Homework 9 - Math 321, Spring 2012

Due on Friday March 23

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1. Let α be continuous and increasing. Given $f \in \mathcal{R}_\alpha[a,b]$ and $\epsilon > 0$, show that there exist
   (a) a step function h on $[a,b]$ with $||h||_\infty \leq ||f||_\infty$ such that $\int_a^b |f-h|\,d\alpha < \epsilon$, and
   (b) a continuous function g on $[a,b]$ with $||g||_\infty \leq ||f||_\infty$ such that $\int_a^b |f-g|\,d\alpha < \epsilon$.
2. Give an example of a sequence of Riemann integrable functions on $[0,1]$ that converges pointwise to a nonintegrable function.
3. Recall the definition of a Riemann-Stieltjes intgral for a general (not necessarily increasing) integrator α. Show that this definition coincides with our previous definition (in terms of upper and lower sums) if α is increasing.
4. (a) Construct a nonconstant increasing function α and a nonzero continuous function $f \in \mathcal{R}_\alpha[a,b]$ such that $\int_a^b |f|\,d\alpha = 0$. Is it possible to choose α to also be continuous? Explain.
   (b) If f is continuous on $[a,b]$ and if $f(x_0) \neq 0$ for some x_0, show that $\int_a^b |f(x)|\,dx \neq 0$. Conclude that $||f|| = \int_a^b |f(x)|\,dx$ defines a norm on $C[a,b]$. Does it define a norm on all of $\mathcal{R}[a.b]$? Explain.