Math 321 Assignment 2
Due Wednesday, January 16 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series, though not necessarily absolutely convergent. Prove that the Dirichlet series given by
$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$
converges uniformly on the half-infinite interval $0 \leq s < \infty$. Use this to prove that
$$\lim_{s \to 0+} \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n.$$
Hint: You may want to review the Abel summation formula (Theorem 3.41, equation (20) in Rudin).

2. (a) For which values of x does the series $\sum_{n=1}^{\infty} ne^{-nx}$ converge? On which intervals is the convergence uniform?

   (b) Conclude that
$$\int_1^2 \left[\sum_{n=1}^{\infty} ne^{-nx}\right] dx = \frac{e}{e^2 - 1}.$$

3. This problem contains a review of important material from Math 320, but its real purpose is to place Problem 4 in context. Determine whether each of the following statements is true or false, with adequate justification.

   (a) There exists a bijection between $[0, 1]$ and $[0, 1] \times [0, 1]$.

   (b) There exists a continuous injection from $[0, 1]$ into $[0, 1] \times [0, 1]$.

   (c) There exists a continuous injection from $[0, 1] \times [0, 1]$ into $[0, 1]$.

   (d) There exists a continuous surjection from $[0, 1] \times [0, 1]$ onto $[0, 1]$.

4. Problem 3 leaves the following question unanswered: does there exist a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$? In this problem, we answer this question in the affirmative, by providing an example of a *space-filling curve.*

   To begin with, define a map $f : \mathbb{R} \to [0, 1]$ as follows: for $t \in [0, 1]$, let
$$f(t) = \left\{ \begin{array}{ll} 0 & \text{for } 0 \leq t \leq \frac{1}{3}, \\ 3t - 1 & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ 1 & \text{for } \frac{2}{3} \leq t \leq 1. \end{array} \right\}$$

Extend f to all of $\mathbb{R}$ by taking f to be even and periodic of period 2. Set

$$x(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k} t), \qquad y(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k+1} t).$$

(a) Show that x and y are continuous on all of $\mathbb{R}$, and maps $\mathbb{R}$ into $[0,1]$.

(b) Remember our old friend the middle-third Cantor set C? (If not, review its definition in page 41 of the textbook). Show that given any $x_0, y_0 \in [0,1]$, there exists a point $t_0 \in C$ such that

$$x(t_0) = x_0, \qquad y(t_0) = y_0.$$

Argue that the curve maps C, and hence $[0,1]$, onto $[0,1] \times [0,1]$. *Hint:* Starting with the ternary (or base 3) expansion of $t \in C$, determine $f(3^k t)$.