Math 321 Assignment 3
Due Wednesday, January 23 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. In our proof of Weierstrass approximation theorem in class, we used the following identity concerning the binomial expansion: for $x \in [0, 1]$, and any integer $n \geq 1$,

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n}.$$

Prove this identity.

2. In Math 320 we learnt about the property of total boundedness in a metric space, which was intricately connected with the important notion of compactness. Recall that a subset of a metric space is said to be *totally bounded* if it can be covered by finitely many ϵ-balls, for every $\epsilon > 0$. Let us revisit this property in the context of the continuous function spaces that we are exploring now.

Determine whether the following statements are true or false, with adequate justification.

(a) Every totally bounded subset of $C[0, 1]$ is equicontinuous.

(b) Every equicontinuous family of functions in $C[0, 1]$ is totally bounded.

3. The notion of uniform convergence is a powerful tool in constructing functions with esoteric properties that are not readily obtainable by standard methods. You saw an example of this in the last homework set (HW 2, Problem 4), where we constructed a space-filling curve. Here is another application of uniform convergence. We will use it to find a function that is *continuous everywhere but differentiable nowhere!*

(a) Let g denote the function that represents "distance to the nearest integer", i.e.

$$g(x) = \min\{|x - n| : n \in \mathbb{Z}\}, \qquad x \in \mathbb{R}.$$

(You may want to plot g to find out what it looks like. It is a periodic function with period 1. Between two consecutive integers, its graph looks like an isosceles triangle, with the peak of height 1/2 attained at the midpoint). Set

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} g(2^n x), \qquad x \in \mathbb{R}.$$

Prove that the function f is well-defined, and continuous on $\mathbb{R}$.

(b) Now fix any $x \in \mathbb{R}$ and any integer $n \geq 1$. Set $u_n = i/2^n$ and $v_n = (i+1)/2^n$, where i is the unique integer such that $u_n \leq x < v_n$. Show that for every integer $0 \leq k < n$, the two numbers $2^k u_n$ and $2^k v_n$ must lie in the same half-period for g, i.e., there is some $m \in \mathbb{Z}$ for which

$$\text{either both } 2^k u_n, 2^k v_n \in \big[m, m + 1/2\big], \qquad \text{or both } 2^k u_n, 2^k v_n \in \big[m + 1/2, m + 1\big].$$

(c) Use properties of the function g and the observation in part (b) to show that for every $n \geq 1$, the quantity

$$d_n = \frac{f(v_n) - f(u_n)}{v_n - u_n}$$

is a sum of n integers, with each summand being either 1 or -1.

(d) Show that the sequence $\{d_n : n \geq 1\}$ cannot converge. Use this fact to show that $f'(x)$ cannot exist for any $x \in \mathbb{R}$.

Remark: Our textbook has a different construction of a continuous but nowhere differentiable function (Theorem 7.18). You may want to compare the construction there wth the one above. Look for common features as well as distinguishing ones.

4. Weierstrass approximation theorem says that polynomials are dense in $C[a, b]$. Are there other useful function classes that enjoy the same property of being dense in $C[a, b]$? In fact, there are many. Let us explore one of them.

(a) Show that $L[a, b]$, the space of *continuous and piecewise linear functions* on $[a, b]$, is dense in $C[a, b]$. Recall that a function f is piecewise linear on $[a.b]$ if there exist finitely many points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ such that f is linear on each interval $[x_j, x_{j+1}]$, $0 \leq j < n$.

(b) Use part (a) to give yet another proof that $C[a, b]$ is separable, i.e., admits a countable dense subset.