example 3.3. [Binary Markov Chains Corrupted by Special Gilbert-Elliot Channel]

Consider a binary Gilbert-Elliot channel, whose channel state (denoted by $C = \{C_n\}$) varies as an i.i.d. binary stochastic process with $p_C(0) = q_0$, $p_C(1) = q_1$ (here the channel state varies as an i.i.d. process, rather than a generic Markov process). At time n the channel can be characterized by the following equation

$$Z_n = X_n \oplus E_n,$$

where $\{X_n\}$ denotes the input process, \oplus denotes binary addition, $\{E_n\}$ denotes the i.i.d. binary noise with $p_{E|C}(0|0) = 1 - \varepsilon_0$, $p_{E|C}(0|1) = 1 - \varepsilon_1$, $p_{E|C}(1|0) = \varepsilon_0$, $p_{E|C}(1|1) = \varepsilon_1$ and $\{Z_n\}$ denotes the corrupted output. For such a channel, $p_{Z|(X,C)}(1|1, c) = 1$, $p_{Z|(X,C)}(0|0, c) = 1$ at $\varepsilon = 0$ for any channel state c . So it fits in the alternative framework described in the beginning of Section 3.

To see this in more detail, we consider the special case when the input X is a first order irreducible Markov chain with transition probability matrix

$$\Pi = \begin{bmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{bmatrix},$$

and let Z denote the output process. Then $Y = (X, C, E)$ is jointly Markov with (the column and row indices of the following matrix are ordered alphabetically)

$$\Delta = \begin{bmatrix} \pi_{00q_0}(1-\varepsilon_0) & \pi_{00q_0\varepsilon_0} & \pi_{00q_1}(1-\varepsilon_1) & \pi_{00q_1\varepsilon_1} & \pi_{01q_0}(1-\varepsilon_0) & \pi_{01q_0\varepsilon_0} & \pi_{01q_1}(1-\varepsilon_1) & \pi_{01q_1\varepsilon_1} \\ \pi_{00q_0}(1-\varepsilon_0) & \pi_{00q_0\varepsilon_0} & \pi_{00q_1}(1-\varepsilon_1) & \pi_{00q_1\varepsilon_1} & \pi_{01q_0}(1-\varepsilon_0) & \pi_{01q_0\varepsilon_0} & \pi_{01q_1}(1-\varepsilon_1) & \pi_{01q_1\varepsilon_1} \\ \pi_{00q_0}(1-\varepsilon_0) & \pi_{00q_0\varepsilon_0} & \pi_{00q_1}(1-\varepsilon_1) & \pi_{00q_1\varepsilon_1} & \pi_{01q_0}(1-\varepsilon_0) & \pi_{01q_0\varepsilon_0} & \pi_{01q_1}(1-\varepsilon_1) & \pi_{01q_1\varepsilon_1} \\ \pi_{00q_0}(1-\varepsilon_0) & \pi_{00q_0\varepsilon_0} & \pi_{00q_1}(1-\varepsilon_1) & \pi_{00q_1\varepsilon_1} & \pi_{01q_0}(1-\varepsilon_0) & \pi_{01q_0\varepsilon_0} & \pi_{01q_1}(1-\varepsilon_1) & \pi_{01q_1\varepsilon_1} \\ \pi_{10q_0}(1-\varepsilon_0) & \pi_{10q_0\varepsilon_0} & \pi_{10q_1}(1-\varepsilon_1) & \pi_{10q_1\varepsilon_1} & \pi_{11q_0}(1-\varepsilon_0) & \pi_{11q_0\varepsilon_0} & \pi_{11q_1}(1-\varepsilon_1) & \pi_{11q_1\varepsilon_1} \\ \pi_{10q_0}(1-\varepsilon_0) & \pi_{10q_0\varepsilon_0} & \pi_{10q_1}(1-\varepsilon_1) & \pi_{10q_1\varepsilon_1} & \pi_{11q_0}(1-\varepsilon_0) & \pi_{11q_0\varepsilon_0} & \pi_{11q_1}(1-\varepsilon_1) & \pi_{11q_1\varepsilon_1} \\ \pi_{10q_0}(1-\varepsilon_0) & \pi_{10q_0\varepsilon_0} & \pi_{10q_1}(1-\varepsilon_1) & \pi_{10q_1\varepsilon_1} & \pi_{11q_0}(1-\varepsilon_0) & \pi_{11q_0\varepsilon_0} & \pi_{11q_1}(1-\varepsilon_1) & \pi_{11q_1\varepsilon_1} \\ \pi_{10q_0}(1-\varepsilon_0) & \pi_{10q_0\varepsilon_0} & \pi_{10q_1}(1-\varepsilon_1) & \pi_{10q_1\varepsilon_1} & \pi_{11q_0}(1-\varepsilon_0) & \pi_{11q_0\varepsilon_0} & \pi_{11q_1}(1-\varepsilon_1) & \pi_{11q_1\varepsilon_1} \end{bmatrix},$$

and $Z = \Phi(X, C, E)$ is hidden Markov with

$$\Phi(0, 0, 0) = \Phi(0, 1, 0) = \Phi(1, 0, 1) = \Phi(1, 1, 1) = 0,$$

$$\Phi(0, 0, 1) = \Phi(0, 1, 1) = \Phi(1, 0, 0) = \Phi(1, 1, 0) = 1.$$

For some positive k , let $\varepsilon_0 = \varepsilon$, $\varepsilon_1 = k\varepsilon$. If $\varepsilon = 0$, one checks that

$$\Delta_0 = \begin{bmatrix} \pi_{00q_0} & 0 & \pi_{00q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{00q_0} & 0 & \pi_{00q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{00q_0} & 0 & \pi_{00q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{00q_0} & 0 & \pi_{00q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{10q_0} & 0 & \pi_{10q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{10q_0} & 0 & \pi_{10q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{10q_0} & 0 & \pi_{10q_1} & 0 & 0 & 0 & 0 & 0 \\ \pi_{10q_0} & 0 & \pi_{10q_1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \pi_{01q_0} & 0 & \pi_{01q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{01q_0} & 0 & \pi_{01q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{01q_0} & 0 & \pi_{01q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{01q_0} & 0 & \pi_{01q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{11q_0} & 0 & \pi_{11q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{11q_0} & 0 & \pi_{11q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{11q_0} & 0 & \pi_{11q_1} & 0 \\ 0 & 0 & 0 & 0 & \pi_{11q_0} & 0 & \pi_{11q_1} & 0 \end{bmatrix}.$$

So, both Δ_0 and Δ_1 will be rank one matrices and one can check that $\Delta(\varepsilon)$ is normally parameterized by ε . Again, Theorem 2.8 can be applied to the case when the input is an m -th order irreducible Markov chain X to obtain an asymptotic formula for $H(Z)$ around $\varepsilon = 0$.

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