MATH 215/255 QUIZ 3: SOLUTIONS (SUMMER 2015, T1)

QUESTION 1

(i) Set:

$$\begin{vmatrix} 0-r & -5 \\ 1 & \alpha - r \end{vmatrix} = 0$$

(ii) Characteristic equation:

$$r^2 - \alpha r + 5 = 0$$

(iii) Solve for $r_{1,2}$:

$$r_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 20}}{2}$$

(iv) The phase portrait changes as α varies as follows:

$\alpha \leq -\sqrt{20}$: Stable node

$-\sqrt{20} < \alpha < 0$: Stable spiral

$\alpha = 0$: Center

$0 < \alpha < \sqrt{20}$: Untable spiral

$\alpha \geq \sqrt{20}$: Unstable node

(v) Therefore Answer is (D)

QUESTION 2

(a) Find Homogeneous solution

Find eigen values

$$\begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0$$

$$r^2 - 1 = 0$$

$$r_1 = -1, r_2 = 1$$

Find eigen vectors for the corresponding eigen values

$r_1 = -1$:

$$(A - r_1 I)\vec{v_1} = 0$$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_{1y} = 3v_{1x}$$

$$\vec{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$r_2 = 1$:

$$(A - r_2 I)\vec{v_2} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_{2y} = v_{2x}$$

$$\vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The homogeneous solution is:

$$\underline{\underline{\vec{x_h} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}}}$$

(b) Find Particular solution

Guess the form of the particular solution:

$$\vec{x_p} = \vec{a}e^t + \vec{b}te^t$$

$$\vec{x_p}' = (\vec{a} + \vec{b})e^t + \vec{b}te^t$$

Plug into initial equation and collect like terms

te^t terms:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2b_1 - b_2 \\ 3b_1 - 2b_2 \end{pmatrix}$$

e^t terms:

$$\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} = \begin{pmatrix} 2a_1 - a_2 \\ 3a_1 - 2a_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Combining the two we have the following sets of equations:

$$-a_1 + a_2 + b_1 = 1$$

$$-3a_1 + 3a_2 + b_2 = -1$$

$$-b_1 + b_2 = 0$$

$$-3b_1 + 3b_2 = 0$$

Put it in matrix form and find the row reduced echelon form:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ -3 & 3 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 1 \\ -3 & 3 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right)$$

Row Reduced Echelon Form (RREF):

$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore we get:

$$b_1 = b_2 = 2$$

$$a_1 = a_2 + 1$$

$$\vec{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\vec{a} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Any "c" would work, but let us pick c=0, then our particular solution would be:

$$\underline{\underline{\vec{x_p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t}}$$