Math 121: Homework 7 solutions

1. (a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 - 2\sqrt{n} + 1}{1 - n - 3n^2} = \lim_{n\to\infty} \frac{1 - \frac{2}{n\sqrt{n}} + \frac{1}{n^2}}{\frac{1}{n^2} - \frac{1}{n} - 3} = -\frac{1}{3}.$$

(b)

$$\lim_{n\to\infty}(n - \sqrt{n^2 - 4n}) = \lim \frac{n^2 - (n^2 - 4n)}{n + \sqrt{n^2 - 4n}} = \lim \frac{4n}{n + \sqrt{n^2 - 4n}} = \lim \frac{4}{1 + \sqrt{1 - 4/n}} = 2.$$

(c)

$$a_n = \frac{(n!)^2}{(2n)!} = \frac{(1\cdot 2\cdot 3....n)(1\cdot 2\cdot 3....n)}{1\cdot 2\cdot 3...n\cdot(n+1)..(2n)} = \frac{1}{n+1}\frac{2}{n+2}...\frac{n}{n+n} \le (1/2)^n.$$

So $\lim_{n\to\infty} a_n = 0$.

2. Let $a_1 = 3$ and $a_{n+1} = \sqrt{15 + 2a_n}$ for $n = 1, 2, 3, ...$ Then we have $a_2 = \sqrt{21} > 3$. If $a_{k+1} > a_k$ for some k, then

$$a_{k+2} = \sqrt{15 + 2a_{k+1}} > \sqrt{15 + 2a_k} = a_{k+1}.$$

Thus, $\{a_n\}$ is increasing by induction. Observe that $a_1 < 5$ and $a_2 < 5$. If $a_k < 5$ then

$$a_{k+1} = \sqrt{2a_k + 15} < \sqrt{15 + 2(5)} = 5.$$

Therefore, $a_n < 5$ for all n, by induction. Since $\{a_n\}$ is increasing and bounded above, it converges. Let $\lim a_n = a$. Then

$$a = \sqrt{15 + 2a}.$$

$$a^2 - 2a - 15 = 0.$$

$$a = -3, \quad a = 5.$$

Since $a > a_1$, we have $\lim a_n = 5$.

3. (a) Let

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + ...$$

Since

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right),$$

the partial sum is

$$\begin{aligned} s_n &= \frac{1}{2}(1 - 1/3) + \frac{1}{2}(1/3 - 1/5) + ... + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

(b) Since $1+2+3+...+n = \frac{n(n+1)}{2}$, the given series is $\sum_{n=1}^{\infty} = \frac{2}{n(n+1)}$, which converges to 2.

4. The total distance is

$$\begin{aligned} d &= 2+2[2*3/4+2*(3/4)^2+...] \\ &= 2+2*3/2*[1+3/4+(3/4)^2+....] \\ &= 2+3/(1-3/4) = 14. \end{aligned}$$

5. (a) False. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n+1}$. Then $\sum = \infty$ and $0 < b_n \leq 1/2$. But $\sum a_n b_n = \frac{1}{n(n+1)}$ which converges.

(b) True. Since $\sum a_n$ converges, therefore $\lim a_n = 0$. Thus there exists N such that $0 < a_n \leq 1$ for $n \geq N$. Thus $0 < a_n^2 \leq a_n$ for $n \geq N$. If $S_n = \sum_{k=N}^{n} a_k^2$ and $s_n = \sum_{k=N}^{n} a_k$, then $\{S_n\}$ is increasing and bounded above:

$$S_n \leq s_n \leq \sum_{k=1}^{\infty} a_k < \infty.$$

Thus, the statement is true.

(c) False. $a_n = \frac{(-1)^n}{n}$ is a counterexample.

(d) True. Because $|(-1)^n a_n| = |a_n|$.

6. (a) We apply the ratio test, we have

$$\rho = \lim \frac{2^{2n+2}((n+1)!)^2}{(2n+2)!} \frac{(2n)!}{2^{2n}(n!)^2} = 1.$$

Thus the ratio test provides no information. However

$$\begin{aligned} \frac{2^{2n}(n!)^2}{(2n)!} &= \frac{[2n(2n-2)..(6)(4)(2)]^2}{2n(2n-1)(2n-2)...(3)2)(1)} \\ &= \frac{2n}{2n-1}\frac{2n-2}{2n-3}....\frac{4}{3}\frac{2}{1} > 1. \end{aligned}$$

Since the terms exceed 1, the series diverges to infinity.

(b) Since

$$\rho = \lim_{n\to\infty}[(n/(n+1))^{n^2}]^{1/n} = \lim \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1.$$

The sum converges.

(c) The sum converges by the integral test:

$$\int_a^\infty \frac{dt}{t\ln(t)(\ln\ln(t))^2} = \int_{\ln\ln(a)}^\infty \frac{du}{u^2} < \infty,$$

if $\ln\ln(a) > 0$.

7. (a) Apply the ratio test. We have

$$\rho = \lim \left|\frac{(2x+3)^{n+1}}{(n+1)^{1/3}4^{n+1}}\frac{n^{1/3}4^n}{(2x+3)^n}\right| = \frac{|2x+3|}{4}.$$

The series converges absolutely if $|x+\frac{3}{2}| < 2$, that is, if $-7/2 < x < 1/2$. By the alternating series test it converges conditionally at $x = -\frac{7}{2}$. It diverges elsewhere.

(b) Apply the ratio test

$$\rho = \lim \left|\frac{1}{n+1}(1+\frac{1}{1+x})^{n+1}\frac{n}{1}(1+\frac{1}{x})^{-n}\right| = |1+\frac{1}{x}| = 1.$$

if and only if $|x+1| < |x|$, that is, $-2 < \frac{1}{x} < 0$. $x < -\frac{1}{2}$. If $x = -\frac{1}{2}$, then $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{(-1)^n}{n}$, which converges conditionally. Thus, the series converges absolutely if $x < -\frac{1}{2}$, converges conditionally if $x = -\frac{1}{2}$ and diverges elsewhere. It is undefined at $x = 0$.

(c) Apply the ratio test, we obtain

$$\rho = \lim |x|\frac{(2n+2)(2n+1)}{4(n+1)^2} = |x|.$$

Thus, $\sum a_n x^n$ converges absolutely if $-1 < x < 1$, and diverges if $x > 1$ or $x < -1$. In exercise 36 of Section 9.3 it was shown that $a_n \geq \frac{1}{2n}$, so the given series definitely diverges at $x = 1$ and may at most converge conditionally at $x = -1$. To see whether it does converge at -1, we write

$$\begin{aligned}
a_n &= \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1\times 2\times 3\times 4 ...\times 2n}{(2\times 4\times 8 ...\times 2n)^2} \\
&= \frac{1\times 3\times 5 ...\times (2n-1)}{2\times 4\times 6 .... \times (2n-2)\times 2n} \\
&= \frac{1}{2}\frac{3}{4}....\frac{2n-3}{2n-2}\frac{2n-1}{2n} \\
&= (1-1/2)(1-/1/4)...(1-1/(2n-2))(1-1/2n).
\end{aligned}$$

It is evident that a_n decreases as n increases. To see whether $\lim a_n = 0$, take logarithms and use the inequality $\ln(x+1) \leq x$:

$$\begin{aligned}
\ln(a_n) &= \ln(1-1/2) + \ln(1-1/4) + ... + \ln(1-1/2n) \\
&\leq -\frac{1}{2} - \frac{1}{4} - ... - \frac{1}{2n} \\
&= -\frac{1}{2}(1+\frac{1}{2}+...+\frac{1}{n}) \\
&\to \infty,
\end{aligned}$$

as $n \to \infty$. Thus, $\lim a_n = 0$, and the given series converges conditionally at $x = -1$ by the alternating series test.