Math 121: Midterm solutions

1. (a) Since $\lim_{x\to\infty} e^{-x/2}x^{\alpha-1} = 0$, there exits $T > 0$ such that $e^{-x/2}x^{\alpha-1} \leq 1$ if $x \geq T$. Thus,
$$0 \leq \int_T^\infty e^{-x}x^{\alpha-1}dx \leq \int_T^\infty e^{-x/2}dx = 2e^{-T/2}$$
and $\int_T^\infty e^{-x}x^{\alpha-1}dx$ converges by the comparison theorem. If $\alpha > 0$, then
$$0 \leq \int_0^T e^{-x}x^{\alpha-1}dx < \int_0^T x^{\alpha-1}dx$$
converges by Theorem $2(b)$. Thus the integral defining I_α converges.

   (b)
$$I_\alpha = \int_0^\infty e^{-x}x^{\alpha-1}dx = \lim_{c\to 0_+, R\to\infty} \int_c^R e^{-x}x^{\alpha-1}dx,$$
Let $U = x^{\alpha-1}$, $dV = e^{-x}dx$, $dU = (\alpha-1)x^{\alpha-2}dx$, and $V = -e^{-x}$,
$$\begin{aligned} I_\alpha &= \lim_{c\to 0_+, R\to\infty} (-e^{-x}x^{\alpha-1}|_c^R + (\alpha-1)\int_c^R e^{-x}x^{\alpha-2}dx) \\ &= 0 + (\alpha-1)\int_0^\infty e^{-x}x^{\alpha-2}dx = (\alpha-1)I_{\alpha-1}. \end{aligned}$$

2. Let $x = \tan\frac{\theta}{2}$, $d\theta = \frac{2}{1+x^2}dx$, $\cos\theta = \frac{1-x^2}{1+x^2}$, $\sin\theta = \frac{2x}{1+x^2}$. So
$$\begin{aligned} \int \frac{d\theta}{1+\sin\theta+\cos\theta} &= \int \frac{(\frac{2}{1+x^2})dx}{1+(\frac{1-x^2}{1+x^2})+(\frac{2x}{1+x^2})} \\ &= \int \frac{dx}{1+x} = \ln|1+x| + C = \ln|1+\tan\frac{\theta}{2}| + C. \end{aligned}$$

3. The arc length element for the parabola $y = x^2$ is $ds = \sqrt{1+4x^2}dx$, so the required surface area is
$$S = 2\pi\int_0^1 x\sqrt{1+4x^2}dx,$$
let $u = 1+4x^2$ and $du = 8xdx$, so
$$\begin{aligned} S &= \frac{\pi}{4}\int_1^5 u^{1/2}du \\ &= \frac{\pi}{6}u^{3/2}|_1^5 \\ &= \frac{\pi}{6}(5\sqrt{5}-1) squareunits. \end{aligned}$$

4. Pick points $(-\frac{9}{5},0)$, $(\frac{16}{5},0)$ and $(0,\frac{12}{5})$ to construct the right triangle, the hypotenuse lies on the x-axis. A horizontal strip has area

$$dA = \frac{25}{12}(\frac{12}{5} - y)dy,$$

therefore, the mass of the plate is

$$m = \int_0^{\frac{12}{5}} \frac{25}{12}(\frac{12}{5} - y)ydy.$$

And we have

$$M_{x=0} = \frac{1}{2}\int_0^{\frac{12}{5}} [\frac{25}{12}(\frac{12}{5} - y)]^2 ydy,$$
$$M_{y=0} = \int_0^{\frac{12}{5}} \frac{25}{12}(\frac{12}{5} - y)y^2 dy.$$

Thus, the center of mass of plate can be expressed as: $(\frac{M_{x=0}}{m}, \frac{M_{y=0}}{m})$.

5.
$$g(x) = \int_1^{e^x} \cos(x)\sec((\ln t)^2)dt = \cos(x)\int_1^{e^x} \sec((\ln t)^2)dt.$$

$$\begin{aligned} g'(x) &= -\sin(x)\int_1^{e^x} \sec((\ln t)^2)dt + \cos(x)\sec((\ln e^x)^2)e^x \\ &= -\sin(x)\int_1^{e^x} \sec((\ln t)^2)dt + \cos(x)\sec(x^2)e^x, \end{aligned}$$

so we have

$$g'(0) = 0 + 1 = 1.$$