Math 105 - Practice Midterm 2

1 Compute the following:

a $\frac{d}{dx}\int_x^2 \frac{1}{1+t^3}dt$

Solution: We rewrite this as $-\frac{d}{dx}\int_2^x \frac{1}{1+t^3}dt$ and apply the Fundamental Theorem of Calculus to yield $-\frac{1}{1+x^3}$.

b $\int_{-\infty}^{\infty} xdx$

Solution: This needs to be split into two improper integrals because it ranges over $(-\infty, \infty)$. If the integral converges, $I = \int_{-\infty}^{0} xdx + \int_0^{\infty} xdx$. Both must converge for the integral to converge.

Note that $\int_0^{\infty} xdx = \lim_{R\to\infty}\int_0^R xdx = \lim_{R\to\infty}\frac{x^2}{2}|_0^R = \infty$. Thus I diverges and we don't even need to consider the other integral.

c $\int_0^{\infty}\frac{dx}{x^{1/2}+x^{3/2}}$

Solution: This integral is improper for two reasons: the vertical asymptote at $x = 0$ and the infinite range of integration. We split it into $\int_0^1 \frac{dx}{x^{1/2}+x^{3/2}} + \int_1^{\infty}\frac{dx}{x^{1/2}+x^{3/2}}$. Both must converge for the integral to converge.

Making a substitution $u = x^{1/2}$ with $du = \frac{1}{2x^{1/2}}dx$ so $dx = 2udu$ in both integrals, we have $\int \frac{dx}{x^{1/2}+x^{3/2}} = \int \frac{2}{1+u^2}du = 2\arctan u + C = 2\arctan\sqrt{x} + C$.

Thus, $\int_0^{\infty}\frac{dx}{x^{1/2}+x^{3/2}} = \lim_{t\to 0^+} 2\arctan\sqrt{x}|_t^1 + \lim_{R\to\infty} 2\arctan\sqrt{x}|_1^R = 2(\pi/4 - 0) + 2(\pi/2 - \pi/4) = \pi$.

d $\int \frac{dy}{1+\cos(4y)}$

Solution: We write $1 + \cos(4y) = 2\cos^2(2y)$. Thus, $\int \frac{dy}{1+\cos(4y)} = \int \frac{1}{2}\sec^2(2y) = \frac{1}{4}\tan(2y) + C$.

e $\int (\ln(x^8))^5 dx$

Solution: First of all we pull out the 8 to find $\int (8\ln x)^5 dx = 8^5 \int \ln^5 xdx$. This will involve multiple integration-by-parts. We'll first note that $\int \ln^n xdx = \int \ln^n x1dx$ so that if $f = \ln^n x$ with $df = n(\ln^{n-1} x)\frac{1}{x}$ and $dg = dx$ with $g = x$ then $\int \ln^n xdx = x\ln^n x - n\int \ln^{n-1} xdx$. Life is good now:

$8^5\int \ln^5 xdx = 8^5(x\ln^5 x - 5\int \ln^4 xdx) = 8^5(x\ln^5 x - 5(x\ln^4 x - \int 4\ln^3 xdx)) = 8^5(x\ln^5 x - 5x\ln^4 x + 20(\int \ln^3 xdx)) \underbrace{=}_{\text{guess pattern}} 8^5(x\ln^5 x - 5x\ln^4 x + 5\times 4x\ln^3 x - 5\times 4\times 3x\ln^2 x + 5\times 4\times 3\times 2\ln x - 5\times 4\times 3\times 2\times 1x)$
$= 8^5(x\ln^5 x - 5x\ln^4 x + 20x\ln^3 x - 60x\ln^2 x + 120x\ln x - 120x) + C$.

This result can be verified by taking a derivative.

f $\int(\tan^2 t\sec t - \tan^3 t\sec t)dt$

Solution: We split the integral into two, as they require different strategies.

• $I = \int tan^2 t\sec t dt = \int(\sec^2 t - 1)\sec t dt = \int\sec^3 t - \int\sec t dt$.

To find $\int\sec^3 t dt = \int\sec t\sec^2 t dt$ we integrate by parts setting $f = \sec t$ so $df = \sec t\tan t dt$ and $dg = \sec^2 t$ so $g = \tan t$. Then $\int\sec^3 t dt = \sec t\tan t - \int\tan^2 t\sec t dt = \sec t\tan t - \int(\sec^2 t - 1)\sec t dt$ so that $\int\sec^3 t dt = \sec t\tan t - \int sec^3 t dt + \int\sec t dt$. Rearranging and using $\int\sec t dt = \ln|\sec t + \tan t| + C$ gives us $\int\sec^3 t = \frac{1}{2}(\sec t\tan t + \ln|\sec t + \tan t|) + C$.

Therefore part I is $\frac{1}{2}\ln|\sec t + \tan t| - \frac{1}{2}\sec t\tan t + C$.

• Now we consider $II = \int\tan^3 t\sec t dt = \int\tan^2 t\sec t\tan t dt$ (we pulled out a factor of $\sec t\tan t$, the derivative of $\sec t$). We then use $\tan^2 t = \sec^2 t - 1$ so that $II = \int(\sec^2 t - 1)\sec t\tan t dt$ and by letting $u = \sec t$ we have $\int(u^2 - 1)du = \frac{u^3}{3} - u + C = \frac{\sec^3 t}{3} - \sec t + C$.

• The final result is $\frac{1}{2}\ln|\sec t + \tan t| - \frac{1}{2}\sec t\tan t - \frac{\sec^3 t}{3} + \sec t + C$.

g $\int\frac{dy}{y(y^2-1)}$

Solution: This requires partial fractions. We consider $\frac{1}{y(y+1)(y-1)} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{y-1}$. Multiplying by the common denominator yields: $1 = A(y+1)(y-1) + By(y-1) + Cy(y+1)$.

We can find A, B, and C by selecting convenient values of y. Setting $y = 0$ yields $1 = -A$ so $A = -1$. Setting $y = 1$ yields $1 = 2C$ so $C = 1/2$. And setting $y = -1$ yields $1 = 2B$ so $B = 1/2$.

Then $\int\frac{dy}{y(y^2-1)} = \int(\frac{-1}{y} + \frac{1/2}{y+1} + \frac{1/2}{y-1})dy = -\ln|y| + \frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| + C$.

h $\int_{-2}^{3}\frac{dw}{w}$

Solution: This integral is improper due to the vertical asymptote at $x = 0$. To exist we require both $\int_{-2}^{0}\frac{dw}{w}$ and $\int_{0}^{3}\frac{dw}{w}$ to exist.

Looking just at $\int_{-2}^{0}\frac{dw}{w} = \lim_{t\to 0^-}\int_{-2}^{t}\frac{dw}{w} = \lim_{t\to 0^-}\ln|w||_{-2}^{t} = -\infty$ because ln goes to $-\infty$ as its argument goes to zero.

Thus the entire integral diverges.

i $\int(100 - x^2)^{3/2}dx$

Solution: We use trigonometric substitution. We set $x = 10\sin\theta$ so that $dx = 10\cos\theta d\theta$ and $100 - x^2 = 100\cos^2\theta$. This yields: $\int(1000\cos^3\theta)10\cos\theta d\theta = 10^4\int\cos^4\theta d\theta$.

Here we employ double-angle identities where $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.

We have $10^4\int(\frac{1+\cos(2\theta)}{2})^2 d\theta = \frac{10^4}{4}\int(1 + 2\cos(2\theta) + \cos^2(2\theta))d\theta$.

Trivially, $\int d\theta = \theta + C$ and $\int 2\cos(2\theta)d\theta = \sin(2\theta) + C$.

Then $\int\cos^2(2\theta)d\theta = \int\frac{1+\cos(4\theta)}{2}d\theta = \frac{\theta}{2} + \frac{1}{8}\sin(4\theta) + C$.

The net trig integral result is $\frac{10^4}{4}(\sin(2\theta) + \frac{3\theta}{2} + \frac{1}{8}\sin(4\theta)) + C$.

Using $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin\theta\cos\theta(2\cos^2\theta - 1)$ the final result is:

$\frac{10^4}{4}(2\sin\theta\cos\theta + \frac{3\theta}{2} + \frac{1}{2}\sin\theta\cos\theta(2\cos^2\theta - 1)) + C$.

If $x = 10\sin\theta$ then $\sin\theta = x/10$ and $\cos\theta = \sqrt{1 - x^2/100} = \frac{1}{10}\sqrt{100 - x^2}$ and $\theta = \arcsin(x/10)$.

The integral is $\frac{10^4}{4}(\frac{3}{2}\arcsin(x/10) + 2\frac{x}{10}\frac{\sqrt{100-x^2}}{10} + \frac{1}{2}\frac{x}{10}\frac{\sqrt{100-x^2}}{10}(2\frac{100-x^2}{100} - 1)) + C$ which simplifies to $3750\arcsin(x/10) + \frac{150}{4}x\sqrt{100 - x^2} + \frac{1}{4}x(100 - x^2)^{3/2} + C$.

j The cumulative distribution function and standard deviation of X the number of "heads" in 4 flips of a fair coin.

Solution: There are 16 possible outcomes:
$HHHH, HHHT, HHTH, HHTT,$
$HTHH, HTHT, HTTH, HTTT,$
$THHH, THHT, THTH, THTT,$
$TTHH, TTHT, TTTH, TTTT.$

From 16 outcomes, 1 involves no heads, 4 involve one heads, 6 involve two heads, 4 involve three heads, and 1 involves four heads.

Therefore, $\Pr(X = 0) = 1/16$, $\Pr(X = 1) = 4/16 = 1/4$, $\Pr(X = 2) = 6/16 = 3/8$, $\Pr(X = 3) = 4/16 = 1/4$ and $\Pr(X = 4) = 1/16$.

Therefore $\bar{X} = 0 \times 1/16 + 1 \times 1/4 + 2 \times 3/8 + 3 \times 1/4 + 4 \times 1/16 = 2$.

To find the variance (and standard deviation) we will find $\mathbb{E}(X^2) = 0^2 \times 1/16 + 1^2 \times 1/4 + 2^2 \times 3/8 + 3^2 \times 1/4 + 4^2 \times 1/16 = 80/16$. Therefore, $Var(X) = 80/16 - 2^2 = 16/16 = 1$ and $\sigma = \sqrt{1} = 1$.

2 Use Simpson's rule with $n = 6$ to approximate $\int_0^1 \sqrt{1+x}dx$. Use the error bound formula to bound the error in your approximation.

Solution: If $n = 6$ then $\Delta x = \frac{1-0}{6} = 1/6$ and $x_0 = 0, x_1 = 1/6, x_2 = 2/6 = 1/3, x_3 = 3/6 = 1/2, x_4 = 4/6 = 2/3, x_5 = 5/6, x_6 = 1$.

We compute $S(6) = \frac{1/6}{3}(\sqrt{1} + \sqrt{1 + 1/6} + \sqrt{1 + 1/3} + \sqrt{1 + 1/2} + \sqrt{1 + 2/3} + \sqrt{1 + 5/6} + \sqrt{2})$.

To bound the error, we let $f(x) = (1 + x)^{1/2}$, $f'(x) = \frac{1}{2}(1 + x)^{-1/2}$, $f''(x) = \frac{-1}{4}(1 + x)^{-3/2}$, $\frac{3}{8}(1 + x)^{-5/2}$, and $f^{(4)}(x) = \frac{-15}{16}(1 + x)^{-7/2}$.

Over $[0, 1]$, $|f^4(x)| = \frac{15}{16}(1 + x)^{-7/2} \le \frac{15}{16}$. The error is bounded by $\frac{15/16 \times (1-0)^5}{180 \times 6^4}$.

3 You deposit \$10,000 into a bank account that is compounded continuously at rate 0.02. After this initial deposit, you withdraw money at a constant rate of \$$W$ per year. After 11 years, your account is empty. What was your withdrawal rate?

Solution: We will find the amount in the account $A(t)$ after t years. With continuously compounded interest, $A'(t)$ includes a term $0.02A$. With a continuous withdrawal of W, $A'(t)$ also contains a component of $-W$. Therefore $A'(t) = \frac{dA}{dt} = 0.02A - W$.

Separating variables and integrating yields: $\int \frac{dA}{0.02A - W} = \int dt$ or that $50\ln|0.02A - W| = t + C$. Therefore, $|0.02A - W| = e^{t/50}e^{C/50}$. We remove the absolute values and choose a new arbitrary constant $B = \pm e^{C/50}$: $0.02A - W = Be^{t/50}$. Thus $A = 50(W + Be^{t/50})$.

Given $A(0) = 10000$ we must have $10000 = 50(W + B)$ so $B = 200 - W$.

$A(t) = 50(W + (200 - W)e^{t/50})$. If the account has zero balance at $t = 11$, then $W + (200 - W)e^{11/50} = 0$ so $W = 200e^{11/50}/(e^{11/50} - 1)\$/\text{yr}$.

4 Consider $f(x) = A\sin^2 x$ on $[0, 2\pi]$. For what value of A is $f(x)$ a probability density function for a random variable X on $[0, 2\pi]$? For this value of A, what are the mean and variance of X?

Solution:

• To be a PDF, we need $f(x) \geq 0$ and $\int_0^{2\pi} f(x)dx = 1$.

$\int_0^{2\pi} A\sin^2 xdx = A\int_0^{2\pi} \frac{1-\cos(2x)}{2}dx = A(\frac{x}{2} - \frac{1}{2}\sin(2x))|_0^{2\pi} = A\pi = 1$ so $A = 1/\pi$. Note this value of A ensures $f(x) \geq 0$, too.

• To find the mean, we compute $\bar{X} = \int_0^{2\pi} xp(x)dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos(2x))dx = \frac{1}{2\pi}(\int_0^{2\pi} xdx - \int_0^{2\pi} x\cos(2x)dx)$.

We apply integration by parts on the integrand $x\cos(2x)$ letting $u = x$ with $du = dx$ and $dv = \cos(2x)dx$ and $v = \frac{1}{2}\sin(2x)$. Then $\int x\cos(2x)dx = x\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)dx = \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C$.

The mean is therefore $\frac{1}{2\pi}(\frac{x^2}{2} - \frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x)|_0^{2\pi}) = \pi$.

• To find the variance, we will compute $\mathbb{E}(X^2)$ first.

$\mathbb{E}(X^2) = \int_0^{2\pi} x^2 p(x)dx = \frac{1}{2\pi}(\int_0^{2\pi} x^2dx - \int_0^{2\pi} x^2\cos(2x)dx)$

The integrand $x^2\cos(2x)$ also requires integration by parts steps. Setting $u = x^2$ and $dv = \cos(2x)dx$ so that $du = 2xdx$ and $v = \frac{1}{2}\sin(2x)$ gives $\int x^2\cos(2x)dx = \frac{x^2}{2}\sin(2x) - \int x\sin(2x)dx$.

We integrate by parts once more on $\int x\sin(2x)dx$ with $u = x$ and $dv = \sin(2x)dx$ so that $du = dx$ and $v = \frac{-1}{2}\cos(2x)$. Thus, $\int x\sin(2x)dx = \frac{-x}{2}\cos(2x) + \int \frac{1}{2}\cos(2x)dx = \frac{-x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$.

Finally, $\frac{1}{2\pi}\int_0^{2\pi} x^2(1 - \cos(2x))dx = \frac{1}{2\pi}(\frac{x^3}{3} - \frac{x^2}{2}\sin(2x) - \frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x))|_0^{2\pi} = \frac{4\pi^2}{3} - \frac{1}{2}$.

The variance is $\mathbb{E}(X^2) - \bar{X}^2 = \frac{\pi^2}{3} - \frac{1}{2}$.

5 In a large clinical trial, the resting heart rate of the patients was normally distributed with a mean of 70 bpm and standard deviation of 6 bpm. What is the median heart rate? What is the 75th percentile? What fraction of patients had resting heart rates below 60?

Solution: We will consider the table here for illustration.

• By symmetry, the median will be the mean for a normal distribution.

• We let X be the resting heart rate. For the 75th percentile, we seek s so that $\Pr(X \leq s) = 0.75$ i.e. $\Pr(\frac{X-70}{6} \leq \frac{s-70}{6}) = \Pr(Z \leq \frac{s-70}{6}) = 0.75$.

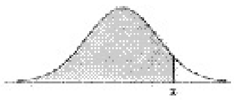
From a $z-$score table, we see $\Pr(Z \leq 0.68) \approx 0.75$ (see green circle). Therefore $\frac{s-70}{6} \approx 0.68$ so the 75th percentile is approximately $s = 74.08$ beats per minute.

• $\Pr(X \leq 60) = \Pr(\frac{X-70}{6} \leq \frac{60-70}{6}) \approx \Pr(Z \leq -1.67)$.

Often tables such as the one included here give $\Pr(Z \leq a)$ for $a > 0$, so let's work with that...

$\Pr(Z \leq -1.67) = \Pr(Z \geq 1.67) = 1 - \Pr(Z \leq 1.67) = 1 - 0.9525 = 0.0475$ (see red circle).

Tables of the Normal Distribution



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990