Math 300 Midterm 2 Solutions

1. (a) *Express the principal value of* $(1-i)^{4i}$ *in the form* $a+ib$.

   Solution.

$$\begin{aligned}(1-i)^{4i} &= e^{4i\mathrm{Log}(1-i)} \\ &= e^{4i(\log\sqrt{2}-i\pi/4)} \\ &= e^{\pi+i\log 4} \\ &= e^{\pi}\cos(\log 4)+ie^{\pi}\sin(\log 4)\end{aligned}$$

$\square$

   (b) *Evaluate the contour integral*

$$\int_C \cos\left(\frac{z}{2}\right)\,dz$$

   where C *is an arc of a parabola starting from its vertex at the origin and ending at* $\pi+2i$. *Express your answer in the form* $a+ib$.

   Solution.

$$\begin{aligned}\int_C \cos\left(\frac{z}{2}\right)\,dz &= 2\sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2i} \\ &= 2\sin\left(\frac{\pi+2i}{2}\right)-2\sin\left(\frac{0}{2}\right) \\ &= 2\sin(\pi/2+i) \\ &= 2\sin(\pi/2)\cos i+2\cos(\pi/2)\sin i \\ &= 2\cos i \\ &= 2\cosh 1 \\ &= e+\frac{1}{e}\end{aligned}$$

$\square$

2. *Determine with complete justification the image of the function*

$$f(z)=\frac{1}{z^4}-1$$

   as z *ranges in the exterior* D *of the unit disc. i.e.,* $D=\{z\in\mathbb{C}:|z|>1\}$. *Sketch the image set and use it to define an analytic branch of the function* $\log f(z)$.

Solution. We can parametrize $z \in D$ as $z = re^{i\theta}$, with $r > 1$ and $0 \leq \theta \leq 2\pi$. For every fixed r, the complex numbers $z^{-4} = r^{-4}e^{4i\theta}$ trace out a circle C_r centred at the origin of radius r^{-4}. As r ranges in the interval $(1, \infty)$, r^{-4} covers all positive numbers < 1. Thus we obtain that

$$\{z^{-4} : z \in D\} = \bigcup_{r>1} C_r = \{v \in \mathbb{C} : v \neq 0, |v| < 1\}.$$

Finally, translating left by one unit gives

$$\text{Image}(f) = \{z^{-4} - 1 : z \in D\} = \{w : w \neq -1, |w + 1| < 1\}.$$

In other words, the image set is the open disc of unit radius centred at -1 and punctured at this point.

Since the image set avoids the positive real line, we can use this line as a branch cut. An analytic branch of $\log f(z)$ may therefore be defined by $\log_0(f(z))$, where

$$\log_0(w) = |w| + i\arg(w), \qquad \text{where } \arg(w) \in (0, 2\pi).$$

Other branch cuts that are disjoint from the image set are also possible.

□

3. (a) *Write down a parametrization of the square* Γ *with vertices at* $0, 1, 1+i$ *and* i, *traversed anticlockwise in that order.*

   Solution. $\Gamma_1 : z = t,$ where $0 \leq t \leq 1$.
   $\Gamma_2 : z = 1 + ti,$ where $0 \leq t \leq 1$.
   $\Gamma_3 : z = i + 1 - t,$ where $0 \leq t \leq 1$.
   $\Gamma_4 : z = i - ti,$ where $0 \leq t \leq 1$. □

   (b) *Use the parametrization in part (a) to evaluate the integral*

$$\int_\Gamma \pi e^{\pi \overline{z}}\, dz.$$

   Here $\overline{z}$ *denotes the complex conjugate of* z.

Solution.

$$
\begin{aligned}
&\int_\Gamma \pi e^{\pi\bar{z}}\,dz.\\
=&\int_0^1 \pi e^{\pi t} + i\pi e^{\pi(1-ti)} - \pi e^{\pi(1-t-i)} - i\pi e^{\pi(ti-i)}dt\\
=&e^{\pi t} - e^{\pi}e^{-\pi ti} - e^{\pi}e^{-\pi t} + e^{\pi ti}\Big|_0^1\\
=&e^{\pi} + e^{\pi} - 1 - 1 - 1 + e^{\pi} + e^{\pi} - 1\\
=&4(e^{\pi} - 1)
\end{aligned}
$$

□

4. Find all roots of the equation $\sinh z = i$.

Solution.

$$
\begin{aligned}
\sinh z &= i\\
\frac{e^z - e^{-z}}{2} &= i\\
e^z - e^{-z} &= 2i\\
e^{2z} - 2ie^z - 1 &= 0\\
(e^z - i)^2 &= 0\\
e^z &= i\\
z &= \left(\frac{\pi}{2} + 2k\pi\right) i, \text{where k is any integer.}
\end{aligned}
$$

□

5. *For each of the statements below, indicate whether they are true or false.*

   (a) *Any loop in* $D = \{z \in \mathbb{C} : |Im(z)| \leq 1\} \setminus \{z \in \mathbb{R} : z \leq 0\}$ *can be continuously deformed to a point.*

   Solution. The statement is true. The domain D has an infinite slit but no holes, and is hence simply connected. By definition of simple connectivity, any closed loop can therefore be deformed into a point.

   □

(b) *Let γ be the circle centred at 2 of radius 1 lying in the punctured plane $D = \mathbb{C} \setminus \{0\}$. If f is an analytic function on D with the property that*

$$\int_\gamma f(z)\, dz = 0,$$

then f must have an analytic antiderivative on D.

Solution. The statement is false. For example, $f(z) = 1/z$ is analytic on D and its integral over γ is zero. The last statement follows from Cauchy's theorem, since γ is contained in the simply connected domain $D' = \{z : |z-2| < \frac{3}{2}\}$ on which f is analytic. However, f does not admit an analytic antiderivative on D; if it did, its integral over *any* closed loop in D would have to be zero. But we have seen that f integrated over any circle centred at the origin (traversed once in the anticlockwise direction) is $2\pi i$.

□

(c) *Let $f(z) = \overline{z}$, the complex conjugate of z. If C_1 and C_2 are any two contours in a domain D with common endpoints, then one can conclude from the "independence of path" theorem that*

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

Solution. The statement is false. The "independence of path" theorem applies only to analytic functions, and $f(z) = \overline{z}$ is analytic nowhere. In particular, if one chooses C_1 to be the constant curve 1, and C_2 to be the unit circle centred at 0, then both C_1 and C_2 have common start and endpoints; however,

$$\int_{C_1} f(z)\, dz = 0, \quad \text{and} \quad \int_{C_2} f(z)\, dz = 2\pi i.$$

□

(d) *There does not exist any analytic branch of the function $f(z) = z^i$ on the domain*

$$D = \mathbb{C} \setminus \left\{ z = x + iy : x \geq 0,\ y = x^2 \right\}.$$

Solution. The statement is false - there does exist such a branch, namely

$$\mathcal{L}(z) = |z| + i\arg(z), \qquad \text{where, for } |z| = r,\ \arg(z) \in (\theta_r, \theta_r + 2\pi).$$

Here θ_r is a solution (continuous in r) of the equation $\sin\theta_r = r\cos^2\theta_r$.

□

(e) *The integral of any polynomial over any closed contour is zero.*

Solution. The statement is true. Any polynomial P of the form

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

admits an analytic antiderivative

$$Q(z) = a_0 z + a_1 \frac{z^2}{2} + a_2 \frac{z^3}{3} + \cdots + a_n \frac{z^{n+1}}{n+1}.$$

Hence by the fundamental theorem for contour integrals, the integral of P over any closed contour is zero.

□