Review worksheet for Midterm 2

Math 300, Section 202, Spring 2015

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1. Compute, with complete justification, the contour integrals:
$$\int_C f(z)\,dz$$
where $f(z) = \tan z$ and $f(z) = \mathrm{Log}(z+2)$, and C is the unit circle traversed once with any orientation.

(Answer: 0 for both integrals)

2. Let $f(z) = \frac{z+2}{\sin\left(\frac{z}{2}\right)}$. Explain why
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$
where C_1 is the positively oriented circle $|z| = 4$ and C_2 is the positively oriented square with vertices at $\pm 1 \pm i$.

3. Show that
$$\int_\Gamma z^i\,dz = \frac{1+e^{-\pi}}{2}(1-i)$$
where z^i denotes the principal branch, and the path of integration Γ is any contour from $z = -1$ to $z = 1$ that, except for its endpoints, lies above the real axis.

4. Use a branch cut of the non-negative real line to define an analytic branch of the function $z^{1/2}$. Now evaluate this function over any contour C from -3 to 3 that, except for the endpoints, lies above the x-axis.

(Answer: $2\sqrt{3}(1+i)$)

5. Find all possible values of $\sin^{-1}(-i)$.

(Answer: $n\pi + i(-1)^{n+1}\ln(1+\sqrt{2})$)

6. Give brief answers to the following questions:
   (a) Find the principal value of $(-i)^i$.

   (Answer: $e^{\frac{\pi}{2}}$)

   (b) Determine, with adequate justification, whether the following statements are true or false.
      (i) $\mathrm{Arg}(z)$ is harmonic in every domain that does not contain the negative half-line.
      (Answer: true)

(ii) $\mathrm{Arg}(z)$ is harmonic in every domain that does not contain the origin.

(Answer: false)

(iii) If a domain is not simply connected, then no curve can be continuously deformed to a point.

(Answer: false)

(iv) In a simply connected domain, every analytic function has an antiderivative.

(Answer: true)

(c) Define an analytic branch of $\log(z^2 - 2z)$ as z ranges in the interior of the unit disk centred at 1.

(Answer: $\log_0(z^2 - 2z)$ *works, where* $\log_0$ *denotes the branch of the logarithm with cut along the non-negative real axis)*