Homework 7 - Math 321, Spring 2015

Due on Friday March 13

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1. Recall Jordan's theorem: a function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if f can be written as the difference of two non-decreasing functions g and h.
   (a) Show that the decomposition $f = g - h$ is by no means unique, and that there are uncountably many ways of writing f in this form.

   (b) The following decomposition of f is often useful. Define the positive and negative variations of f by
   $$p(x) = \frac{1}{2}\big(v(x) + f(x) - f(a)\big), \qquad n(x) = \frac{1}{2}\big(v(x) - f(x) + f(a)\big),$$
   where $v(x) = V_a^x f$ is the variation function defined in class. Show that p and n are nondecreasing functions on $[a,b]$ and use this to give an alternative representation of f as the difference of nondecreasing functions.

   (c) The relevance of p and n is that it injects a certain amount of uniqueness into the Jordan decomposition of f, in the following sense. If g and h are any two non-decreasing functions on $[a,b]$ such that $f = g - h$, then
   $$V_x^y p \le V_x^y g \quad \text{and} \quad V_x^y n \le V_x^y h \text{ for all } x < y \text{ in } [a,b].$$
   Prove this.

2. We stated the "integration by parts" formula in class, using it to highlight the interchangability of integrand and integrator. The purpose of this problem is to fill in the details of its proof. Throughout this problem f and α denote arbitrary real-valued functions on $[a,b]$.

   (a) Given any partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and a collection of points $T = \{t_1, \cdots, t_n\}$ with $t_j \in [x_{j-1}, x_j]$, prove the following identity:
   $$S_f(\alpha, P, T) = f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T').$$
   Here $P' = \{a = t_0, t_1, \cdots, t_n, t_{n+1} = b\}$ and $T' = P$.

   (b) Use part (a) to show that $f \in \mathcal{R}_\alpha[a,b]$ if and only if $\alpha \in \mathcal{R}_f[a,b]$. Show that in either case,
   $$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a).$$
   Note that this is one of those rare instances where one implication implies the other!

3. Let $\alpha \in \mathrm{BV}[a,b]$ and let $\beta(x) = V_a^x \alpha$. Recall that both β and $\beta - \alpha$ are increasing. Show that $\mathcal{R}_\alpha[a,b] = \mathcal{R}_\beta[a,b] \cap \mathcal{R}_{\beta-\alpha}[a,b]$. This identity was instrumental to our conclusion that $\mathcal{R}_\alpha[a,b]$ is a vector space, an algebra and a lattice. (*Hint:* Argue that it suffices to only show that $\mathcal{R}_\alpha[a,b] \subseteq \mathcal{R}_\beta[a,b]$.)

4. Let $\{f_n\}$ be a bounded sequence in $\mathrm{BV}[a,b]$, i.e., suppose that $||f_n||_{\mathrm{BV}} \le K$ for all n. Show that f_n admits a pointwise convergent subsequence whose limit f lies in $\mathrm{BV}[a,b]$ with $||f||_{\mathrm{BV}} \le K$. This is known as Helly's first theorem. (*Hint:* First try out the case when all the functions f_n are non-decreasing, then adapt it for functions of bounded variation.)