Math 421/510 Quiz 1 Solution

1. Consider the vector space of all real bounded sequences. Assign two norms on this space that generate two different topologies. Provide adequate arguments for your choice.

(10 points)

Solution. Given the vector space X consisting of all real bounded sequences $\mathbf{a} = (a_1, a_2, \cdots)$, here are two possible norms we can assign to X:

$$||\mathbf{a}||_\infty := \sup_n |a_n|, \qquad ||\mathbf{a}||_* := \sum_{n=1}^{\infty} 2^{-n}|a_n|.$$

We observe that both these quantities are finite for $\mathbf{a} \in X$, both are positively homogeneous and subadditive. Each is non-negative, and zero if and only if $\mathbf{a} = 0$. So both are well-defined norms.

They generate different topologies. For each $n \geq 1$, define $\mathbf{e}_n$ to be the vector that is one in the n-th coordinate and zero everywhere else. The sequence $\mathbf{e}_n$ is not Cauchy in the $||\cdot||_\infty$ topology, since $||\mathbf{e}_n - \mathbf{e}_m||_\infty = 1$ for all $n \neq m$. In particular, this sequence does not have a limit, On the other hand, $||\mathbf{e}_n||_* = 2^{-n} \to 0$. Hence $\mathbf{e}_n \to \mathbf{0}$ in the $||\cdot||_*$-topology. □