Math 421/510 Take-Home Final, Spring 2018

- *The final exam is due by noon PST on Friday April 13. It should be emailed as a pdf to the instructor. Late submissions will not be accepted.*
- *Each part of a question is worth 10 points, for a total of 100 points.*
- *Your work will be graded both on mathematical content and quality of exposition. Please pay attention to the presentation standards.*
- *Feel free to ask the instructor for hints and clarification, but please do not discuss the exam with each other.*
- *Solutions should be based on the material covered in the course. If you need additional results, state and prove them clearly or provide adequate references. Such solutions will be accepted only if the methodology does not lie significantly beyond the scope of the course. If in doubt, ask the instructor.*

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1. Given finite real numbers $a < b$, consider the integral operator

$$Kf(t) = \int_a^b k(t,s) f(s)\, ds,$$

with integration kernel $k \in L^2([a,b] \times [a,b])$.

(a) Show that K is a bounded linear operator on $L^2[a,b]$, with $||K||_{\text{op}} \le ||k||_2$.

(b) If $||k||_2 < 1$, show that $I - K$ is a bijection of $L^2[a,b]$ onto itself. More precisely, express the inverse of $I - K$ in the form $I + L$, where L is another integral operator. Express the integration kernel of L in terms of k.

(c) Can you envision a mathematical scenario where the above principle could be applied?

2. Let $\mathcal{M}[0,1]$ denote the normed space of complex regular finite Borel measures, and let B denote its unit ball.

(a) Show that the weak star topology on $\mathcal{M}[0,1]$ is not metrizable.

(b) Now show that for $\mu, \nu \in B$,

$$d(\mu,\nu) = \sum_{n=0}^{\infty} 2^{-n} \left| \int_0^1 x^n d\mu - \int_0^1 x^n d\nu \right|$$

is a metric on B that describes the weak star topology.

(c) Explain why the result in (b) does not contradict the one in (a).

3. For each of the following statements, determine whether it is true or false, with supporting arguments.

(a) Let $\{\mathbf{a}_n\}$ be an orthonormal basis in a Hilbert space $\mathbb{H}$. Suppose $\{\mathbf{b}_n\} \subset \mathbb{H}$ is an orthonormal system such that

$$\sum_{n=1}^{\infty} ||a_n - b_n||^2 < \infty.$$

Then $\{\mathbf{b}_n\}$ is an orthonormal basis for $\mathbb{H}$.

(b) Let $\mathcal{F}$ denote the space of trigonometric functions of the form

$$f(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}, \qquad \text{where } a_k, \lambda_k \in \mathbb{R}$$

and n can be any positive integer. Then $\overline{\mathcal{F}}$, the space of all uniform limits of functions in $\mathcal{F}$, cannot be an inner product space.

(c) Given *any* sequence of Borel probability measures $\{\mu_n : n \geq 1\}$ on $[0,1]$, there exists a probability measure μ and a subsequence μ_{n_k} such that for every non-negative r,

$$\int_0^1 x^r d\mu_{n_k}(x) \longrightarrow \int_0^1 x^r d\mu(x) \text{ as } k \to \infty.$$

(d) Let $C^1[0,1]$ denote the space of continuously differentiable functions on $[0,1]$ equipped with the usual norm

$$||f||_{C^1} := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|.$$

Every bounded linear functional ℓ on $C^1[0,1]$ is of the form

$$\ell(f) = cf(t_0) + \int_0^1 f' d\mu,$$

for some $t_0 \in [0,1]$, $c \in \mathbb{R}$ and some finite regular Borel measure μ on $[0,1]$.