Homework 1 - Math 321, Spring 2015

Due on Friday January 16

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1. (a) Does the sequence of functions
$$f_n(x) = \frac{nx}{(1+n^2x^2)}$$
converge pointwise on $[0,\infty)$? Is the convergence uniform on this interval? If yes, give reasons. If not, determine the intervals (if any) on which the convergence is uniform.
   (b) Verify that the sequence
$$f_n(x) = \left(1+\frac{x}{n}\right)^n$$
converges uniformly to $f(x) = e^x$ on every compact interval in $\mathbb{R}$. How does this explain our findings in class, namely that $f_n \to f$ pointwise but not uniformly on $\mathbb{R}$, yet
$$\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx?$$

2. Suppose that (X,d) and (Y,ρ) are metric spaces, that $f_n : X \to Y$ is continuous for each n and that f_n converges pointwise to a function f on X. If there exists a sequence $\{x_n\}$ in X such that $x_n \to x$ in X but $f_n(x_n) \not\to f(x)$, show that $\{f_n\}$ does not converge uniformly to f on X. This can be used as a negative test for uniform convergence.

3. Suppose that $\{f_n\}$ is a sequence of real-valued functions, each having a continuous derivative on $[a,b]$, and suppose that the sequence of derivatives $\{f_n'\}$ converges uniformly to a function g on $[a,b]$. If $\{f_n(x_0)\}$ converges at any point x_0 in $[a,b]$, then, show that $\{f_n\}$ converges uniformly to a differentiable function f on $[a,b]$, and that in fact $f' = g$.

4. (a) If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E.
   (b) If in addition $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.
   (c) Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.