Homework 4 - Math 321, Spring 2012

Due on Friday February 2

---

1. Let p_n be a polynomial of degree m_n, and suppose that $p_n \to f$ uniformly on $[a, b]$, where f is not a polynomial. Show that $m_n \to \infty$.
2. Prove that there is a sequence of polynomials p_n such that $p_n \to 0$ pointwise on $[0, 1]$ but $\int_0^1 p_n(x)\,dx \to 3$.
3. Follow the steps outlined below to prove Weierstrass's second theorem from the first.
   (a) Given an *even* function $F \in \mathcal{C}^{2\pi}$ and $\epsilon > 0$, show that there is an *even* trig polynomial T such that $||F - T||_\infty < \epsilon$. (Hint: Approximate the continuous function $g(y) = F(\arccos y)$ on $[-1, 1]$ by a polynomial.)
   (b) Fix an arbitrary function $f \in \mathcal{C}^{2\pi}$ (to be approximated by a trig polynomial). Applying the result in part (a) to the functions
   $$f(x) + f(-x) \qquad \text{and} \qquad [f(x) - f(-x)] \sin x,$$
   deduce that the function $f(x) \sin^2 x$ is well-approximable by trig polynomials.
   (c) Prove that the function $f(x) \cos^2 x$ is well-approximable by trig polynomials as well.
   (d) Combine parts (b) and (c) to conclude the proof of Weierstrass's second approximation theorem.
4. Now assuming Weierstrass's second theorem, fill in these steps to show it implies the first.
   (a) Given a trig polynomial $T(x)$ of degree n, show that there is an algebraic polynomial $p(t, s)$ of degree exactly n (in two variables) such that $T(x) = p(\cos x, \sin x)$. In fact if T is an even function, show that there is an algebraic polynomial $p(t)$ of degree exactly n such that $T(x) = p(\cos x)$.
   (b) Given $f \in C[-1, 1]$, and $\epsilon > 0$, show that there is a trig polynomial T such that
   $$|f(\cos x) - T(x)| < \epsilon \qquad \text{for all } x \in \mathbb{R}.$$
   (c) Deduce that the *even* trig polynomial
   $$g(x) = \frac{T(x) + T(-x)}{2}$$
   satisfies the same estimate
   $$|f(\cos x) - g(x)| < \epsilon \qquad \text{for all } x \in \mathbb{R}.$$
   (d) Combine parts (a) and (c) to complete the proof of Weierstrass's first theorem.