Review worksheet for Midterm 1

Math 300, Section 202, Spring 2015

---

1. **Find the limit of the function $f(z) = (z/\bar{z})^2$, if it exists, as z tends to zero. If you think the limit does not exist, explain your reasoning for this conclusion.**

   Solution. If $z \to 0$ along the line $y = mx$, then $z = x + imx = x(1+im)$, $\bar{z} = x - imx = x(1-im)$, hence

$$f(z) = f(x+imx) = \Big(\frac{x(1+im)}{x(1-im)}\Big)^2 = \frac{(1-m^2)+2im}{(1-m^2)-2im}.$$

   This last quantity depends on m. For example, it is 1 if $m = 0$, i.e., when $z \to 0$ along the x-axis. The value is $(-3+4im)/(-3-4im) \neq 1$ if $m = 2$. Since the limiting value of f depends on the angle of approach, $\lim_{z\to 0} f(z)$ does not exist.

   Caution! This problem is not about the differentiability of the function f, so please do not use the dependence on $\bar{z}$ to deduce that the limit does not exist. □

2. **Describe geometrically the collection of points z satisfying the equation $|z-1| = |z+i|$. Sketch this set of points in the complex plane.**

   Solution. Recall that $|z - z_0|$ is the distance of the point z from z_0. Thus the equation $|z-1| = |z+i|$ represents all points z which are equidistant from 1 and $-i$. Such points lie of the perpendicular bisector of the line segment joining 1 and $-i$. Thus the collection of z satisfying the equation is the infinite line passing through the point $(0,0)$ with slope -1.

   An alternative strategy: You could also try to simplify the equation $(x-1)^2 + y^2 = x^2 + (y+1)^2$. □

3. **Express the complex number $(-1+i)^7$ in the form $a + ib$.**

   Solution. We write $-1+i$ in polar form: $-1+i = \sqrt{2}e^{\frac{3\pi i}{4}}$. Therefore

$$(-1+i)^7 = (\sqrt{2})^7 e^{\frac{21\pi i}{4}} = 8\sqrt{2}e^{5\pi i + \frac{\pi i}{4}} = -8\sqrt{2}\frac{1+i}{\sqrt{2}} = -8(1+i).$$

   □

4. **Decide whether the set $\{z : 0 \leq \arg(z) \leq \frac{\pi}{4}\}$ is bounded. Give reasons for your answer.**

   Solution. The set $\{z : 0 \leq \arg(z) \leq \frac{\pi}{4}\}$ is the infinite triangular region in the first quadrant bounded by the lines $y = 0$ and $y = x$. This region cannot be contained within a ball of any finite radius, and is hence unbounded. □

5. **Describe the domain of definition of the function $f(z) = z/(z + \bar{z})$.**

Solution. The functions z and $z + \bar{z}$ are well-defined on the whole complex plane. Their ratio is defined whenever the denominator is nonzero. But $z+\bar{z} = 0$ if and only if $z = -\bar{z}$, i.e., $x + iy = -x + iy$ or $x = \operatorname{Re}(z) = 0$. Therefore the domain of definition of f is $\{z \in \mathbb{C} : \operatorname{Re}(z) \neq 0\}$. □

6. **Find and sketch the images of the hyperbolas**

$$x^2 - y^2 = -1 \qquad \textbf{and} \qquad xy = -2$$

under the transformation $w = z^2 = (x + iy)^2$**.**

Solution. Observe that

$$z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy = u + iv,$$

so the set of $z = x+iy$ with $x^2 - y^2 = -1$ maps to $u+iv = -1+2ixy$, which is contained in the vertical line $u = -1$ in the (u, v) plane. Conversely, given any point of the form $-1 + ik$ on this line, there exist values of (x, y) satisfying

$$x^2 - y^2 = -1 \qquad \text{and} \qquad 2xy = k.$$

This can be seen by substituting $y = k/(2x)$ from the second equation into the first, obtaining a quadratic equation in x^2, namely

$$x^2 - \left(\frac{k}{2x}\right)^2 = -1, \qquad \text{or} \qquad 4x^4 + 4x^2 - k^2 = 0.$$

The last equation has a non-negative solution $x^2 = (-4+\sqrt{16 + 16k^2})/8$. Thus the image of the hyperbola $x^2 - y^2 = -1$ under the squaring map is the entire line $u = -1$.

Similarly, the set of $z = x + iy$ with $xy = -2$ maps to $p + iq = x^2 - y^2 - 4i$ which is a point on the horizontal line $q = -4$ in the (p, q) plane. As above, one can show that every point $k - 4i$ on this line is in fact the image of some (x, y) on the hyerbola $xy = -2$. To see this, we need to show that there exist (x, y) that satisfy the two equations

$$xy = -2 \qquad x^2 - y^2 = k.$$

Upon eliminating y, this reduces to solving the equation $x^4 - kx^2 - 4 = 0$, which admits a real solution in x. Thus the image of the hyperbola in the entire line $q = -4$. □

7. **Show that the function** $f(z) = x^2 + iy^2$ **is differentiable at the origin but analytic nowhere.**

Solution. Set $u(x, y) = x^2$ and $v(x, y) = y^2$. Then $u_x = 2x$, $u_y = v_x 0$ and $v_y = 2y$. Thus there is no open set on which the Cauchy-Riemann equations hold. Therefore f is not analytic on any open set in the complex plane.

We will now show that f is differentiable at the origin and that $f'(0) = 0$.

$$\left|\lim_{(x,y)\to(0,0)} \frac{f(x + iy) - 0}{x + iy}\right| = \lim_{(x,y)\to(0,0)} \left|\frac{x^2 + iy^2}{x + iy}\right| = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^4 + y^2}}{\sqrt{x^2 + y^2}}.$$

Since the expression above is symmetric in x and y, we may assume without loss of generality that $|x| \geq |y|$. With this assumption, we see that

$$\frac{\sqrt{x^4 + y^2}}{\sqrt{x^2 + y^2}} \leq \frac{2x^4}{x^2} = 2x^2 \to 0,$$

hence the limit exists, and its value is zero. □

8. **Find the harmonic conjugate of the function $u(x, y) = y^3 - 3x^2y$ if it exists. If the answer is yes, determine the analytic function f whose real part is u.**

Solution. If v is the harmonic conjugate of u, then by definition $f = u + iv$ is analytic. Therefore u, v must satisfy the Cauchy-Riemann equations

$$u_x = -6xy = v_y \qquad \text{and} \qquad u_y = 3y^2 - 3x^2 = -v_x.$$

This implies that $v = -3xy^2 + A(x) = -3xy^2 + x^3 + B(y)$. Thus $v = -3xy^2 + x^3 + C$, where C is an arbitrary constant.

Notice that if $f = u + iv$, then $f(x, 0) = u(x, 0) + iv(x, 0) = i(x^3 + C)$. This suggests the possibility that $f(z) = i(z^3 + C)$, which one can now easily verify:

$$f(z) = y^3 - 3x^2y + i(-3xy^2 + x^3 + C) = i(z^3 + C).$$

□

9. **State whether each of the following statements is true or false. If the statement is true, give a short proof of it. If not, give a counterexample to show that it is false.**

(a) **The function $f(z) = e^z$ is harmonic.**

Solution. True. The function f is entire, i.e., satisfies the Cauchy-Riemann equations. Since Laplace's equation follows from the Cauchy-Riemann equations, f is harmonic. □

(b) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$.

Solution. True. $|(2\bar{z} + 5)(\sqrt{2} - i)| = |(2\bar{z} + 5)| \times |(\sqrt{2} - i)| = \sqrt{3}|(2\bar{z} + 5)| = \sqrt{3}|\overline{(2\bar{z} + 5)}| = \sqrt{3}|2z + 5|$. □

(c) **There exists a complex number z_0 whose fourth roots z_1, z_2, z_3, z_4 have the property that**

$$\mathbf{arg}(z_1) = \frac{\pi}{4}, \quad \mathbf{arg}(z_2) = \frac{\pi}{2}, \quad \mathbf{arg}(z_3) = \frac{2\pi}{3}, \quad \mathbf{arg}(z_4) = \pi.$$

Solution. False. The fourth roots of any complex number are equispaced points on a circle centred at the origin. The argument of each root has to be separated from that of its nearest neighbour by $\pi/2$. This is not the case here. □

(d) **The equation $(z^2 + z + 1)e^z = 0$ has exactly two complex roots.**

Solution. True. $e^z = e^x(\cos y + i \sin y)$ has no zero in $\mathbb{C}$, so the equation reduces to finding the roots of the quadratic polynomial $z^2 + z + 1$. By the fundamental theorem of algebra, this polynomial has exactly two complex roots. □

(e) **If a rational function R has a pole at the point a, then then residue of R at a must be a nonzero complex number.**

Solution. False. The function $R(z) = 1/z^2$ has a pole at $z = 0$, but its residue at that point is 0. □