Math 321 Midterm 2 Solutions

1. (a) *When is a function $\alpha : [a,b] \to \mathbb{R}$ said to be of bounded variation?*

Solution. A function $\alpha : [a,b] \to \mathbb{R}$ is said to be of bounded variation if its total variation $V_a^b \alpha$ is finite. The total variation is defined to be

$$V_a^b \alpha = \sup_P \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})|,$$

where the supremum is taken over all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$. □

(b) *Determine whether the function $\alpha : [0,1] \to \mathbb{R}$ given by*

$$\alpha(x) = \begin{cases} \log(1+x)\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is of bounded variation.

Solution. The given function α is not of bounded variation. To see this, let us choose, for every large integer N, a partition P_N of $[0,1]$ of the form

$$P_N = \{1 = t_0 > t_1 > t_2 > \cdots t_{2N} > t_{2N+1} = 0\}, \text{ where } t_k = \frac{2}{k\pi}, 1 \leq k \leq 2N.$$

Since $\sin(1/t_k)$ vanishes for even k and equals ± 1 for odd k, one of the terms in any pair $(\alpha(t_k), \alpha(t_{k+1}))$ must vanish. This means that

$$|\alpha(t_k) - \alpha(t_{k+1})| = \log(1 + s_k) \text{ where } s_k = \begin{cases} t_k & \text{if } k \text{ is odd }, \\ t_{k+1} & \text{if } k \text{ is even.} \end{cases}$$

In other words,

$$\sum_{k=1}^N |\alpha(t_k) - \alpha(t_{k-1})| \geq \sum_{k=1}^N \log(1 + t_{2k-1}) = \sum_{k=1}^N \log\left(1 + \frac{2}{\pi(2k-1)}\right). \tag{1}$$

We know that

$$\frac{\log(1+x)}{x} \to 1 \text{ as } x \to 0.$$

Therefore by limit comparison test, the last series in (1) is comparable to the partial sum $\sum_{k=1}^N 1/k$ of the harmonic series, which diverges to ∞ as $N \to \infty$. □

(c) *A linear functional $L : C[0,1] \to \mathbb{R}$ obeys the following property: for every continuously differentiable $g : [0,1] \to \mathbb{R}$,*

$$L(g) = -\int_0^1 g'(x)\cos(\pi x)\, dx.$$

Does there exist $\alpha \in BV[0,1]$ such that

$$L(f) = \int_0^1 f(x) d\alpha(x), \text{ for every } f \in C[0,1]?$$

If yes, find such a function α. If not, explain why not. Clearly state any result you need to use.

Solution. Integrating by parts, we find that for every continuously differentiable function g,

$$\begin{aligned} L(g) &= -\cos(\pi x)g(x)\Big|_{x=0}^{x=1} + \int_0^1 (-\pi)\sin(\pi x)g(x)\,dx \\ &= g(1) + g(0) - \pi \int_0^1 g(x)\sin(\pi x)\,dx \\ &= g(1) + g(0) + \int_0^1 g(x)d(\cos(\pi x)). \end{aligned}$$

The last expression is linear in g, is meaningful for every *continuous* function g (not merely continuously differentiable), and is bounded above in absolute value by a constant multiple of $||g||_\infty$. Thus by the Riesz representation theorem, L is given by a Riemann-Stieltjes integral with respect to an integrator $\alpha \in BV[0,1]$. In this case, one possible choice of α is the following:

$$\alpha(x) = \cos(\pi x) + \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x < 1, \\ 2 & \text{if } x = 1. \end{cases}$$

□

2. *For each of the following statements, determine whether it is true or false. The answer should be in the form of a short proof or an example, as appropriate.*

(a) *There exists a bounded function on $[a,b]$ that fails to be Riemann-Stieltjes integrable with respect to every nondecreasing non-constant integrator α.*

Proof. True. The function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is such a function. For any nondecreasing α such that $\alpha(a) < \alpha(b)$, and any partition P of $[a,b]$,

$$L_\alpha(P,f) = 0, \text{ but } U_\alpha(P,f) = \alpha(b) - \alpha(a).$$

Hence Riemann's condition fails. □

(b) *The class $C[a,b]$ consists of all functions that are Riemann-Stieltjes integrable on $[a,b]$ with respect to every nondecreasing integrator α.*

Proof. True. Let $\mathcal{R}_\alpha[a,b]$ denote the class of functions that are Riemann-Stieltjes integrable with respect to the integrator α. Using Riemann's condition, we have shown in class that $C[a,b] \subseteq \mathcal{R}_\alpha[a,b]$ for every nondecreasing α. Conversely, suppose f is discontinuous at a point $x_0 \in [a,b]$. This means that

$$\sup\{|f(x) - f(y)| : x, y \in (x_0 - \delta, x_0 + \delta)\} \to \epsilon_0 > 0 \text{ as } \delta \to 0+\,.$$

Let α be a nondecreasing step function with a unit jump only at the point x_0, with the same-sided discontinuity as f. Then for any sufficiently fine partition P with x_0 as a partition point, we have

$$U_\alpha(P,f) - L_\alpha(P,f) \geq \epsilon_0,$$

which violates Riemann's condition. □

(c) *The Fourier series of a continuous 2π-periodic function f converges to f in the L^1 norm $||\cdot||_1$. Recall*

$$||f||_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|\,dx.$$

Proof. True. By the Cauchy-Schwarz inequality, $||g||_1 \le ||g||_2$ for any Riemann-integrable function g. Let $S_N f$ denote the Nth partial Fourier sum of f. Since we know that $||S_N f - f||_2 \to 0$ by Plancherel's theorem, it follows from the inequality above that $||S_N f - f||_1 \to 0$ as $N \to \infty$. □

(d) *For any bounded, Riemann integrable function f, the sequence of Fourier coefficients $\{\widehat{f}(k) : k \ge 0\}$ converges to zero.*

Proof. True. Since a bounded Riemann-integrable function is square-integrable, we know by Plancherel's theorem that

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty$$

Thus the left hand side is a summable series, and hence the kth summand goes to zero as $k \to \infty$. □

(e) Let f be a bounded Riemann integrable function on $[-\pi,\pi]$. Then $||\sigma_N f - f||_1 \to 0$ as $N \to \infty$. Here $\sigma_N f$ denotes the Nth partial Cesaro sum of f.

Solution. Fix $\epsilon > 0$. By HW 7 Problem 4(a) we know that there exists a continuous 2π-periodic function g such that $||f - g||_2 < \epsilon$. By Fejer's theorem, we know that $||\sigma_N g - g||_\infty \to 0$ as $N \to \infty$. Combining these steps together and using the triangle inequality, we find that

$$\begin{aligned} ||\sigma_N f - f||_1 &\le ||\sigma_N(f-g)||_1 + ||f-g||_1 + ||\sigma_N g - g||_1 \\ &\le 2||f-g||_1 + ||\sigma_N g - g||_\infty \\ &\le 2\epsilon + ||\sigma_N g - g||_\infty \to 2\epsilon \text{ as } N \to \infty. \end{aligned}$$

The estimate $||\sigma_N(f-g)||_1 \le ||f-g||_1$ used in the second step follows from the fact that for any function h,

$$\begin{aligned} ||\sigma_N h||_1 = ||K_N * h||_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y)h(y)\,dy\right| dx \\ &\le ||K_N||_1||h||_1 = ||h||_1, \end{aligned}$$

where K_N denotes the Fejer kernel. □

3. *Let $\alpha, \beta > 0$. Evaluate the sum*

$$S = \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} \cos\bigl(m(x+y)\bigr)\,dy\,dx.$$

Solution. Let us define 2π-periodic functions f_α and f_β as follows:

$$f_\alpha(x) = (-x)^{2\alpha}, \qquad f_\beta(x) = x^{2\beta}, \qquad x \in [-\pi,\pi).$$

We first simplify S as follows:

$$\begin{aligned}
S &= \mathrm{Re}\left[\sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} e^{im(x+y)}\, dy\, dx\right] \\
&= \mathrm{Re}\left[\sum_{m\in\mathbb{Z}} \left\{\int_{-\pi}^{\pi} x^{2\alpha} e^{imx}\, dx\right\} \overline{\left\{\int_{-\pi}^{\pi} y^{2\beta} e^{-imy}\, dy\right\}}\right] \\
&= \mathrm{Re}\left[\sum_{m\in\mathbb{Z}} \left\{\int_{-\pi}^{\pi} (-x)^{2\alpha} e^{-imx}\, dx\right\} \overline{\left\{\int_{-\pi}^{\pi} y^{2\beta} e^{-imy}\, dy\right\}}\right] \\
&= 4\pi^2 \mathrm{Re} \sum_{m\in\mathbb{Z}} \widehat{f}_\alpha(m) \overline{\widehat{f}_\beta(m)} \\
&= 4\pi^2 \mathrm{Re}\left[\langle \widehat{f}_\alpha, \widehat{f}_\beta\rangle_{\ell^2}\right] = 4\pi^2 \mathrm{Re}\left[\langle f_\alpha, f_\beta\rangle_{L^2}\right] = 2\pi \int_{-\pi}^{\pi} f_\alpha(x)\overline{f_\beta(x)}\, dx \\
&= \frac{2\pi^{2\alpha+2\beta+1}}{2\alpha+2\beta+1} \mathrm{Re}(1-(-1)^{2\alpha+2\beta+1}).
\end{aligned}$$

The third last inequality is a consequence of the fact that inner product is preserved under the Fourier transform. □

Remark: If we assume that α, β are positive integers, then the proof permits an additional simplification. Now the functions $f_\alpha(x) = x^{2\alpha}$, $f_\beta(x) = x^{2\beta}$ are even, and hence their Fourier series do not contain any terms involving sines. Combining this fact with the trig identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we find that

$$\begin{aligned}
S &= \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} \cos(mx)\cos(my)\, dy\, dx \\
&= 4\pi^2 \sum_{m\in\mathbb{Z}} \widehat{f}_\alpha(m)\widehat{f}_\beta(m) = 4\pi^2 \langle \widehat{f}_\alpha, \widehat{f}_\beta\rangle_{\ell^2} \\
&= 4\pi^2 \langle f_\alpha, f_\beta\rangle_{L^2} = 2\pi \int_{-\pi}^{\pi} f_\alpha(x) f_\beta(x)\, dx = \frac{4\pi^{2(\alpha+\beta+1)}}{2(\alpha+\beta)+1}.
\end{aligned}$$

The fourth inequality is a consequence of the fact that inner product is preserved under the Fourier transform.

4. *(Extra credit) Define the* frequency support *of a function f to be*

$$supp(\widehat{f}) := \left\{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\right\},$$

where $\widehat{f}(n)$ denotes the n-th Fourier coefficient. Let $\mathcal{F}$ denote the class of all continuous functions whose frequency support is contained in $[-10, 10]$. Given any "gap" sequence $\{d_k : k \geq 1\} \subseteq \mathbb{N}$, find a continuous function g with the following frequency-replicating feature: for every $f \in \mathcal{F}$,

$$\begin{aligned}
&supp\big[\widehat{(fg)}\big] = \bigcup_{k=1}^{\infty} A_k, \ \text{with} \\
&A_k := \{a_k + n : n \in supp(\widehat{f})\} \text{ for some integer } a_k, \text{ and} \\
&dist(A_k, A_{k'}) \geq d_k + \cdots + d_{k'-1} \text{ for all } k < k'.
\end{aligned}$$

Solution. For a sequence $\{a_k\}$ specified by

$$a_1 = 0, \quad a_2 = 20 + d_1, \quad a_3 = 40 + d_1 + d_2, \cdots,$$
$$a_k = 20 + a_{k-1} + d_{k-1} = 20(k-1) + d_1 + \cdots d_{k-1},$$

set

$$g(t) = \sum_{k=1}^{\infty} \frac{e^{ia_k t}}{k^2}.$$

By the Weierstrass M-test, g is a continuous function. By construction, $\operatorname{supp}(\widehat{g}) = \{a_k : k \geq 1\}$. We will now show that g has the required properties.

Since every $f \in \mathcal{F}$ is a trigonometric polynomial, it matches its Fourier series:

$$f(x) = \sum_{m \in \mathbb{Z} \cap [-10,10]} \widehat{f}(m) e^{imx}.$$

Substituting this into the integral expression for $\widehat{fg}$ we find that

$$\widehat{fg}(n) = 2\pi \sum_{m \in \mathbb{Z}} \widehat{f}(m) \overline{\widehat{g}(n-m)}.$$

For this last expression to be nonzero, there must exist $m \in \operatorname{supp}(\widehat{f})$ such that $n - m \in \operatorname{supp}(\widehat{g}) = \{a_k : k \geq 1\}$. This means that $n = (n-m) + m \in a_k + \operatorname{supp}(\widehat{f}) = A_k$ for some k, as desired. Finally we verify that for $k < k'$,

$$\begin{aligned}
\operatorname{dist}(A_k, A_{k'}) &\geq \operatorname{dist}(a_k + [-10,10], a_{k'} + [-10,10]) \\
&\geq (a_{k'} - 10) - (a_k + 10) \\
&= a_{k'} - a_k - 20 = 20(k' - k - 1) + d_k + \cdots + d_{k'-1} \\
&\geq d_k + \cdots + d_{k'-1}.
\end{aligned}$$

$\square$