Midterm Review - Math 321, Spring 2015

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1. *Give an example of an equicontinuous family of non-constant functions that is not totally bounded.*

   Sketch of solution. The function class $\mathrm{Lip}_K[0,1] \setminus \{\text{constant functions}\}$ for any fixed K provides an example. This family is equicontinuous because the continuity parameter δ can be chosen to be ϵ/K independent of the functions in this class. On the other hand, any totally bounded set must be bounded, whereas $\mathrm{Lip}_K([0,1])$ contains the unbounded collection of all functions of the form $x + C$, with C an arbitrary real number. □

2. *Find a uniformly convergent sequence of polynomials whose derivatives are not uniformly convergent.*

   Sketch of solution. The sequence $p_k(x) = x^{k+1}/(k+1)$ is uniformly convergent because $||p_K||_\infty \leq \frac{1}{k+1} \to 0$. On the other hand, $p_k'(x) = x^k$, which converges uniformly to the function q which takes the value 1 at $x = 1$ and is zero elsewhere. Since each p_k' is continuous and q is not, the sequence $\{p_k'\}$ cannot be uniformly convergent. □

3. *Let α be continuous and non-decreasing (I forgot to write this down on the board - sorry about that). Given $f \in \mathcal{R}_\alpha[a,b]$ and $\epsilon > 0$ does there exist a step function g such that $\int_a^b |f - g|\, d\alpha < \epsilon$?*

   Sketch of solution. Since $f \in \mathcal{R}_\alpha[a,b]$, by Riemann's condition there exists a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that

$$U_\alpha(f;P) - L_\alpha(f;P) = \sum_{i=1}^{n} \omega(f;I_i)\omega(\alpha;I_i) < \epsilon, \quad \text{where } I_i = [x_{i-1}, x_i], \tag{1}$$

and $\omega(f;I)$ denotes the oscillation of f on I as defined in class. Set g to be the step function defined by $g(x) = f(x_i)$ if $x \in [x_{i-1}, x_i)$. We will prove shortly that $g \in \mathcal{R}_\alpha[a,b]$. Assuming this, we have that $|f - g| \in \mathcal{R}_\alpha[a,b]$, since $\mathcal{R}_\alpha[a,b]$ is a vector space and a lattice. Now,

$$\int_a^b |f-g|\, d\alpha = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f-g| d\alpha \leq \sum_{i=1}^n |f(x) - f(t_i)|\, d\alpha \leq \sum_{i=1}^n \omega(f;I_i)\omega(\alpha;I_i) < \epsilon,$$

where the last step follows from (1).

It remains to show that $g \in \mathcal{R}_\alpha[a,b]$. We will apply Riemann's condition again. Since α is continuous (hence uniformly continuous) there exists for any $\kappa > 0$, some $\delta > 0$ such that $\omega(\alpha; I) < \frac{\kappa}{2n||f||_\infty}$ for all intervals I of length $< \delta$. We pick a partition Q of $[a,b]$ generated by subintervals of the form $J_i = x_i + [-\frac{\delta}{3}, \frac{\delta}{3}]$. Since f is constant on all other subintervals of this partition, these intervals do not contribute to $U_\alpha(g;Q) - L_\alpha(g;Q)$. Thus,

$$U_\alpha(g;Q) - L_\alpha(g;Q) \leq \sum_{i=1}^n |f(t_{i+1}) - f(t_i)|\omega(\alpha;J_i) < \frac{\kappa}{2n||f||_\infty} \sum_{i=1}^n |f(t_{i+1}) - f(t_i)| \leq \kappa.$$

□

4. *Let $\mathcal{S}[0,1] \subseteq \mathcal{B}[0,1]$ be the space of step functions with finitely many jumps. Show that $\mathcal{C}[0,1] \subseteq \overline{\mathcal{S}[0,1]}$. Does there exist a discontinuous function in $\overline{\mathcal{S}[0,1]} \setminus \mathcal{S}[0,1]$?*

Sketch of solution. If $f \in \mathcal{C}[0,1]$ and $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a uniform partition of $[0,1]$ into subintervals of length $\frac{\delta}{2}$. Define g be the step function that takes the value $f(x_{i-1})$ on the interval $I_i = [x_{i-1}, x_i)$ and the value $f(x_{n-1})$ at $x = b$. Then $||f - g||_\infty \leq \sup_{1 \leq i \leq n} \sup_{x \in I_i} |f(x) - f(x_{i-1})| < \epsilon$, as claimed.

Yes. Consider the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ x + 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Approximate each piece of f by step functions. □