Solutions to MATH 300 Homework 4

EXERCISES 4.1

3. By observing the equation of the ellipse, we take $x(t) = a\cos t, y(t) = b\sin t$ with $0 \leqslant t \leqslant 2\pi$. Then $z(t) = x(t) + iy(t) = a\cos t + ib\sin t, 0 \leqslant t \leqslant 2\pi$. Moreover, $z'(t) = -a\sin t + ib\cos t$ is continuous and never vanishes on $t \in [0, 2\pi]$. Also, $z(0) = z(2\pi) = a, z'(0) = z'(2\pi) = ib$. Hence, $z(t) = x(t) + iy(t) = a\cos t + ib\sin t$ with $0 \leqslant t \leqslant 2\pi$ gives an admissible parametrization of the ellipse.

9. The barbell-shaped contour has three components of different shapes, say, a circle γ_1 centered at -2 and of radius 1 traversed once clockwise, a line segment γ_2 of $x-$axis and another circle γ_3 centered at 2 and of radius 1 traversed once counterclockwise. Γ_1 starts from -1 while Γ_3 starts from 1. In **Example 1**, we have already seen the admissible parametrizations for circles and straight lines. So the following functions are admissible parametrizations for γ_1, γ_2, and γ_3, consistent with their directions:

$$\begin{cases} \gamma_1 : z_1(t) = -2 + e^{-2\pi it} & (0 \leqslant t \leqslant 1), \\ \gamma_2 : z_2(t) = -1 + t(1 - (-1)) & (0 \leqslant t \leqslant 1), \\ \gamma_3 : z_3(t) = 2 - e^{2\pi it} & (0 \leqslant t \leqslant 1). \end{cases}$$

Now we rescale so that γ_1 is traced for $0 \leqslant t \leqslant \frac{1}{3}$, γ_2 is traced for $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$, and γ_3 is traced for $\frac{2}{3} \leqslant t \leqslant 1$. Then for γ_1, the range of $z_1(t) = -2 + e^{-2\pi it}, 0 \leqslant t \leqslant 1$ is the same as that of $z_I(t) = -2 + e^{-6\pi it}, 0 \leqslant t \leqslant \frac{1}{3}$. For γ_2, the range of $z_2(t) = -1 + t(1 - (-1)), 0 \leqslant t \leqslant 1$ is the same as that of $z_{II}(t) = -1 + 3(t - \frac{1}{3})(1 - (-1)) = -1 + 6(t - \frac{1}{3}), \frac{1}{3} \leqslant t \leqslant \frac{2}{3}$. Also, for γ_3, the range of $z_3(t) = 2 - e^{2\pi it}, 0 \leqslant t \leqslant 1$ is the same as that of $z_{III}(t) = 2 - e^{6\pi i(t - \frac{2}{3})}, \frac{2}{3} \leqslant t \leqslant 1$. Therefore, we have

$$z(t) = \begin{cases} -2 + e^{-6\pi it} & 0 \leqslant t \leqslant \frac{1}{3}, \\ -1 + 6(t - \frac{1}{3}) & \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ 2 - e^{6\pi i(t - \frac{2}{3})} & \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

11. The function $z = z(t) = 5e^{3it}$ parametrizes a circle centered at the origin and of radius 5. The circle is traversed one and a half times counterclockwise. The starting point returns to itself, which is $(5,0)$ after time $t = 2\pi/3$. And then it starts moving around again for time $t = \pi/3$, which is a half of the period $2\pi/3$. So the length of the contour is one and a half of its perimeter, i.e., $l(\Gamma) = \frac{3}{2} \cdot 2\pi \cdot 5 = 15\pi$. Or directly compute the integral expression of the length $\ell(\Gamma) = \int_0^\pi |15ie^{3it}|\,dt = \int_0^\pi 15\,dt = 15\pi$.

EXERCISES 4.2

3.(b)

$$
\begin{aligned}
&\int_0^2 \frac{t}{(t^2+i)^2}\,dt \\
&= \int_0^2 \frac{1}{(t^2+i)^2}\,d(t^2+i) \\
&= -\frac{1}{2}\frac{1}{(t^2+i)^2}\Big|_0^2 \\
&= \frac{-2-8i}{17}.
\end{aligned}
$$

5. From **Example 2** we know

$$
\int_{C_r} (z-z_0)^n dz = \begin{cases} 0 & \text{for } n \neq -1, \\ 2\pi i & \text{for } n = -1, \end{cases}
$$

where $C_r = \{z \in \mathbb{C} : |z - z_0| = r\}$. Then for $C = \{z \in \mathbb{C} : |z - i| = 4\}$ traversed once counterclockwise,

$$
\begin{aligned}
&\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2\right] dz \\
&= 6\int_C \frac{1}{(z-i)^2}\,dz + 2\int_C \frac{1}{z-i}\,dz + \int_C 1dz - 3\int_C (z-i)^2\,dz \\
&= 0 + 2 \cdot 2\pi i + 0 - 0 \\
&= 4\pi i.
\end{aligned}
$$

10. According to **Definition 4** we have

$$\int_C \bar{z}^2\,dz = \int_{C_1} \bar{z}^2\,dz + \int_{C_2} \bar{z}^2\,dz + \int_{C_3} \bar{z}^2\,dz + \int_{C_4} \bar{z}^2\,dz,$$

where $C_i, i = 1, 2, 3, 4$ can be parametrized as

$$\begin{aligned}
&C_1 : z_1(t) = t && (0 \leqslant t \leqslant 1),\\
&C_2 : z_2(t) = 1 + it && (0 \leqslant t \leqslant 1),\\
&C_3 : z_3(t) = i + 1 - t && (0 \leqslant t \leqslant 1),\\
&C_4 : z_4(t) = i(1 - t) && (0 \leqslant t \leqslant 1).
\end{aligned}$$

By **Theorem 4** we have

$$\begin{aligned}
\int_{C_1} \bar{z}^2\,dz &= \int_0^1 \overline{z_1(t)}^2 z_1'(t)\,dt\\
&= \int_0^1 t^2\,dt = \frac{t^3}{3}\bigg|_0^1 = \frac{1}{3},
\end{aligned}$$

$$\begin{aligned}
\int_{C_2} \bar{z}^2\,dz &= \int_0^1 \overline{z_2(t)}^2 z_2'(t)\,dt\\
&= \int_0^1 (1 - it)^2 i\,dt\\
&= it - \frac{it^3}{3} + t^2\bigg|_0^1 = 1 + \frac{2i}{3},
\end{aligned}$$

$$\begin{aligned}
\int_{C_3} \bar{z}^2\,dz &= \int_0^1 \overline{z_3(t)}^2 z_3'(t)\,dt\\
&= \int_0^1 -(1 - t - i)^2\,dt\\
&= -\frac{t^3}{3} + t^2 - it^2 + 2it\bigg|_0^1 = \frac{2}{3} + i,
\end{aligned}$$

$$\begin{aligned}
\int_{C_4} \bar{z}^2\,dz &= \int_0^1 \overline{z_4(t)}^2 z_4'(t)\,dt\\
&= \int_0^1 -(t - 1)^2(-i)\,dt\\
&= i\frac{t^3}{3} - it^2 + it\bigg|_0^1 = \frac{i}{3},
\end{aligned}$$

Therefore

$$\int_C \bar{z}^2\, dz = \frac{1}{3} + 1 + \frac{2i}{3} + \frac{2}{3} + i + \frac{i}{3} = 2 + 2i.$$

12. Since on the unit circle $|z| = 1$, $z\bar{z} = 1$, it follows that $\bar{z} = \dfrac{1}{z}$. Therefore, the two integrals are equal.

14. **(c)** On the arc of the unit circle lying in the first quadrant,

$$|\mathrm{Log}\, z| \leqslant |\mathrm{Log}\, 1 + i\mathrm{Arg}\, z| \leqslant \pi/2.$$

On the other hand, the length of the arc is $\frac{1}{4} \cdot 2\pi \cdot 1 = \pi/2$. Hence, by **Theorem 5**,

$$\left| \int_\Gamma \mathrm{Log}\, z\, dz \right| \leqslant \frac{\pi^2}{4}.$$

EXERCISE 4.3

1. **(b)** Since the integrand has the antiderivative $F(z) = e^z$ for all z. By **Theorem 6**, we have

$$\int_\Gamma e^z\, dz = e^z \Big|_1^{-1} = e^{-1} - e = -2\sinh 1.$$

(g) The principal branch of z^α is the same as that of $\mathrm{Log}\, z$, $\mathbb{C} \setminus (-\infty, 0]$. By the discussion on p.133, we know that z^α has derivative $\alpha z^\alpha \frac{1}{z}$ on each branch, including the principal branch. Thus, using **Theorem 6**, we can directly compute the integral.

$$\int_\Gamma z^{1/2}\, dz = \frac{2}{3} z^{\frac{3}{2}} \Big|_\pi^i = \frac{2}{3} e^{\frac{3\pi i}{4}} - \frac{2}{3}\pi^{\frac{3}{2}} = -\frac{\sqrt{2}}{3} - \frac{2}{3}(\sqrt{\pi})^3 + \frac{i\sqrt{2}}{3}.$$

(h) Since $(\mathrm{Log}\, z)^2$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$, we can directly compute the integral.

$$\int_\Gamma (\mathrm{Log}\, z)^2\, dz = z(\mathrm{Log}\, z)^2 - 2z\mathrm{Log}\, z + 2z \Big|_1^i = i(\mathrm{Log}\, i)^2 - 2i\,\mathrm{Log}\, i + 2i - 2 = \pi - 2 + i(2 - \frac{\pi^2}{4}).$$

2. Let the polynomial be of the form $P(z) = \sum_{i=0}^n a_i z^i$. Then $P(z)$ is continuous and has an antiderivative throughout $\mathbb{C}$ containing Γ. That is,

$$\mathcal{P}(z) = \sum_{i=0}^n \frac{a_i}{i+1} z^{i+1}.$$

Then by **Corollary 2**

$$\int_\Gamma P(z)\,dz = 0.$$

4. False. Let Γ be the unit circle $|z| = 1$ and $1/z$ the function. We know that $1/z$ is analytic at each point of the unit circle, but the integral

$$\oint_{|z|=1} \frac{1}{z}\,dz = 2\pi i.$$

5. By **Theorem 7**, $f(z) = 1/z$ has antiderivative in $\mathbb{C} \setminus \{0\}$ if and only if $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$, where Γ_1 and Γ_2 are any two contours in $\mathbb{C} \setminus \{0\}$ sharing the same initial and terminal points. But **Example 2** shows that for the two contours the integrals are not equal.

12. Let the contour Γ be the line segment from z_1 to z_2. Then by **Theorem 5** in **Sec. 4.2**,

$$|f(z_1) - f(z_2)| = \left| \int_\Gamma f'(z)\,dz \right| \leqslant M|z_2 - z_1|.$$

Exercise 4.4

3. **(a)**, **(b)**, **(d)**, **(e)**. For **(a)**, **(b)**, **(d)**, it is easy to observe that they are continuously deformable to Γ in D (by first deformed into a point, then to Γ). For **(e)**, the deformation function will be

$$z(s,t) = 3 + e^{(2-6s)\pi i t},$$

where $0 \leqslant s \leqslant 1$, $0 \leqslant t \leqslant 1$. For **(c)**, it is impossible to continuously deform it to Γ, since any deformation will pass through the unit disk $|z| \leqslant 1$, which is not contained in D.

4. We construct Γ_0 first. The unit circle can be parametrized by

$$z(t) = \begin{cases} e^{4\pi i t} & 0 \leqslant t \leqslant 1/2, \\ e^{4\pi i(1-t)} & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Also, the point can be simply written as $e^{2\pi i(0)}$. Then we have

$$z(s,t) = \begin{cases} e^{4\pi i(1-s)t} & (0 \leqslant t \leqslant 1/2), \\ e^{4\pi i(1-s)(1-t)} & (1/2 \leqslant t \leqslant 1), \end{cases}$$

where $0 \leqslant s \leqslant 1$. It is immediate to verify that

$$\begin{aligned}\int_{\Gamma_0} f(z)\,dz &= \oint_{|z|=1} f(z)\,dz + \oint_{|z|=1} f(z)\,dz \\ &= \oint_{|z|=1} f(z)\,dz - \oint_{|z|=1} f(z)\,dz = 0 = \int_{z=1} f(z)\,dz.\end{aligned}$$

15. Since Γ is the circle $|z| = 4$ traversed twice counterclockwise, $\displaystyle\int_\Gamma \frac{1}{z-a}\,dz = 2\cdot(-2\pi i)$ for $|a| \leqslant 4$. Therefore,

$$\begin{aligned}\int_\Gamma \frac{z}{(z+2)(z-1)}\,dz &= \frac{1}{3}\int_\Gamma \frac{1}{z-1}\,dt + \frac{2}{3}\int_\Gamma \frac{1}{z+2}\,dz \\ &= \frac{1}{3}(-4\pi i) + \frac{2}{3}(-4\pi i) \\ &= -4\pi i.\end{aligned}$$

18. (a) For all $R > 2$, the circle $|z| = 2$ can be continuously deformed into $|z| = R$. Then by **Theorem 8**, for $f(z) := \dfrac{1}{z^2(z-1)^3}$ we have

$$I(R) = \oint_{|z|=R} f(z)\,dz = \oint_{|z|=2} f(z)\,dz = I.$$

(b) By **Theorem 5** in **Sec. 4.2**,

$$\begin{aligned}|I(R)| &= \left|\oint_{|z|=R} \frac{1}{z^2(z-1)^3}\,dz\right| \\ &\leqslant \frac{1}{|z|^2(|z|-1)^3}\,\ell(|z|=R) \\ &= \frac{1}{R^2(R-1)^3}\,2\pi R \\ &= \frac{2\pi}{R(R-1)^3}.\end{aligned}$$

(c) It is evident that

$$\lim_{R\to+\infty} \frac{2\pi}{R(R-1)^3} = 0.$$

Then $\displaystyle\lim_{R\to+\infty} I(R) = 0$.

(d) As $R \to +\infty$, the circle $|z| = 2$ can also be continuously deformed into $|z| = R$. Therefore, $I = \displaystyle\lim_{R\to+\infty} I(R) = 0$.