Math 300 Homework 7 Solution

1. (Section 4.5 Q3(a)) Integral equals
$$2\pi i \sin\left(3\frac{\pi}{2}\right) = -2\pi i.$$

2. (Section 4.5 Q3(e)) Integral equals
$$2\pi i \frac{d}{dz} e^{-z}\Big|_{z=-1} = -2\pi e i.$$

3. (Section 4.5 Q3(f)) Integral equals
$$2\pi i \frac{d}{dz}\frac{\sin z}{z-4}\Big|_{z=0} = -\frac{\pi i}{2}.$$

4. (Section 4.5 Q5) Let $g(\zeta) = \zeta^2 - \zeta + 2$. Then
$$G(1) = 2\pi i g(1) = 4\pi i.$$
$$G'(i) = 2\pi i g'(i) = -4\pi - 2\pi i.$$
$$G''(-i) = 2\pi i g''(-i) = 4\pi i.$$

5. (Section 4.5 Q6) For $r < 2$, the integral equals
$$\oint_{|z-i|=r} \frac{e^{iz}/(z+i)^2}{(z-i)^2}\,dz + \oint_{z+i=r} \frac{e^{iz}/(z-i)^2}{(z+i)^2}\,dz$$
$$=2\pi i\left(-\frac{e^{-1}i}{2}\right) + 2\pi i(0) = \frac{\pi}{e}.$$

6. (Section 4.5 Q7) Integral equals
$$2\pi i \frac{d}{dz}\frac{\cos z}{z-3}\Big|_{z=0} = -\frac{2\pi i}{9}.$$

7. (Section 4.5 Q11) $\dfrac{\partial^2 u}{\partial x^2}$ is the real part of the analytic function $f''(z)$ (Theorem 16), so it is harmonic (Theorem 7, Section 2.5).

8. (Section 4.6 Q3) Fix z. The circle $|\zeta - z| = r - |z|$ lies inside the disk $|\zeta| \le r$, so $\max\limits_{|\zeta - z| = r - |z|} |f(\zeta)| \le M$ (Theorem 24). Applying Theorem 20 to the circle of radius $r - |z|$ centered at z yields

$$\left| f^{(n)}(z) \right| \le \frac{n! M}{(r - |z|)^n}.$$

9. (Section 4.6 Q4) Applying Theorem 20,

$$\left| p^{(k)}(0) \right| = |k! a_k| \le \frac{k! M}{1}.$$

   Hence $|a_k| \le M$.

10. (Section 4.6 Q6) By Liouville's theorem, $f^{(5)}$ is a constant function. One can anti-differentiate five times to obtain f equal to a polynomial of degree at most five. (Theorem 6, Sec. 2.4 guarantees that each antiderivative is unique up to a constant of integration.)

11. (Section 4.6 Q11) Suppose $f(z_0) = 0$ for some z_0 inside C. Let $g(z) = f(z) - 1$. Then $|g(z_0)| = 1$. By Theorem 23, $g(z) \equiv -1$. This contradicts $|g(z)| < 1$ for all z on C. Thus f can have no zeros inside C.

12. (Section 5.1 Q2(b))

$$\lim_{k \to \infty} \left| \frac{(3+i)^{k+1}}{(k+1)!} \frac{k!}{(3+i)^k} \right| = \lim_{k \to \infty} \left| \frac{3+i}{k+1} \right| = 0.$$

13. (Section 5.1 Q2(d))

$$\lim_{k \to \infty} \left| \frac{(k+1)!}{(k+1)^{k+1}} \frac{k^k}{k!} \right| = \lim_{k \to \infty} \left(\frac{k}{k+1} \right)^k = \frac{1}{e}.$$

14. (Section 5.1 Q7(d)) By ratio test, it diverges because

$$\lim_{j \to \infty} \left| \frac{(j+1)!}{5^{j+1}} \frac{5^j}{j!} \right| = \lim_{j \to \infty} \frac{j}{5} = +\infty.$$

15. (Section 5.1 Q7(e)) By ratio test, it converges because

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+1)^3}{(1+i)^{k+1}}\frac{(1+i)^k}{(-1)^k k^3}\right| = \lim_{k\to\infty}\frac{1}{\sqrt{2}}\left(\frac{k+1}{k}\right)^3 = \frac{1}{\sqrt{2}} < 1.$$

16. (Section 5.1 Q11(a))

$$\lim_{j\to\infty}\left|\frac{(j+1)z^{j+1}}{jz^j}\right| = |z|.$$

The series converges when $|z| < 1$.

17. (Section 5.1 Q11(b))

$$\lim_{k\to\infty}\left|\frac{(z-i)^{k+1}}{2^{k+1}}\frac{2^k}{(z-i)^k}\right| = \left|\frac{z-i}{2}\right|.$$

The series converges when $|z-i| < 2$.

18. (Section 5.1 Q12) Choose $\varepsilon > 0$ and suppose $|z| \le R$. For $n > \dfrac{R+3}{\varepsilon}$,

$$|F_n(z) - z| = \left|\frac{nz}{n+1} + \frac{3}{n} - z\right| = \left|\frac{-z}{n+1} + \frac{3}{n}\right| \le \frac{|z|}{n} + \frac{3}{n} \le \frac{R+3}{n} < \varepsilon.$$

Thus $F_n(z)$ converges uniformly to $F(z) = z$ on $|z| \le R$.

19. (Section 5.2 Q1(e))

$$\frac{d^j}{dz^j}\operatorname{Log}(1-z) = \frac{-(j-1)!}{(1-z)^j}$$

for $j \ge 1$, so

$$\frac{f^{(j)}(0)}{j!} = \frac{-1}{j}$$

for $j \ge 1$.

20. (Section 5.2 Q5(d)) For all z,

$$\sum_{k=0}^{\infty}\frac{2(-1)^k z^{2k}}{(2k)!} - \sum_{j=0}^{\infty} iz^j j! = \sum_{j=0}^{\infty}\frac{i^j + (-i)^j - i}{j!}z^j.$$

21. (Section 5.2 Q5(g)) For all $|z| < 1$,

$$\frac{z}{(1-z)^2} = z\frac{d}{dz}\left(\frac{1}{1-z}\right) = z\sum_{j=0}^{\infty}\frac{d}{dz}\left(z^j\right) = \sum_{j=1}^{\infty} jz^j.$$

22. (Section 5.2 Q8(c))

$$\begin{aligned} e^{-iz} &= \sum_{j=0}^{\infty}\frac{(-iz)^j}{z!} \\ &= \sum_{k=0}^{\infty}\frac{(-i)^{2i}z^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(-i)^{2k+1}z^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!} - i\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!} \\ &= \cos z - i\sin z. \end{aligned}$$

23. (Section 5.2 Q11(a))

$$\begin{aligned} e^z\cos z &= \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots\right) \\ &= 1 + z - \frac{1}{3}z^3 + \cdots \end{aligned}$$

24. (Section 5.2 Q11(b))

$$\begin{aligned} -e^z\left(\frac{1}{1-z}\right) &= \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)\left(1 + z + z^2 + z^3 + \cdots\right) \\ &= -1 - 2z - \frac{5}{2}z^2 + \cdots \end{aligned}$$

25. (Section 5.2 Q12) Suppose there is another such polynomial q_n. Then $F = p_n - q_n$ is a polynomial of degree at most n for which $F^{(j)}(z_0) = p_n^{(j)}(z_0) - q_n^{(j)}(z_0) = 0$ as long as $0 \leq j \leq n$. Examining the Taylor series for F reveals that $F \equiv 0$ and $p_n \equiv q_n$.