Homework 5 - Math 321, Spring 2015

Due on Friday February 13

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1. The classical Weierstrass approximation theorem says that the class of polynomials is dense in $\mathcal{C}[a, b]$. The Stone-Weierstrass theorem, on the other hand, provides a necessary and sufficient condition for a subalgebra of $\mathcal{C}(X)$ to be dense if X is compact, but does not seem to furnish a concrete dense class of functions akin to the polynomials. While we do not have anything so convenient as polynomials at our disposal for a general compact X, we do have a familiar collection of functions to work with that for many purposes serves as an adequate replacement. Here it is:

   Given a metric space (X, d) and a constant $0 \leq K < \infty$, let $\mathcal{L}_K(X)$ denote the collection of all real-valued Lipschitz functions on X with Lipschitz constant at most K; in other words, $f : X \to \mathbb{R}$ is in $\mathcal{L}_K(X)$ if

   $$|f(x) - f(y)| \leq K d(x, y) \qquad \text{for all } x, y \in X.$$

   We write $\mathcal{L}(X)$ to denote the set of functions that are in $\mathcal{L}_K(X)$ for some finite K; i.e., $\mathcal{L}(X) = \bigcup_{K=1}^{\infty} \mathcal{L}_K(X)$.

   (a) Clearly all constant functions are Lipschitz. Show that $\mathcal{L}(X)$ contains non-constant functions as well.

   (b) Show that for an arbitrary metric space (X, d), $\mathcal{L}(X)$ is a subspace of $\mathcal{C}(X)$.

   (c) If X is compact, show that $\mathcal{L}(X)$ is in fact a subalgebra of $\mathcal{C}(X)$.

   (d) Given an arbitrary metric space X, show that $\mathcal{L}(X)$ separates points of X and vanishes at no point of X. If X is compact, deduce from this that $\mathcal{L}(X)$ is dense in $\mathcal{C}(X)$.

2. Show that any compact subset of a metric space is separable, and deduce from it that a countable union of compact sets is separable as well.

3. One of the applications of the classical Weierstrass theorem was to prove that $\mathcal{C}[a, b]$ is separable. Likewise, the Stone-Weierstrass theorem can be used to show that $\mathcal{C}(X)$ is separable where X is a compact metric space. This exercise aims to give a proof of this statement.

   (a) Recall the class $\mathcal{L}(X)$ of Lipschitz functions introduced in Problem 1. Use Problem 2 to argue that $\mathcal{L}(X)$ is separable if X is compact. (*Hint:* You may want to look at bounded subsets of $\mathcal{L}_K(X)$.)

   (b) Conclude from part (a) that $\mathcal{C}(X)$ is separable.

4. You have seen examples showing that pointwise convergence does not imply uniform convergence. However, pointwise convergence combined with equicontinuity does imply convergence in this stronger sense. Prove this. More precisely, show that if X is a compact metric space, then any equicontinuous sequence of functions in $\mathcal{C}(X)$ that is pointwise convergent is in fact uniformly convergent.

5. Let $\{f_n\}$ be an equicontinuous sequence in $\mathcal{C}[a, b]$ such that $\{f_n\}$ converges pointwise at every rational in $[a, b]$. Prove that $\{f_n\}$ converges uniformly on $[a, b]$.