Midterm Exam II

November 5, 2014

No books. No notes. No calculators. No electronic devices of any kind.

Name ________________ **Student Number** ________________

Problem 1. (3 points)
Find all values of i^i.

$$i^i = e^{\log(i^i)} = e^{i\log(i)} = e^{i(\ln|i| + i\arg(i))}$$

$$= e^{i(0 + i\arg i)} = e^{-\arg(i)} = e^{-(\pi/2 + 2\pi k)}, \quad k \in \mathbb{Z}$$

Surprise: all values of i^i are positive real!

1	2	3	4	5	6	total/25

Problem 2. (3 points)
Carefully sketch the branch cut(s) of the principal branch of the multi-valued function

$$f(z) = \sqrt[3]{z^3 - 27}.$$

(The principal branch is defined in terms of the principal branch of the logarithm.)

$f(z) = (z^3-27)^{1/3} = e^{\frac{1}{3}\text{Log}(z^3-27)}$.

So the branch cuts of $f(z)$ are where $z^3-27 = -r$ ($r \geq 0$ real).

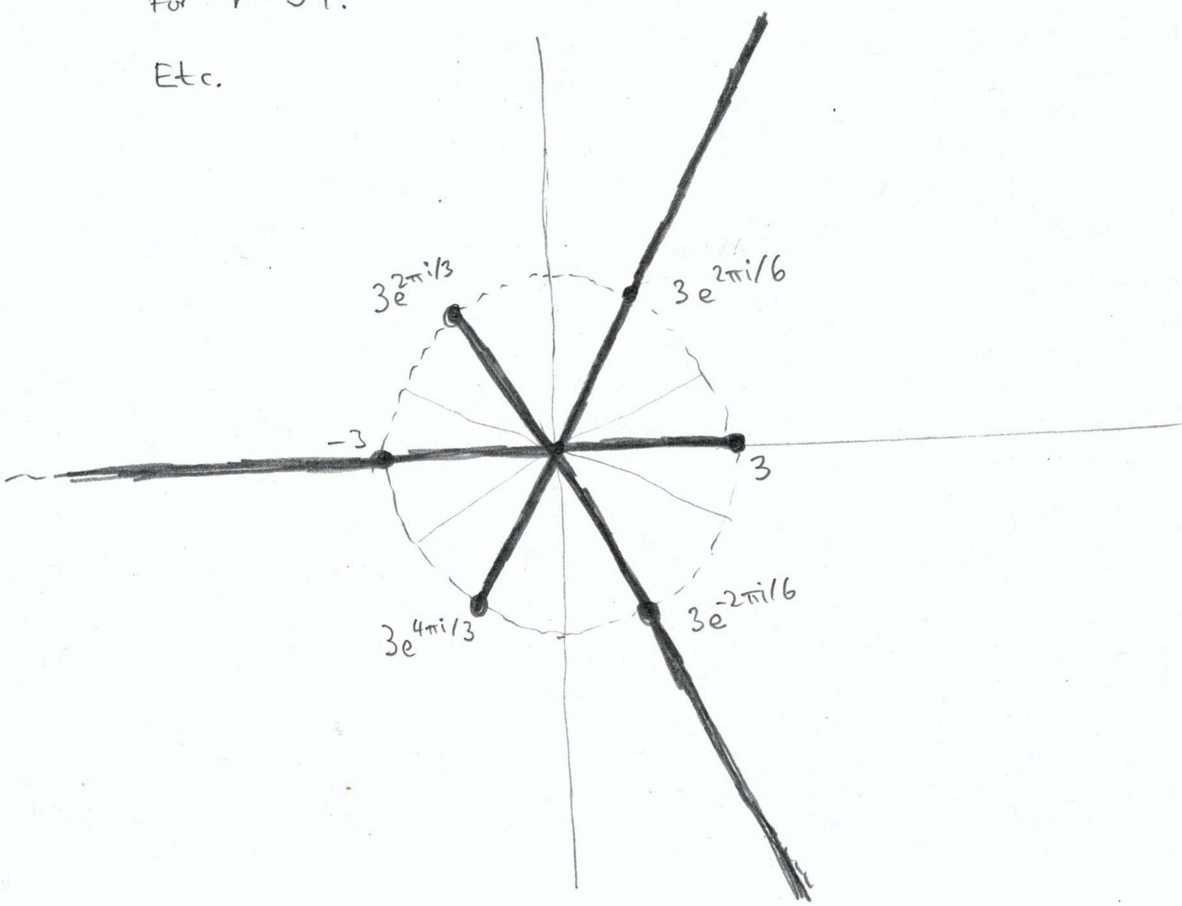
or $z^3 = 27-r$

For $r=0$: $z^3=27$ so $z = 3e^{2\pi i k/3}$ $k=0,1,2$

For $r=27$: $z^3=0$ so $z=0$

For $r=54$: $z^3=-27$ so $z = 3e^{2\pi i/6 + 2\pi i k/3}$ $k=0,1,2$

Etc.



Problem 3. (5 points)
True or false? (No reasons necessary.)

(a) Every branch of the multi-valued function $f(z) = i^z$ is an entire function.

(b) The principal branch of the function $f(z) = z^i$ is analytic at $z = i$.

(c) The function $f(z) = \sinh(z)$ is periodic, with period $2\pi i$.

(d) Let Γ be a circle centered at the origin, traversed once in the counterclockwise direction. Then, for every complex number z not on Γ, we have

$$\oint_\Gamma \frac{w}{w-z}\,dw = \begin{cases} 2\pi i z & \text{if } z \text{ is inside } \Gamma, \\ 0 & \text{if } z \text{ is outside } \Gamma. \end{cases}$$

(e) For every closed contour Γ, which avoids the origin, we have

$$\oint_\Gamma \frac{1+z^{10}}{z^{100}}\,dz = 0\,.$$

(a) $f(z) = i^z = e^{z\log(i)}$. For every value of $\log(i)$, for example $\mathrm{Log}(i) = \frac{\pi i}{2}$, this gives an entire function, for example $e^{z i\pi/2}$. Another value of $\log(i)$ is $\frac{5\pi i}{4}$, this gives $i^z = e^{z\cdot 5\pi i/4}$ another entire function. $\boxed{\text{TRUE.}}$

(b) $f(z) = e^{i\log(z)}$. Principal branch is $f(z) = e^{i\,\mathrm{Log}(z)}$. $\mathrm{Log}(z)$ is analytic at $z=i$, hence $f(z)$ is, too. $\boxed{\text{TRUE.}}$

(c) $f(z) = \sinh(z) = \frac{1}{2}(e^z - e^{-z})$.

$f(z+2\pi i) = \sinh(z+2\pi i) = \frac{1}{2}\left(e^{z+2\pi i} - e^{-(z-2\pi i)}\right) = \frac{1}{2}(e^z - e^{-z}) = \sinh(z) = f(z)$.

$\boxed{\text{TRUE.}}$

(d) Cauchy's Integral Formula for $f(z)=z$ gives $z = f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(w)}{w-z}\,dw$ for z inside Γ. If z is outside Γ, the function $\frac{w}{w-z}$ has no pole in a domain containing Γ, so $\oint_\Gamma \frac{w}{w-z}\,dw = 0$. $\boxed{\text{TRUE.}}$

(e) $\frac{1+z^{10}}{z^{100}} = z^{-100} + z^{-90}$ has an antiderivative, for example $-\frac{1}{99}z^{-99} - \frac{1}{89}z^{-89}$

So the integral over every closed contour vanishes. $\boxed{\text{TRUE.}}$

Problem 4. (4 points)
Compute the contour integral

$$\int_\Gamma \bar{z}\, dz\,,$$

where Γ is the circle of radius 2, centered at $z = i$, traversed once in the clockwise direction. Simplify your answer.

clockwise!

parametrize the circle: $z(t) = i + 2e^{-2\pi i t}$ $\quad 0 \le t \le 1$.

$$\oint_\Gamma \bar{z}\, dz = \int_0^1 \overline{\left(i + 2e^{-2\pi i t}\right)}\, d\left(i + 2e^{-2\pi i t}\right)$$

$$= \int_0^1 \left(-i + 2e^{2\pi i t}\right)(-4\pi i)\, e^{-2\pi i t}\, dt$$

$$= -4\pi \int_0^1 e^{-2\pi i t}\, dt - 8\pi i \int_0^1 dt$$

$$= -4\pi \frac{1}{-2\pi i} e^{-2\pi i t}\Big|_0^1 - 8\pi i t\Big|_0^1$$

$$= \frac{4\pi}{2\pi i}(1-1) - 8\pi i(1-0)$$

$$= -8\pi i$$

Problem 5. (5 points)
Compute the contour integral

$$\int_i^{-i} \frac{z}{(z+1)^2} dz\,,$$

along the straight path from i to $-i$. Simplify your answer.

Try partial fractions:

$$f(z) = \frac{z}{(z+1)^2} = \frac{A}{(z+1)^2} + \frac{B}{z+1}$$

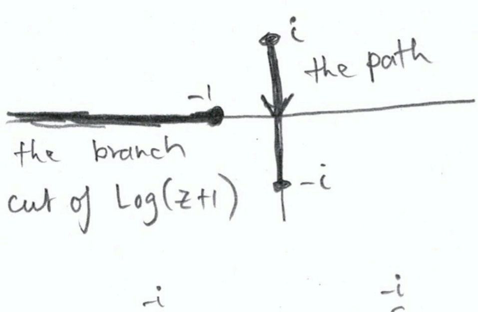
$$A = \lim_{z\to -1} f(z)(z+1)^2 = \lim_{z\to -1} z = -1$$

$$B = \lim_{z\to -1} \frac{d}{dz}\left(f(z)(z+1)^2\right) = \lim_{z\to -1} 1 = 1$$

<u>Check:</u> $\frac{-1}{(z+1)^2} + \frac{1}{z+1} = \frac{-1+z+1}{(z+1)^2} = \frac{z}{(z+1)^2} = f(z)$ ✓

antiderivative of $\frac{-1}{(z+1)^2} = -(z+1)^{-2}$ is $(z+1)^{-1} = \frac{1}{z+1}$

antiderivative of $\frac{1}{z+1}$ is $\mathrm{Log}(z+1)$. The branch cut of $\mathrm{Log}(z+1)$ is where $z+1 = -r$. ($r \geq 0$ real.) $\iff$ $z = -1-r$ $\iff$ $z \leq -1$ real.



Since the path avoids the branch cut, we can use $\mathrm{Log}(z+1)$ as antiderivative of $\frac{1}{z+1}$.

$$\int_i^{-i} \frac{z}{(z+1)^2} dz = \int_i^{-i} \frac{-1}{(z+1)^2} dz + \int_i^{-i} \frac{1}{z+1} dz = \frac{1}{z+1}\Big|_i^{-i} + \mathrm{Log}(z+1)\Big|_i^{-i}$$

$$= \frac{1}{-i+1} - \frac{1}{i+1} + \mathrm{Log}(-i+1) - \mathrm{Log}(i+1)$$

$$= \frac{(1+i)-(1-i)}{(1+i)(1-i)} + \ln|1-i| + i\,\mathrm{Arg}(1-i) - \left(\ln|1+i| + i\,\mathrm{Arg}(1+i)\right) = \ldots$$

Overflow space.

$$\ldots = \frac{2i}{2} + \ln\sqrt{2} + i\left(-\frac{\pi}{4}\right) - \left(\ln\sqrt{2} + i\frac{\pi}{4}\right)$$

$$= i - \frac{\pi}{4}i - \frac{\pi}{4}i$$

$$= \left(1 - \frac{\pi}{2}\right)i$$

Problem 6. (5 points)
Compute the contour integral

$$\oint_\Gamma \frac{z^3}{(z-1)^3} dz\,,$$

where Γ is a simple closed curve winding around $z = 1$ once in the clockwise direction. Simplify your answer.

By the residue theorem for rational functions,

$$\oint_\Gamma \frac{z^3}{(z-1)^3}\,dz = -2\pi i \ \operatorname{res}\left(\frac{z^3}{(z-1)^3}\,;\,1\right)$$

$\uparrow$ clockwise

$$= -2\pi i \ \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2}\left(\frac{z^3}{(z-1)^3}(z-1)^3\right) = -\pi i \lim_{z\to 1} 6z$$

$$= -6\pi i$$

Alternatively, use $\quad f''(z) = \frac{2!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^3}\,dw$

with $f(w) = w^3$ and $z = 1$:

$$f''(1) = \frac{1}{\pi i}\int_\Gamma \frac{w^3}{(w-1)^3}\,dw$$

$f'(w) = 3w^2$

$f''(w) = 6w \qquad 6\pi i = \int_\Gamma \frac{z^3}{(z-1)^3}\,dz$

$f''(1) = 6$

but the path goes the wrong way, so $\boxed{\int_\Gamma \frac{z^3}{(z-1)^3}\,dz = -6\pi i}$.