Practice Problem Set for the final exam

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1. Let $\mathcal{S}$ denote the set of functions in $\mathcal{C}[-\pi,\pi]$ of the form
$$f(x) = a\sin x + b\sin 2x$$
where a and b are arbitrary real numbers. Let $g(x) = x$ for $x \in [-\pi,\pi]$. Find $f \in \mathcal{S}$ for which $||g-f||_2$ is smallest.

   (*Answer:* $f(x) = 2\sin x - \sin 2x$.)

2. Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0,1]$. Assume that the sequence f_n converges uniformly to f. Answer true or false:
$$\int_0^{1-\frac{1}{n}} f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx.$$

   (*Answer: True.*)

3. Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be the function
$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   (a) Compute the lower and upper Riemann integrals
$$\underline{\int_0^1} f(x,y)\,dx \quad \text{and} \quad \overline{\int_0^1} f(x,y)\,dx$$
   in terms of y.

   (b) Show that
$$\int_0^1 f(x,y)\,dy \text{ exists for each fixed } x.$$
   Compute
$$\int_0^t f(x,y)\,dy \text{ in terms of } (x,t) \in [0,1]\times[0,1].$$

   (c) Define
$$F(x) = \int_0^1 f(x,y)\,dy.$$
   Show that $\int_0^1 F(x)\,dx$ exists and find its value.

   (d) There must be a moral to this long-winded story. What is it?

4. A certain Riemann-integrable function $f : [-\pi,\pi] \to \mathbb{C}$ and a complex sequence $\{c_k\}$ obey
$$\left|\left| f(t) - \sum_{k=-n}^{n} c_n e^{ikt} \right|\right|_2 \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$
   Prove the following statements:

(a) For any $g : [-\pi, \pi] \to \mathbb{C}$ with $g \in \mathcal{R}[-\pi, \pi]$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt = \sum_{k=-\infty}^{\infty} c_k \overline{\widehat{g}(k)}, \text{ where } \widehat{g}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ikt}\, dt.$$

(b) $c_k = \widehat{f}(k)$ and $\sum_k |c_k|^2 < \infty$.

5. If f is a positive continuous function on $[a, b]$, does

$$\lim_{n\to\infty} \left[\int_a^b (f(x))^n\, dx \right]^{\frac{1}{n}}$$

exist? If not, explain why not. If it does, find its value.

6. Evaluate the following, with careful justification of all steps:

(a)

$$\sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} t^5 e^{-int}\, dt \right|^2$$

(*Answer:* $\frac{4\pi^{12}}{11}$.)

(b)

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{n} \quad \text{where } x \in (-1, 1).$$

(*Answer:* $-\ln(1 + x)$.)

(c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

(*Answer:* $-\ln 2$)

7. Let $g : [0, 1] \to \mathbb{R}$ be bounded and $\alpha : [0, 1] \to \mathbb{R}$ be nondecreasing. Assume that $g \in \mathcal{R}_\alpha[\delta, 1]$ for every $\delta > 0$.

(a) Show that $g \in \mathcal{R}_\alpha[0, 1]$ if α is continuous at 0.

(b) Given an examples of a pair (g, α) which shows that the conclusion of part (a) is false if α is not assumed to be continuous at 0.

8. Suppose that $\alpha, \beta : [0, 1] \to \mathbb{R}$ are two right-continuous non-decreasing functions with $\alpha(0) = \beta(0) = 0$ and such that

$$\int_0^1 f(x)\, d\alpha(x) = \int_0^1 f(x)\, d\beta(x) \quad \text{for all } f \in \mathcal{C}[0, 1].$$

Show that $\alpha \equiv \beta$.

9. Let
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
be the Fourier series of a function $f \in \mathrm{BV}[-\pi, \pi]$. Show that $\{na_n\}$ and $\{nb_n\}$ are bounded sequences.

10. Determine whether or not the following functions f are of bounded variation on $[0, 1]$.
    (a) $f(x) = x^2 \sin(\frac{1}{x})$ if $x \neq 0$, $f(0) = 0$.
    (b) $f(x) = \sqrt{x} \sin(\frac{1}{x})$ if $x \neq 0$, $f(0) = 0$.

11. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy a Lipschitz or Hölder condition of order $\alpha > 0$ if there exists $M > 0$ such that
$$|f(x) - f(y)| < M|x - y|^{\alpha} \text{ for all } x, y \in [a, b].$$
    (a) If f is such a function, show that $\alpha > 1$ implies that f is constant on $[a, b]$, whereas $\alpha = 1$ implies $f \in \mathrm{BV}[a, b]$.
    (b) Give an example of a function not of bounded variation satisfying a Hölder condition of order $\alpha < 1$.
    (c) Given an example of a function of bounded variation on $[a, b]$ that satisfies no Lipschitz condition on $[a, b]$.