Math 121 Assignment 2
Due Friday January 22

1. Find the function f that satisfies the equation
$$2f(x) + 1 = 3\int_x^1 f(t)\,dt.$$
2. Find the area of
   (a) the plane region bounded between the two curves $y = \frac{4}{x^2}$ and $y = 5 - x^2$.
   (b) Find the area of the closed loop of the curve $y^2 = x^4(2+x)$ that lies to the left of the origin.
3. Evaluate the integrals
$$(a)\ \int_0^4 \sqrt{9t^2 + t^4}\,dt.$$
$$(b)\ \int \cos^2\left(\frac{t}{5}\right)\sin^2\left(\frac{t}{2}\right)\,dt.$$
$$(c)\ \int \cos^4 x\,dx.$$
$$(d)\ \int \frac{dx}{e^x + 1}.$$
$$(e)\ \int_0^2 \frac{x\,dx}{x^4 + 16}.$$
$$(f)\ \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin(\theta)}\,d\theta.$$
4. Use mathematical induction to show that for every positive integer k,
$$\sum_{j=1}^{n} j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + P_{k-1}(n),$$
where P_{k-1} is a polynomial of degree at most $k-1$. Deduce from this that
$$\int_0^a x^k\,dx = \frac{a^{k+1}}{k+1}.$$
5. Does the function
$$F(x) = \int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right)\,dt$$
have a maximum or minimum value? Justify your answer.

6. (a) If m, n are integers, compute the integrals

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx, \; \int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx, \; \int_{-\pi}^{\pi} \sin mx \, \cos nx \, dx.$$

(b) Suppose that for some positive integer k,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos nx + b_n \sin nx)$$

holds for all $x \in [-\pi, \pi]$. Find integral formulas for the coefficients a_0, a_n, b_n with the integrand involving f of course. (Remark: The coefficients a_0, a_n, b_n are called the *Fourier coefficients* of f. Fourier coefficients arise in a variety of contexts, such as communications and signal processing. If f is a musical note, then the integers n for which a_n or b_n are nonzero are precisely the frequencies comprising the note.)