Math 121: Homework 6 solutions

1. The spring force is $F(x) = kx$, where x is the amount of compression. The work done to compress the spring 3 cm is

$$100N.cm = W = \int_0^3 kxdx = \frac{1}{2}kx^2|_0^3 = \frac{9}{2}k.$$

Hence, $k = \frac{200}{9}$ N/cm. The work necessary to compress the spring a further 1 cm is

$$W = \int_3^4 kxdx = (\frac{200}{9})\frac{1}{2}x^2|_3^4 = \frac{700}{9}N.cm.$$

2. Let the time required to raise the bucket to height h m be t minutes. Given that the velocity is 2 m/min, then $t = \frac{h}{2}$. The weight of the bucket at time t is

$$16kg - (1kg/min)(tmin) = 16 - \frac{h}{2}kg.$$

Therefore, the work done required to move the bucket to a height of 10 m is

$$\begin{aligned} W &= g\int_0^{10}(16 - \frac{h}{2})dh \\ &= 9.8(16h - \frac{h^2}{4})|_0^{10} = 1323N.m. \end{aligned}$$

3. (a)

$$y(x) = 3 + \int_0^x e^{-y}dy,$$

so we have $y(0) = 3$.

$$\frac{dy}{dx} = e^{-y},$$
$$e^y = x + C,$$
$$y = \ln(x + C).$$

Based on the initial condition, $C = e^3$. So $y = \ln(x + e^3)$.

(b)

$$x^2y' + y = x^2e^{1/x}, \quad y(1) = 3e.$$
$$y' + \frac{1}{x^2}y = e^{1/x},$$
$$\mu = \int \frac{1}{x^2}dx = -\frac{1}{x},$$
$$\frac{d}{dx}(e^{-1/x}y) = e^{-1/x}(y' + \frac{1}{x^2}y) = 1,$$
$$e^{-1/x}y = \int 1dx = x + C,$$

since $y(1) = 3e$, so $3 = 1 + C$, $C = 2$.

$$y = (x + 2)e^{1/x}.$$

4.

$$\frac{dy}{dx} = \frac{3y}{x-1},$$

$$\begin{aligned} \int \frac{dy}{y} &= 3\frac{dx}{x-1} \\ \ln|y| &= \ln|x-1|^3 + \ln|C| \\ y &= C(x-1)^3. \end{aligned}$$

Since $y = 4$ when $x = 2$, we have $4 = C(2-1)^3 = C$, so the equation of the curve is $y = 4(x-1)^3$.

5. The balance in the account after t years is $y(t)$ and $y(0) = 1000$. The balance must satisfy

$$\begin{aligned} \frac{dy}{dt} &= 0.1y - \frac{y^2}{1000000} \\ \frac{dy}{dt} &= \frac{10^5 y - y^2}{10^6} \\ \int \frac{dy}{10^5 y - y^2} &= \int \frac{dt}{10^6} \\ \frac{1}{10^5}\int (\frac{1}{y} + \frac{1}{10^5 - y})dy &= \frac{t}{10^6} - \frac{C}{10^5} \\ \ln|y| - \ln|10^5 - y| &= \frac{t}{10} - C \\ \frac{10^5 - y}{y} &= e^{C-(t/10)} \\ y &= \frac{10^5}{e^{C-(t/10)} + 1}. \end{aligned}$$

Since $y(0) = 1000$, we have

$$1000 = y(0) = \frac{10^5}{e^C + 1},$$

so $C = \ln 99$, and

$$y = \frac{10^5}{99e^{-t/10} + 1}.$$

The balance after 1 year is

$$y = \frac{10^5}{99e^{-1/10} + 1} \approx 1104.01$$

As $t \to \infty$, the balance can grow to

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{10^5}{e^{4.6-0.1t} + 1} = 100000.$$

For the account to grow to 50000, t must satisfy

$$50000 = y(t) = \frac{100000}{99e^{-t/10} + 1},$$

so $t = 10 \ln 99 \approx 46$ years.

6.

$$x = \cosh t, \quad y = \sinh^2 t.$$

Parabola $x^2 - y = 1$ or $y = x^2 - 1$, traversed left to right.

$$x = \cos t + \sin t, \quad y = \cos t - \sin t.$$

The circle $x^2 + y^2 = 2$, traversed clockwise, starting and ending at $(1, 1)$.

7. $x = \frac{4}{1+t^2}$, $y = t^3 - 3t$.

$$\frac{dx}{dt} = -\frac{8t}{(1+t^2)^2}, \quad \frac{dy}{dt} = 3(t^2 - 1).$$

Horizontal tangent at $t = \pm 1$, i.e. at $(2, \pm 2)$. Vertical tangent at $t = 0$, i.e. at $(4, 0)$. Self-intersection at $t = \pm\sqrt{3}$, i.e. at $(1, 0)$.

$$x = t^3 - 3t, \quad y = t^3 - 12t.$$

$$\frac{dx}{dt} = 3(t^2 - 1), \quad \frac{dy}{dt} = 3(t^2 - 4).$$

Horizontal tangent at $t = \pm 2$, i.e. at $(2, -16)$ and $(-2, 16)$. Vertical tangent at $t = \pm 1$, i.e. at $(2, 11)$ and $(-2, -11)$. Slope $\frac{dy}{dx} = \frac{t^2-4}{t^2-1}$, positive if $|t| > 2$ or $|t| < 1$; negative if $1 < |t| < 2$. Slope $\to 1$ as $t \to \pm\infty$.

8.

$$\begin{aligned} L &= \int_0^2 \sqrt{(e^t - 1)^2 + 4e^t} dt \\ &= \int_0^2 \sqrt{(e^t + 1)^2} dt = \int_0^2 (e^t + 1) dt \\ &= (e^t + 1)|_0^2 = e^2 + 1 units. \end{aligned}$$

9. The area of a smaller loop:

$$\begin{aligned} A &= 2 \times \frac{1}{2} \int_{\pi/3}^{\pi/2} (1 + 2\cos(2\theta))^2 d\theta \\ &= \int_{\pi/3}^{\pi/2} [1 + 4\cos(2\theta) + 2(1 + \cos(4\theta))] d\theta \\ &= (3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta))|_{\pi/3}^{\pi/2} \\ &= \frac{\pi}{2} - \frac{3\sqrt{3}}{4} sq.units. \end{aligned}$$

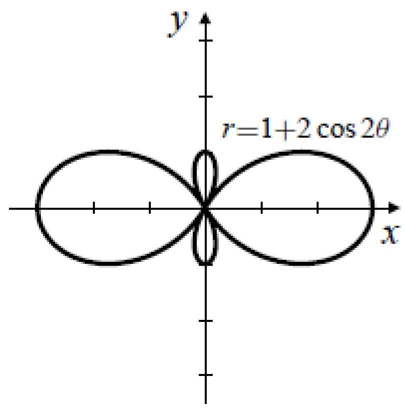

Figure 1: 9

10. $r\cos\theta = x = 1/4$ and $r = 1+\cos\theta$ intersect where

$$1+\cos\theta = \frac{1}{4\cos\theta},$$
$$4\cos^2\theta + 4\cos\theta - 1 = 0$$
$$\cos\theta = \frac{\pm\sqrt{2}-1}{2}.$$

Only $(\sqrt{2}-1)/2$ is between -1 and 1, so is a possible value of $\cos\theta$. Let $\theta_0 = \cos^{-1}\frac{\sqrt{2}-1}{2}$. Then

$$\sin\theta_0 = \sqrt{1-(\frac{\sqrt{2}-1}{2})^2} = \frac{\sqrt{1+2\sqrt{2}}}{2}.$$

By symmetry, the area inside $r = 1+\cos\theta$ to the left of the line $x = 1/4$ is

$$\begin{aligned} A &= 2\times\frac{1}{2}\int_{\theta_0}^{\pi}(1+2\cos\theta+\frac{1+\cos(2\theta)}{2})d\theta + \cos\theta_0\sin\theta_0 \\ &= \frac{3}{2}(\pi-\theta_0) + (2\sin\theta+\frac{1}{4}\sin(2\theta))|_{\theta_0}^{\pi} + (\sqrt{2}-1)\sqrt{1+2\sqrt{2}}/4 \\ &= \frac{3}{2}(\pi-\cos^{-1}\frac{\sqrt{2}-1}{2}) + \sqrt{1+2\sqrt{2}}(\frac{\sqrt{2}-9}{8})sq.units. \end{aligned}$$

11. Let S_1 and S_2 be two spheres inscribed in the cylinder, one on each side of the plane that intersects the cylinder in the curve C that we are trying to s show in an ellipse. Let the spheres be tangent to the cylinder around the circles C_1 and C_2, and suppose they are also tangent to the plane at the points F_1 and F_2, respectively, as shown in the figure.

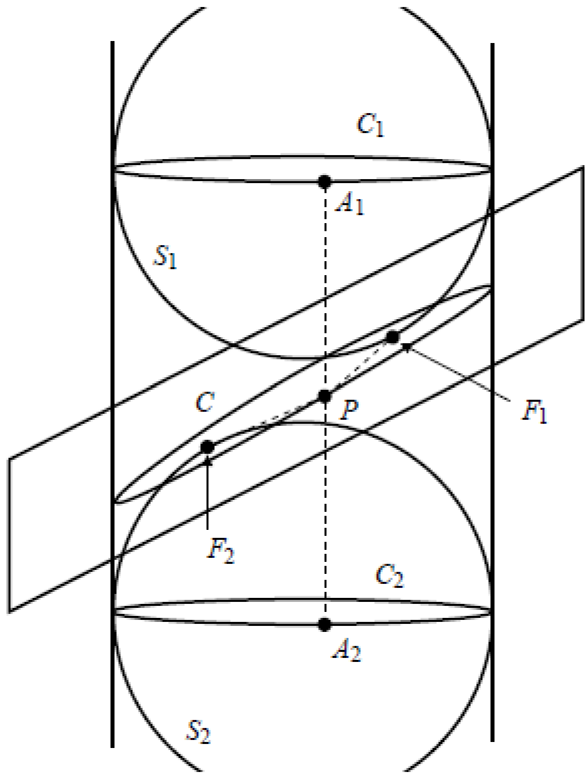

Figure 2: 11

Let P be any points on C. Let A_1A_2 be the line through P that lies on the cylinder, with A_1 on C_1 and A_2 on C_2. Then $PF_1 = PA_1$ because both lengths are of tangents drawn to the sphere S_1 from the same exterior point P. Similarly, $PF_2 = PA_2$. Hence,

$$PF_1 + PF_2 = PA_1 + PA_2 = A_1A_2,$$

which is constant, the distance between the centers of the two spheres. Thus C must be an ellipse, with foci at F_1 and F_2.

12. The two curves $r^2 = 2\sin(2\theta)$ and $r = 2\cos\theta$ intersects where

$$\begin{aligned} 2\sin(2\theta) &= 4\cos^2(\theta) \\ 4\sin\theta\cos\theta &= 4\cos^2\theta \\ (\sin\theta - \cos\theta)\cos\theta &= 0 \\ \sin\theta &= \cos\theta or \cos\theta = 0, \end{aligned}$$

i.e. at $P_1 = [\sqrt{2}, \pi/4]$ and $P_2 = (0, 0)$.
For $r^2 = 2\sin\theta$ we have $2r\frac{dr}{d\theta} = 4\cos(2\theta)$. At P_1 we have $r = \sqrt{2}$ and $dr/d\theta = 0$. Thus the angle ϕ between the curve and the radial line $\theta = \pi/4$ is $\phi = \pi/2$. For $r = 2\cos\theta$ we have $dr/d\theta = -2\sin\theta$, so the angle between this curve and the radial line $\theta = \pi/4$ satisfies $\tan\phi = \frac{r}{dr/d\theta}|_{\theta=\pi/4} = -1$ and $\phi = 3\pi/4$. The two curves

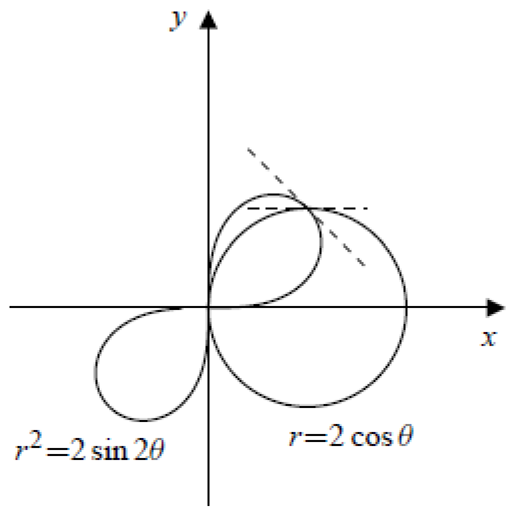

Figure 3: 12

intersect at P_1 at angle $\frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$. The Figure shows that at the origin, P_2, the circle meets the lemniscate twice, at angle 0 and $\pi/2$.

13. If $Q = (0, Y)$, then the slope of PQ is

$$\frac{y - Y}{x - 0} = f'(x) = \frac{dy}{dx}.$$

Since $|PQ| = L$, we have $(y - Y)^2 = L^2 - x^2$. Since the slope dy/dx is negative at P, $dy/dx = -\sqrt{L^2 - x^2}/x$. Thus,

$$y = -\int \frac{\sqrt{L^2 - x^2}}{x} dx = L \ln(\frac{L + \sqrt{L^2 - x^2}}{x}) - \sqrt{L^2 - x^2} + C.$$

Since $y = 0$ when $x = L$, we have $C = 0$ and the equation of the tractrix ix

$$y = L \ln(\frac{L + \sqrt{L^2 - x^2}}{x}) - \sqrt{L^2 - x^2}.$$

Note that the first term can be written in an alternate way:

$$y = L \ln(\frac{x}{L - \sqrt{L^2 - x^2}}) - \sqrt{L^2 - x^2}.$$

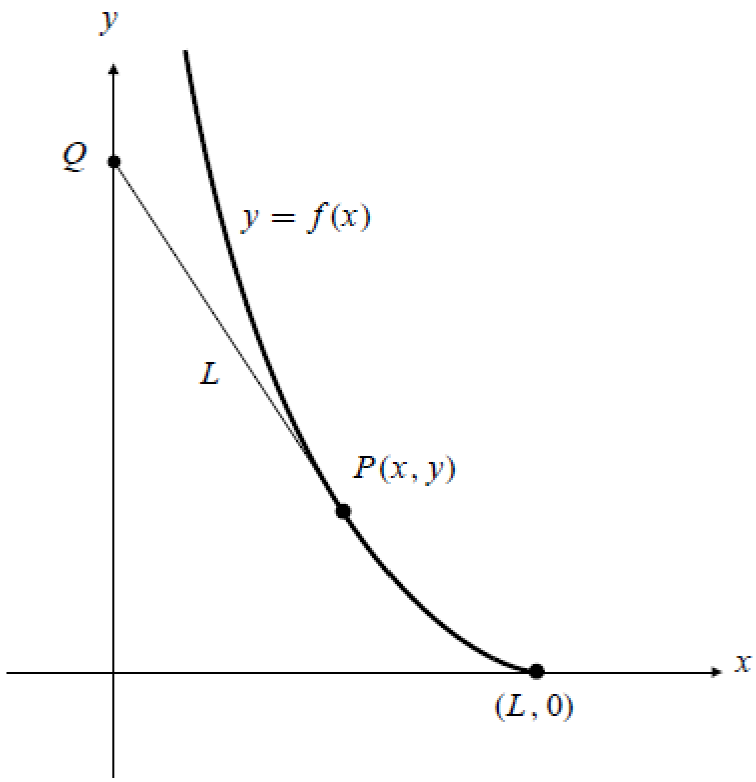

Figure 4: 13