Solutions to MATH 300 Homework 4

EXERCISES 2.4

4. Let $u(x,y) = \dfrac{x^{\frac{4}{3}}y^{\frac{5}{3}}}{x^2+y^2}, v(x,y) = \dfrac{x^{\frac{5}{3}}y^{\frac{4}{3}}}{x^2+y^2}$. Then

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,y=0} = \lim_{\Delta x\to 0}\frac{u(\Delta x,0)-u(0,0)}{\Delta x} = \lim_{\Delta x\to}\frac{0}{\Delta x} = 0$$

and

$$\left.\frac{\partial u}{\partial y}\right|_{x=0,y=0} = \lim_{\Delta y\to 0}\frac{u(0,\Delta y)-v(0,0)}{\Delta y} = \lim_{\Delta y\to}\frac{0}{\Delta y} = 0.$$

Similarly $\left.\dfrac{\partial v}{\partial x}\right|_{x=0,y=0} = 0$ and $\left.\dfrac{\partial v}{\partial y}\right|_{x=0,y=0} = 0$. Hence the Cauchy-Riemann equations holds at $z=0$. However, when $\Delta z \to 0$ through the real values ($\Delta z = \Delta x$),

$$\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x} = 0,$$

while along the real line $y = x$ ($\Delta z = \Delta x + i\Delta x$)

$$\lim_{\Delta x\to 0}\frac{f(0+\Delta z)-f(0)}{\Delta z} = \lim_{\Delta x\to 0}\frac{\frac{(\Delta x)^{4/3}(\Delta x)^{5/3}+(\Delta x)^{5/3}(\Delta x)^{4/3}}{2(\Delta x)^2}}{\Delta x(1+i)} = \frac{1}{2}.$$

Therefore f is not differentiable at $z=0$.

5. Let $u(x,y) = e^{x^2-y^2}\cos(2xy), v(x,y) = e^{x^2-y^2}\sin(2xy)$. Then

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)] = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}[y\cos(2xy) - x\sin(2xy)] = -\frac{\partial v}{\partial x}.$$

f is entire since these first partials exist and are continuous for all x and y. Since

$$f(z) = e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)] = e^{(x+iy)^2} = e^{z^2},$$

it follows that $f'(z) = 2ze^{z^2}$.

8. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Suppose that $u(x,y) = c$ in D, where c is a constant. Then $\frac{\partial u}{\partial x} = 0$ and by Cauchy-Riemann $-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$. It follows that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0.$$

Therefore, f is constant in D.
If we suppose that $\mathrm{Im}(f) = c$, then by a similar argument we can prove the result.

11. If both f and $\bar{f}$ are analytic in D, then $g := \mathrm{Re}(f) = \frac{1}{2}(f + \bar{f})$ is analytic and real-valued. That is, $\mathrm{Im}(g) = 0$. Hence it follows from Exercise 8 that $g = \mathrm{Re}(f)$ is constant in D. So f is constant by Exercise 8 again.

EXERCISES 2.5

3. **(b)** Since $\frac{\partial^2 u}{\partial x^2} = e^x \sin y$, and $\frac{\partial^2 u}{\partial y^2} = \frac{\partial(e^x \cos y)}{\partial y} = -e^x \sin y$, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

u is harmonic. Next, we find the harmonic conjugate of u, denoted by v, which satisfies the Cauchy-Riemann equations. It follows from

$$\frac{\partial u}{\partial x} = e^x \sin y = \frac{\partial v}{\partial y}$$

that $v(x,y) = -e^x \cos y + \phi(x)$, where $\phi(x)$ is a differentiable real-valued function of x. Also, from

$$\frac{\partial u}{\partial y} = e^x \cos y = -\frac{\partial v}{\partial x}$$

we have $e^x \cos y = e^x \cos y - \phi'(x)$, i.e., $\phi(x)$ is constant. Therefore, $v(x,y) = -e^x \cos y + c$, $c \in \mathbb{C}$, is the harmonic conjugate of u.
Or: Since $u(x,y) = e^x \sin y$ can be written as the real part of the function

$-ie^{x+iy} = -ie^z$, which is entire, it follows that the u is harmonic and a harmonic conjugate of u is $\text{Im}(-ie^z + c) = -e^x \cos y + c, c \in \mathbb{C}$.

(c) Since $u(x, y) = xy - x + y$ can be written as $\text{Re}(-\frac{i}{2}z^2 - iz - z)$, which is entire, it follows that the u is harmonic and a harmonic conjugate of u is $\text{Im}(-\frac{i}{2}z^2 - iz - z + c) = -\frac{1}{2}(x^2 - y^2) - x - y + c, c \in \mathbb{C}$.

5. If $f(x + iy) = u(x, y) + iv(x, y)$ is analytic, then $-if(x + iy) = v(x, y) - iu(x, y)$ is analytic. Thus $-u$ is a harmonic conjugate of v.

7. Take $z = x + iy$. Since the region is bounded by $x = -1$ and $x = 3$, naïvely, we consider the analytic function $f(z) = z$, whose real part is x. But it does not satisfy the boundary conditions. Hence we take $af(z) + b$ with $a, b \in \mathbb{R}$, which is also analytic. Then $\text{Re}(af + b) = ax + b$ is harmonic. Using the boundary values we have

$$\begin{cases} a \cdot (-1) + b &= 0 \\ a \cdot (3) + b &= 4 \end{cases},$$

which gives $a = b = 1$. Then $\phi(x) = x + 1$.

8. **(a)** Yes, because $\nabla^2(u + v) = \nabla^2 u + \nabla^2 v = 0$ if both u and v are harmonic.

(b) No. Set $\frac{\partial f}{\partial x} = f_x$. Then $\nabla^2(uv) = v(u_{xx} + u_{yy}) + u(v_{xx} + u_{yy}) + 2(u_x v_x + u_y v_y) = 2(u_x v_x + u_y v_y)$. So unless $u_x v_x + u_y v_y = 0$, uv is not harmonic. Take $u = x$, $v = xy$ as an example. Both of them are harmonic in $\mathbb{R}^2$. But $\nabla^2(uv) = 2y$, which means that it is harmonic only on the line $y = 0$.

(c) Yes, because $\Delta(u_x) = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx} = \frac{\partial}{\partial x}(\Delta u) = \frac{\partial}{\partial x}(0) = 0$.

15. Take $z = re^{i\theta}$. Denote the annulus $\{z \in \mathbb{C} \big| 1 \leqslant |z| \leqslant 2\}$ by $\mathcal{A}$. We consider on $\mathcal{A}$ the analytic function $f(z) = az^n + bz^{-n} + c$ with $a, b, c \in \mathbb{R}, n \in \mathbb{N}$, whose real part is $ar^n \cos n\theta + br^{-n} \cos(-n\theta) + c$. Set $n = 3$ to agree with the cosine argument on $|z| = 2$.

$$\begin{aligned} \phi(re^{i\theta}) &= ar^3 \cos 3\theta + br^{-3} \cos(-3\theta) + c. \\ &= (ar^3 + br^{-3}) \cos 3\theta + c \end{aligned}$$

If $r = 1$, then $\phi(e^{i\theta}) = 0 \Rightarrow a + b = 0, c = 0$. If $r = 2$, then $\phi(2e^{i\theta}) = 5\cos 3\theta \Rightarrow (8a + b/8)\cos 3\theta = 5\cos 3\theta$. So $a = 40/64, b = -40/63$ and

$$\begin{aligned}\phi(re^{i\theta}) &= \frac{40}{63}(r^3 - r^{-3})\cos 3\theta,\\ \phi(z) &= \frac{40}{63}\mathrm{Re}(z^3 - z^{-3}).\end{aligned}$$

EXERCISE 3.1

2. **(a)** Since $\deg(fg) = \deg(f) + \deg(g)$, it follows that

$$n = \deg(p(z)) = \sum_{i=1}^{r} \deg((z - z_i)^{d_i}) = \sum_{i=1}^{r} d_i.$$

(b) a_{n-1} is the coefficient corresponding to z^{n-1}. Observe from the expression

$$p(z) = a_n \underbrace{(z - z_1)\cdots(z - z_1)}_{d_1 \text{ copies}}\underbrace{(z - z_2)\cdots(z - z_2)}_{d_2 \text{ copies}}\cdots\underbrace{(z - z_r)\cdots(z - z_r)}_{d_r \text{ copies}}$$

that we need to choose $-z_i$ from one of the factors once and z from the rest of the factors $n - 1$ times, in order to give a term of degree $n - 1$. Repeat the process for $i = 1, 2, \ldots, r$. Then since there are d_i copies of $(z - z_i)$, we have d_i $(-z_i)$'s. Therefore,

$$a_{n-1} = a_n((-z_1)d_1 + (-z_2)d_2 + \cdots + (-z_r)d_r) = -a_n(d_1 z_1 + d_2 z_2 + \cdots + d_r z_r).$$

(c) By a similar consideration, a_0 is the constant term. Hence we need to choose $-z_i$ from each of the factors d_i times and z without once. Therefore,

$$\begin{aligned}a_0 &= a_n(-z_1)^{d_1}(-z_2)^{d_2}\cdots(-z_r)^{d_r} = a_n(-1)^{d_1+d_2+\cdots+d_r} z_1^{d_1} z_2^{d_2}\cdots z_r^{d_r}\\ &= a_n(-1)^n z_1^{d_1} z_2^{d_2}\cdots z_r^{d_r}.\end{aligned}$$

3. **(b)** $z^4 - 16 = (z^2 + 4)(z^2 - 4) = (z + 2i)(z - 2i)(z + 2)(z - 2)$.
(c)

$$\begin{aligned}1 + z + z^2 + z^3 + z^4 + z^5 + z^6 &= \frac{z^7 - 1}{z - 1}\\ &= (z - \zeta_7)(z - \zeta_7^2)\cdots(z - \zeta_7^6),\end{aligned}$$

where $\zeta_7 = e^{2\pi i/7}$.

5. **(c)** $(z-1)(z-2)^3 = ((z-2)+1)(z-2)^3 = (z-2)^4 + (z-2)^3$.

13. **(b)**

$$\begin{aligned}\frac{2z+i}{z^3+z} &= \frac{2z+i}{z(z+i)(z-i)} \\ &= \frac{A_0^{(1)}}{z} + \frac{A_0^{(2)}}{z+i} + \frac{A_0^{(3)}}{z-i} \\ &= \frac{i}{z} + \frac{i/2}{z+i} - \frac{3i/2}{z-i},\end{aligned}$$

since by using (21), we have $A_0^{(1)} = \lim_{z\to 0} z \cdot \frac{2z+i}{z^3+z} = i$, $A_0^{(2)} = \lim_{z\to -i}(z+i)\cdot \frac{2z+i}{z^3+z} = \frac{i}{2}$, $A_0^{(3)} = \lim_{z\to i}(z-i)\cdot\frac{2z+i}{z^3+z} = -\frac{3i}{2}$.

(d)

$$\begin{aligned}\frac{5z^4+3z62+1}{2z^2+3z+1} &= \frac{5}{2}z^2 - \frac{15}{4}z + \frac{47}{8} + \frac{-\frac{111}{8}z - \frac{39}{8}}{(z+1)(2z+1)} \\ &= \frac{5}{2}z^2 - \frac{15}{4}z + \frac{47}{8} + \frac{1}{2}\cdot\frac{A_0^{(1)}}{z+\frac{1}{2}} + \frac{A_0^{(2)}}{z+1} \\ &= \frac{5}{2}z^2 - \frac{15}{4}z + \frac{47}{8} + \frac{\frac{33}{16}}{z+\frac{1}{2}} - \frac{9}{z+1},\end{aligned}$$

since by using (21), we have $A_0^{(1)} = \lim_{z\to -\frac{1}{2}}(z+\frac{1}{2})\cdot\frac{-\frac{111}{8}z-\frac{39}{8}}{(z+1)(2z+1)} = \frac{33}{16}$, $A_0^{(2)} = \lim_{z\to -1}(z+1)\cdot\frac{-\frac{111}{8}z-\frac{39}{8}}{(z+1)(2z+1)} = -9$.