Homework 5 - Math 321, Spring 2012

Due on Friday February 9

---

1. Show that a subset of a metric space is compact if and only if it is complete and totally bounded.
2. Let X be a compact metric space and let $\{f_n : n \geq 1\}$ be an equicontinuous sequence in $C(X)$. If $\{f_n\}$ is pointwise convergent, prove that in fact $\{f_n : n \geq 1\}$ is uniformly convergent. (In other words, pointwise convergence + equicontinuity $\implies$ uniform convergence).
3. Recall the vector space $C^1[a,b]$ of all functions $f : [a,b] \to \mathbb{R}$ having a continuous first derivative on $[a,b]$. The space $C^1[a,b]$ is complete under the norm

$$||f||_{C^1} = \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} |f'(x)| .$$

   (You should check this but need not submit a proof of it). Show that a bounded subset of $C^1[a,b]$ is equicontinuous.
4. Let $K(x,t)$ be a continuous function on the square $[a,b] \times [a,b]$.
   (a) Show that the mapping T defined by

$$Tf(x) = \int_a^b f(t)K(x,t)\,dt$$

   maps $C[a,b]$ to $C[a,b]$, and in fact maps bounded sets to equicontinuous sets. Use this to conclude that T is continuous.
   (b) Let $\{f_n : n \geq 1\}$ be a sequence in $C[a,b]$ with $||f_n||_\infty \leq 1$ for all n. Define

$$F_n(x) = \int_a^x f_n(t)\,dt.$$

   Show that some subsequence of $\{F_n : n \geq 1\}$ is uniformly convergent.