Math 300 Homework 5 Solution

1. (Section 3.2 Q5d) By definition,

$$\begin{aligned}\cos(1-i) &= \frac{1}{2}(e^{i(1-i)} + e^{-i(1-i)}) \\ &= \frac{1}{2}(e(\cos(1) + i\sin(1)) + e^{-1}(\cos(-1) + i\sin(-1)) \\ &= \frac{1}{2}(e + e^{-1})\cos(1) + \frac{i}{2}(e - e^{-1})\sin(1) \\ &= \cosh(1)\cos(1) + i\sinh(1)\sin(1).\end{aligned}$$

2. (Section 3.2 Q11) As e^z is never zero, the function $\frac{\cos z}{e^z}$ is entire. Thus the real part of this function is harmonic in the whole plane.

3. (Section 3.2 Q18)

   (a)
   $$\lim_{z\to 0}\frac{\sin z}{z} = \lim_{z\to 0}\frac{\sin z - \sin 0}{z} = \cos z|_{z=0} = 1$$

   (b)
   $$\lim_{z\to 0}\frac{\cos z - 1}{z} = \lim_{z\to 0}\frac{\cos z - \cos 0}{z} = -\sin z|_{z=0} = 0$$

4. (Section 3.2 Q19) Let z_1, z_2 lie in an open disk of radius π. Then,
   $$e^{z_1} = e^{z_2} \implies e^{z_1 - z_2} = 1 \implies z_1 - z_2 = 2k\pi i$$
   for some integer k. But $|z_1 - z_2| < 2\pi \implies k = 0$. Hence, $z_1 = z_2$.

5. (Section 3.2 Q21)

   (a) If $\sin(z_2) = \sin(z_1)$, then $2\cos(\frac{z_1+z_2}{2})\sin(\frac{z_1-z_2}{2}) = 0$. So either $z_2 = -z_1 + (2k+1)\pi$ or $z_2 = z_1 + 2k\pi$ where k is an integer.
   If $z_2 = -z_1 + (2k+1)\pi$, then $y_2 = -y_1$. This implies that z_1 and z_2 cannot be both in the semi-infinite strip.
   If $z_2 = z_1 + 2k\pi$, both z_1 and z_2 are in the semi-infinite strip only when $k = 0$. So $z_2 = z_1$ and the mapping is one-to one.
   Note that $w = \sin z = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$. The image of this strip is $\mathbb{C}\backslash\{(x,y)|x = 0, y < 0\}\backslash\{(x,y)| -1 \leq x \leq 1, y = 0\}$

(b) The image is $\{(x,y)|y>0\}$, i.e. the upper half plane.

6. (Section 3.3 Q1(d))

$$\begin{aligned}\mathrm{Log}\,(\sqrt{3}+i) &= \mathrm{Log}\,|\sqrt{3}+i| + i\mathrm{Arg}\,(\sqrt{3}+i)\\ &= \mathrm{Log}\,2 + i\frac{\pi}{6}\end{aligned}$$

7. (Section 3.3 Q5(b))

$$\begin{aligned}\mathrm{Log}\,(z^2-1) &= \frac{i\pi}{2}\\ z^2-1 &= i\\ z^2 &= 1+i = \sqrt{2}e^{i\pi/4}\\ z &= 2^{1/4}e^{i\pi/8} \text{ or } 2^{1/4}e^{9i\pi/8}\end{aligned}$$

8. (Section 3.3 Q8) $\mathrm{Log}\,|z|$ is harmonic on $\mathbb{C}\backslash\{x\le 0, y=0\}$ since it is the real part of $\mathrm{Log}\,z$. Similarly, $\mathrm{Log}\,|z|$ is harmonic on $\mathbb{C}\backslash\{x\ge 0, y=0\}$ because it is the real part of $\mathcal{L}_0(z)$. Putting these together, $\mathrm{Log}\,|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$.

9. (Section 3.3 Q9) Domain of analyticity is $\mathbb{C}\backslash\{x\ge 4, y=1\}$.
By chain rule, $f'(z) = \frac{-1}{4+i-z}$.

10. (Section 3.3 Q11) Choose the principal branch.

$$\left.\frac{d}{dz}\mathrm{Log}\,(z^2+2z+3)\right|_{z=-1} = \left.\frac{2z+2}{z^2+2z+3}\right|_{z=-1} = 0$$

11. (Section 3.3 Q15) $w = \frac{1}{\pi}\mathrm{Log}\,z$.

12. (Section 3.5 Q4) No. $1^\alpha = e^{\alpha\log 1} = e^{2\alpha k\pi i}$, where $k = 0, \pm 1, \ldots$
For example, take $k=1, \alpha=\frac{1}{2}$. Then, $1^\alpha = e^{\pi i} = -1$.

13. (Section 3.5 Q7) $(1+i)i^i = (1+i)e^{-\pi/2}$

14. (Section 3.5 Q15)

(b) $z\exp\left[\frac{1}{2}\mathrm{Log}\,(1+\frac{4}{z^2})\right]$

(d) $z \exp\left[\frac{1}{3}\text{Log}\,(1 - \frac{1}{z^3})\right]$

15. (Section 3.5 Q19) $w = 2e^z + e^{2z}$ if and only if $e^{2z} + 2e^z - w = 0$.
$e^z = -1 \pm \sqrt{1+w}$
$z = \log(-1 \pm \sqrt{1+w})$
When $w = 3$, $z = i2\pi k$ or $z = \text{Log}\,(3) + i\pi(2k+1)$.