Math 121 Assignment 8

Due Friday April 1

1. Find the centre, radius and interval of convergence of each of the following power series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}.$$

2. Expand
   (a) $1/x^2$ in powers of $x+2$.

   (b) $x^3/(1-2x^2)$ in powers of x.

   (c) e^{2x+3} in powers of $x+1$.

   (d) $\sin x - \cos x$ about $\frac{\pi}{4}$.

   (e) the Maclaurin series of $\ln(e + x^2)$.

   (f) the Maclaurin series of $\cos^{-1} x$.

   For each expansion above, determine the interval on which the representation is valid.

3. Find the sums of the following series.

   $$\text{(a)} \sum_{n=0}^{\infty} \frac{(n+1)^2}{\pi^n} \quad \text{(b)} \sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{2^n} \quad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n},$$

   (d) $x^3 - \dfrac{x^9}{3! \times 4} + \dfrac{x^{15}}{5! \times 16} - \dfrac{x^{21}}{7! \times 64} + \dfrac{x^{27}}{9! \times 256} - \cdots$

   (e) $1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \dfrac{x^6}{7!} + \dfrac{x^8}{9!} + \cdots$

   (f) $1 + \dfrac{1}{2 \times 2!} + \dfrac{1}{4 \times 3!} + \dfrac{1}{8 \times 4!} + \cdots$

   (g)$1 - \dfrac{x}{2!} + \dfrac{x^2}{4!} - \cdots$

4. This problem outlines a srategy for verifying whether a function f is real-analytic. Recall the nth order Taylor polynomial of f centred at c:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

and set $E_n = f(x) - P_n(x)$.

(a) Use mathematical induction to show that

$$E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt,$$

provided $f^{(n+1)}$ exists on an interval containing c and x. The formula above is known as Taylor's formula with integral remainder.

(b) Use Taylor's formula with integral remainder to prove that $\ln(1+x)$ is real analytic at $x = 0$; more precisely, that the Maclaurin series of $\ln(1+x)$ converges to $\ln(1+x)$ for $-1 < x \leq 1$.

5. Find the Maclaurin series for the functions:

(a)

$$L(x) = \int_1^{1+x} \frac{\ln t}{t-1}\,dt$$

(b)

$$M(x) = \int_0^x \frac{\tan^{-1} t^2}{t^2}\,dt$$

6. Evaluate the limits

(a)

$$\lim_{x\to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1+x^2)}$$

(b)

$$\lim_{x\to 0} \frac{\sin(\sin x) - x}{x(\cos(\sin x) - 1)}$$

(c)

$$\lim_{x\to 0} \frac{x^3 - 3S(x)}{x^7} \quad \text{where } S(x) = \int_0^x \sin(t^2)\,dt.$$

(d)

$$\lim_{x\to 0} \frac{(x - \tan^{-1} x)(e^{2x} - 1)}{2x^2 - 1 + \cos(2x)}.$$

7. (a) Estimate the size of the error if the Taylor polynomial of degree 4 about $x = \pi/2$ for $f(x) = \ln\sin x$ is used to approximate $\ln\sin(1.5)$.

(b) How many nonzero terms of the Maclaurin expansion of e^{-x^4} are needed to evaluate $\int_0^{1/2} e^{-x^4}\,dx$ correct to five decimal places? Evaluate the integral to that accuracy.

8. Find the Fourier series of the 3-periodic function

$$f(x) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2 \\ 3 - t & \text{if } 2 \le t < 3. \end{cases}$$

9. Verify that if f is an even function of period T, then the Fourier sine coefficients b_n of f are all zero and the Fourier cosine coefficients a_n of f are given by

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\,cos(n\omega t)\,dt, \qquad n = 0, 1, 2, \cdots$$

where $\omega = 2\pi/T$. State and verify the corresponding result for odd functions f.

10. Prove that the binomial coefficients satisfy:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$