# Math 215/255 Final Exam (Dec 2011)

**Last Name**: ______________________________ **First name**: ______________________________

**Student #**: ______________________________ **Signature**: ______________________________

**Circle your section #**:

Rozada=101, Li=102, Magyar=103, Kitagawa=104, Karli=105

I have read and understood the instructions below:

**Please sign:**

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**Instructions:**

1. The exam is closed-book exam. No notes or books or calculators are allowed.
2. Justify every answer whenever is necessary, and show your work. Unsupported answers will receive no credit.
3. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.
4. At the end of the hour you will be given the instruction "Put away all writing implements and remain seated." *Continuing to write after this instruction will be considered as cheating.*
5. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

| Question | grade | value |
|---|---|---|
| 1 | | 8 |
| 2 | | 10 |
| 3 | | 7 |
| 4 | | 20 |
| 5 | | 17 |
| 6 | | 18 |
| 7 | | 20 |
| **Total** | | **100** |

**Problem I.** **[8 marks]**

(a) (4 marks) Solve the following linear ODE:

$$y' - \frac{4}{t}\, y = -\frac{2}{t^2}, \qquad t > 0.$$

(b) (4 marks) Solve the following nonlinear ODE:
(Hint: the substitution $w = v^{-1/2}$ will probably simplify matters)

$$t^2 w' + 2tw - w^3 = 0, \qquad t > 0.$$

**Problem II.** **[10 marks]**

(a) (4 marks) Find all values of $a$ and $b$ for which the ODE below is an exact equation. (It is not necessary to solve the ODE.)

$$ax^3e^{x+y} + bx^4e^{x+y} + 2x + (x^4e^{x+y} + 2y)\,\frac{dy}{dx} = 0$$

(b) (6 marks) Solve the following ODE:

$$3x^2y + 8xy^2 + (x^3 + 8x^2y + 12y^2)\,\frac{dy}{dx} = 0$$

**Problem III.** **[7 marks]**

For the following ODE

$$\frac{dy}{dt} = (y^2 - 1)(y - 2)^2$$

(a) (2 marks) Determine the equilibrium (fixed) points.

(b) (2 marks) Determine the sign of $y'$ in the regions separated by the equilibrium points)

(c) (3 marks) Determine the $\lim_{t\to\infty} y(t)$ and sketch roughly the graph of the solutions for each of the following initial conditions: $y(0) = 1.5$, $y(0) = -3$, and $y(0) = 4$.

**Problem IV.** **[20 marks]**

Suppose the motion of a spring-mass system is described by the following differential equation

$$2u'' + 2\gamma u' + 18u = 0, \qquad u(0) = 1, \quad u'(0) = -\frac{11}{2}$$

(a) (4 marks) Find all values of the parameter for which $\gamma$ for which the system is overdamped or critically damped (that is the mass cannot pass its equilibrium position more than once, so there are no oscillatory solutions).

(b) (8 marks) Solve the following equation for $u(t)$ in explicit form.

$$u'' + 6u' + 25u = 15\cos(5t), \qquad u(0) = 1, \quad u'(0) = -\frac{11}{2}.$$

(c) (8 marks) Find the general solution of

$$t^2y'' - 4ty' + 6y = 3t, \quad (t > 0)$$

provided that $y_1(t) = t^2$ and $y_2(t) = t^3$ are two linearly independent solutions of the homogeneous equation: $t^2y'' - 4ty' + 6y = 0$.

**Problem V.** **[17 marks]**

Given the function

$$f(t) = \begin{cases} 0 & , t < 1 \\ 2\pi & , 1 \leq t < 2 \\ \pi & , t \geq 2 \end{cases}$$

(a) (5 marks) Find the Laplace transform of $f(t)$.

(b) (8 marks) Solve the initial value problem

$$y'' - 5y' + 6y = f(t), \qquad y(0) = 1, \quad y'(0) = 0.$$

(c) (4 marks) Let $f$ and $g$ be two functions satisfying the following conditions:

$$f(0) = 2, \quad f'(0) = 3, \quad f''(1) = 2$$

$$G(s) = e^{-4s}(s^2F(s) - 2s - 3)$$

where $F(s)$ and $G(s)$ are the Laplace transforms of the funcions $f(t)$ and $g(t)$, respectively. Find the value of $g(5)$.

**Problem VI.** **[18 marks]**

Let

$$A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}.$$

(a) (3 marks) Find the eigenvalues of $A$.

(b) (6 marks) Find the general solution of the system of equations

$$\mathbf{x}' = A\mathbf{x}$$

Here $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

(c) (9 marks) Find the solution to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2e^{2t} \\ 1 \end{pmatrix}$$

and

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

**Problem VII.** **[20 marks]**

Consider the system of equations

$$x' = x(2 - x - y)$$
$$y' = y(1 - x)$$

(a) (4 marks) Find and plot the critical (or fixed) points. Find and plot both nullclines, that is the set of points where $x' = 0$ and where $y' = 0$.

(b) (4 marks) Find the Jacobian matrix of the system.

(c) (9 marks) Classify each critical (fixed) point, and sketch the phase portrait of the linearized system near each critical point.

(d) (3 marks) Sketch the phase portrait of the non-linear system.

# Table of Laplace Transforms

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}, \quad s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| $t^n, \quad n =$ positive integer | $\frac{n!}{s^{n+1}}, \quad s > 0$ |
| $t^p, \quad p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$ |
| $\sin at$ | $\frac{a}{s^2+a^2}, \quad s > 0$ |
| $\cos at$ | $\frac{s}{s^2+a^2}, \quad s > 0$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$ |
| $\cosh at$ | $\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$ |
| $e^{at}\sin bt$ | $\frac{b}{(s^2-a^2)+b^2}, \quad s > a$ |
| $e^{at}\cos bt$ | $\frac{s-a}{(s^2-a^2)+b^2}, \quad s > a$ |
| $t^n e^{at}, \quad n =$ positive integer | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| $u_c(t)$ | $\frac{e^{-cs}}{s}, \quad s > 0$ |
| $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ |
| $e^{ct}f(t)$ | $F(s-c)$ |
| $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$ |
| $\int_0^t f(t-\tau)g(\tau)d\tau$ | $F(s)G(s)$ |
| $\delta(t-c)$ | $e^{-cs}$ |
| $f^{(n)}(t)$ | $s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$ |
| $(-t)^n f(t)$ | $F^{(n)}(s)$ |