## Practice Problem Set 1 - Sequences and Series of functions

**More problems may be added to this set. Stay tuned.**

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1. Evaluate with justification
$$\lim_{n\to\infty}\int_0^\pi \frac{n+\sin nx}{3n-\sin^2 nx}\,dx.$$

2. For each $n \in \mathbb{N}$, you are given a differentiable function $f_n : \mathbb{R} \to \mathbb{R}$ that satisfies
$$f_n(0) = 2000, \qquad |f_n'(t)| \le 321 + |t|^{201} \text{ for all } t \in \mathbb{R}.$$
Prove that there exists $f : \mathbb{R} \to \mathbb{R}$ and a subsequence $f_{n_k}$ with the following property: for every compact subset $K$ of $\mathbb{R}$, $f_{n_k} \to f$ uniformly on $K$. Clearly identify the principal theorems and methods that you apply.

3. State whether the following statements are true or false, with adequate justification.
   (a) There exists a function $f : \mathbb{R} \to \mathbb{R}$, which is differentiable at exactly one point.
   (b) In every metric space, a closed and bounded set is compact.
   (c) For any nonempty subset of $\mathcal{C}[0,1]$ that is closed, bounded and equicontinuous, there exists $g \in \mathcal{F}$ such that
   $$\int_0^1 g(x)\,dx \le \int_0^1 f(x)\,dx \qquad \text{for all } f \in \mathcal{F}.$$
   (d) The function $F(x) = \sum_{n=1}^\infty (nx)^{-n}$ is continuous but not differentiable on $\mathbb{R} \setminus \{0\}$.
   (e) There exists a dense subset of $\mathcal{C}[0,1]$ with empty interior.

4. Let $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ be bounded and continuous. For each $n \in \mathbb{N}$, let $F_n : [0,1] \to \mathbb{R}$ satisfy
$$F_n(0) = 0, \qquad F_n'(t) = \phi(t, F_n(t)) \quad \text{for } t \in [0,1].$$
Prove that there is a subsequence $\{F_{n_k}\}$ that converges uniformly to a function $F$ that is a solution of the differential equation
$$y(0) = 0, \qquad y'(t) = \phi(t, F(t)) \quad \text{for } t \in [0,1].$$

5. Let $\alpha > 0$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Hölder continuous with exponent $\alpha$ if the quantity
$$||f||_\alpha := \sup_{x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$
is finite.
   (a) Show that the only Hölder continuous functions of exponent $\alpha > 1$ are the constant functions. For each $0 < \alpha \le 1$, give examples of nonconstant Hölder continuous functions of exponent $\alpha$.

(b) Let $\{f_n\}$ be a sequence of Hölder continuous real-valued functions on $\mathbb{R}$ such that obey

$$\sup_{x\in\mathbb{R}} |f_n(x)| \le 1 \quad \text{and} \quad ||f_n||_\alpha \le 1 \quad \text{for all } n \in \mathbb{N}.$$

Prove that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a subsequence of $\{f_n : n \in \mathbb{N}\}$ that converges pointwise to $f$ and that furthermore converges uniformly to $f$ on $[-M, M]$ for every $M > 0$.

6. Let $\{f_n : n \in \mathbb{N}\}$ be a uniformly convergent sequence of continuous real-valued functions defined on a metric space $M$ and let $g$ be a continuous function on $\mathbb{R}$. Define for each $n \in \mathbb{N}$, $h_n(x) = g(f_n(x))$.

   (a) Let $M = [0, 1]$. Prove that the sequence $\{h_n : n \in \mathbb{N}\}$ converges uniformly on $[0, 1]$.

   (b) Let $M = \mathbb{R}$. Either prove that the sequence $\{h_n : n \in \mathbb{N}\}$ converges uniformly on $\mathbb{R}$ or provide a counterexample.

7. Give an example of each of the following, together with a brief explanation of your example. If such an example does not exist, explain why not.

   (a) A sequence of functions that converges to zero pointwise on $[0, 1]$ and uniformly on $[\epsilon, 1 - \epsilon]$ for every $\epsilon > 0$, but does not converge uniformly on $[0, 1]$.

   (b) A continuous function $f : (-1, 1) \to \mathbb{R}$ that cannot be uniformly approximated by a polynomial.

   (c) A monotonically decreasing sequence of functions $f_n : [0, 1] \to \mathbb{R}$ which converges pointwise, but not uniformly, to zero.

8. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions that obey $|f_n(y)| \le 1$ for all $n \in \mathbb{N}$ and all $y \in [0, 1]$. Let $T : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Define

$$g_n(x) = \int_0^1 T(x, y) f_n(y)\, dy.$$

   Prove that the sequence $\{g_n\}$ has a uniformly convergent subsequence.

9. Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : x, y \ge 0,\ x^2 + y^2 \le 1\}$.

   (a) Prove that for any $\epsilon > 0$ and any continuous function $f : \mathbb{H} \to \mathbb{R}$ there exists a function $g(x, y)$ of the form

$$g(x, y) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{mn} x^{2m} y^{2n}, \qquad N = 0, 1, 2, \cdots, \quad a_{mn} \in \mathbb{R}$$

   such that

$$\sup_{(x,y)\in\mathbb{H}} |f(x, y) - g(x, y)| < \epsilon.$$

   (b) Does the result in (a) hold if $\mathbb{H}$ is replaced by the disk $\mathbb{D} = \{(x, y) : x^2 + y^2 \le 1\}$?

10. Give examples of the following (provide brief justifications):

(a) A right-continuous increasing function on $[-1, 1]$ which is discontinuous at 0 and continuous elsewhere.

(b) A sequence $\{f_n\}$ of real-valued differentiable functions on $[0, 4]$ which converges uniformly to a differentiable function $f$ but such that $f_n'$ does not converge pointwise to $f'$.

11. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{x^2}{1+x^2} \right)^{\frac{1}{n}}$$

is uniformly convergent on $\mathbb{R}$.

12. Let $f : [0, 1] \to \mathbb{R}$ have left limits at each point in $(0, 1]$ and right limits at all points in $[0, 1)$. Prove that $f$ is a bounded function.

13. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x/(1 + nx^2)$, $n \in \mathbb{N}$. Show that $\{f_n : n \in \mathbb{N}\}$ converges uniformly to a function $f$ and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

14. Let $f$ and $\{f_n : n \in \mathbb{N}\}$ be continuous real-valued functions on $[0, 1]$.

(a) If $\{f_n\}$ converges uniformly to $f$, show that for each $k \in \{0, 1, 2, \cdots\}$

$$\lim_{n\to\infty} \int_0^1 f_n(x)x^k\, dx = \int_0^1 f(x)x^k\, dx.$$

(b) Prove that the converse of the statement in (a) does not hold in general.

(c) However, there does exist a partial converse to (a), under some extra assumptions. Suppose that each $f_n$ is continuously differentiable, with $f_n(0) = 0$ and $\{f_n' : n \in \mathbb{N}\}$ uniformly bounded. If for every $k \in \{0, 1, 2, \cdots\}$,

$$\lim_{n\to\infty} \int_0^1 f_n(x)x^k\, dx \text{ exists and equals } m_k \text{ for some } m_k \in \mathbb{R},$$

prove that $f_n$ converges uniformly to a function $f \in C[0, 1]$ for which

$$m_k = \int_0^1 x^k f(x)\, dx.$$