Homework 2 - Math 321, Spring 2012

Due on Friday January 20

---

1. Consider the metric space $C[a, b]$ equipped with the sup norm metric $|| \cdot ||_\infty$. Is $(C[a, b], || \cdot ||_\infty)$ complete?
2. Define a metric on $C(\mathbb{R})$ by setting

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(f, g)}{1 + d_n(f, g)} \qquad \text{where} \qquad d_n(f, g) = \max_{|t| \leq n} |f(t) - g(t)| \, .$$

   Convince yourself that d is a metric (but you do not have to submit the proof for it).
   (a) Prove that f_n converges to f in the above metric of $C(\mathbb{R})$ if and only if f_n converges uniformly to f on every compact subset of $\mathbb{R}$. For this reason, convergence in $C(\mathbb{R})$ is sometimes called *uniform convergence on compacta.*
   (b) Show that $C(\mathbb{R})$ is complete.
3. (Exercise 8, Chapter 7 of the textbook) Define the function I as follows,

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

   If $\{x_n\}$ is a sequence of distinct points of (a, b) and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad a \leq x \leq b$$

   converges uniformly, and that f is continuous for every $x \neq x_n$.
4. Show that both

$$\sum_{n=1}^{\infty} x^n (1 - x) \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n x^n (1 - x)$$

   are convergent on $[0, 1]$, but only one converges uniformly. Which one? Why?
5. Show that

$$\sum_{n=1}^{\infty} \frac{x^2}{(1 + x^2)^n}$$

   converges for all $|x| \leq 1$, but that the convergence is not uniform.