MATH 105 Practice Problem Set 2 Solutions

1. Parts (a)–(d) are `TRUE` or `FALSE`, **plus explanation**. Give a full-word answer `TRUE` or `FALSE`. If the statement is true, explain why, using concepts and results from class to justify your answer. If the statement is false, give a counterexample.

   (a) [5 marks] The level curves of the plane $ax + by + cz = d$, where $a, b, c, d \neq 0$, are parallel lines in the xy-plane.

   Solution: TRUE. Consider any two distinct level curves of the plane corresponding to $z = z_0$ and $z = z_1$, where $z_0 \neq z_1$. Then, the equation of the level curve corresponding to $z = z_0$ is: $ax + by + cz_0 = d$, or equivalently, $y = \frac{-a}{b}x + \frac{d-cz_0}{b}$. Similarly, the equation of the level curve corresponding to $z = z_1$ is: $y = \frac{-a}{b}x + \frac{d-cz_1}{b}$. They are both lines in the xy-plane being linear equations in x and y. Moreover, they are parallel having the same slope $\frac{-a}{b}$.

   (b) [5 marks] The domain of the function $g(x, y) = \ln((x+1)^2 + (y-2)^2 - 1)$ consists of all points (x, y) lying strictly in the interior of a circle centered at $(-1, 2)$ of radius 1.

   Solution: FALSE. The domain of the function is: $D_g = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + (y-2)^2 > 1\}$, which consists of all points (x, y) lying strictly in the exterior of the circle centered at $(-1, 2)$ of radius 1.

   (c) [5 marks] There exists a function $f(x, y)$ defined on $\mathbb{R}^2$ such that $f_x(x, y) = \cos(3y)$ and $f_y(x, y) = x^4$.

   Solution: FALSE. Suppose that such a function $f(x, y)$ exists. Then, $f_{xy}(x, y) = -3\sin(3y)$, and $f_{yx}(x, y) = 4x^3$, which are both continuous. By Clairaut's theorem, we should have that $f_{xy} = f_{yx}$. However, since $-3\sin(3y) \neq 4x^3$, it follows that such f cannot exist.

   (d) [5 marks] If $f(x, y)$ is a function on $\mathbb{R}^2$ such that $f_x(x, y) = 2x + y$ and $f_y(x, y) = x + 1$, then f is differentiable at every point in $\mathbb{R}^2$.

   Solution: TRUE. Since $f_x(x, y)$ and $f_y(x, y)$ are both continuous on all of $\mathbb{R}^2$ (being polynomials), it follows from the Condition of Differentiability that f is differentiable at every point in $\mathbb{R}^2$.

2. Consider the surface $zx^2 = z^2 - y^2$.

   (a) 10 marks Find the equations and sketch the level curves for $z = -1, 0, 1$ on the same set of axes.

   Solution:

   - For $z = -1$, the equation for the level curve is: $y^2 - x^2 = 1$, which is a hyperbola.
   - For $z = 0$, the equation for the level curve is: $y = 0$, which is the x-axis.
   - For $z = 1$, the equation for the level curve is: $x^2 + y^2 = 1$, which is a circle centered at $(0, 0)$ of radius 1.

   (b) 5 marks Find all values of z which correspond to a level curve containing the point $(x, y) = (2, 0)$.

   Solution: Let z_0 be a value of z which corresponds to a level curve containing the point $(x, y) = (2, 0)$. Then, the equation for the level curve at the height $z = z_0$ is: $z_0 x^2 = z_0^2 - y^2$. If the level curve contains the point $(2, 0)$, then we get $4z_0 = z_0^2$. So, $z_0 = 4$ or $z_0 = 0$.

3. 10 marks Show that $u(x,t) = t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$ is a solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Solution: Compute the first order partial derivatives, we get:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{-1}{2} t^{\frac{-3}{2}} e^{-\frac{x^2}{4t}} + t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \left(\frac{4x^2}{16t^2}\right) \\ &= \frac{-1}{2} t^{\frac{-3}{2}} e^{-\frac{x^2}{4t}} + \frac{1}{4} x^2 t^{-\frac{5}{2}} e^{-\frac{x^2}{4t}} \\ \frac{\partial u}{\partial x} &= t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \left(\frac{-2x}{4t}\right)\end{aligned}$$

Compute $\frac{\partial^2 u}{\partial x^2}$:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \left(\frac{-2x}{4t}\right)^2 + t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \left(\frac{-2}{4t}\right) \\ &= \frac{1}{4} x^2 t^{\frac{-5}{2}} e^{\frac{-x^2}{4t}} - \frac{1}{2} t^{\frac{-3}{2}} e^{\frac{-x^2}{4t}}.\end{aligned}$$

Thus, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

4. Consider the Body Mass Index function being calculated by $b(w,h) = \frac{w}{h^2}$, where w is the weight in kilograms and h is the height in meters.

   (a) [5 marks] Compute b_w and b_h.

   Solution:

   $$b_w = \frac{1}{h^2}$$
   $$b_h = \frac{-2w}{h^3}$$

   (b) [5 marks] For a fixed weight, as the height increases, how does the Body Mass Index change? Explain using the answers in part a.

   Solution: Since we are interested in the change in the Body Mass Index as the height increases and the weight is fixed, we want to look at the partial derivative b_h. For any $h > 0, w > 0$, we have that $b_h < 0$, which means that the Body Mass Index would decreases if we fix the weight and let the heigh increase.

5. Let $f(x, y) = y^3 \sin(4x)$.

   (a) 5 marks Explain in your own words what it means for the function $f(x, y)$ to be differentiable at a point (a, b).

   Solution: A function $f(x, y)$ is differentiable at a point (a, b) if as we move from the point (a, b) to a point $(a + \Delta x, b + \Delta y)$, then the change $f(a + \Delta x, b + \Delta y) - f(a, b)$ can be well approximated by $f_x(a, b)\Delta x + f_y(a, b)\Delta y$.

   (b) 5 marks Show that f is differentiable at every point in $\mathbb{R}^2$.

   Solution: We have that the first order partial derivatives of f are:

   $$f_x(x, y) = 4y^3 \cos(4x) \qquad f_y(x, y) = 3y^2 \sin(4x)$$

   We have that f_x and f_y are continuous at every point in $\mathbb{R}^2$ being products of trigonometric functions and polynomials, both of which are continuous everywhere. It follows from the Condition of Differentiability that $f(x, y)$ is differentiable at every point in $\mathbb{R}^2$.

6. 15 marks Find the maximum and minimum values of the function $f(x,y) = ye^x - e^y$ in the area bounded by the triangle whose vertices are $(4,1)$, $(1,1)$ and $(4,4)$.

Solution: We first find the critical points of f in the given area:

$$f_x(x,y) = ye^x, f_y(x,y) = e^x - e^y$$

If $f_x(x,y) = 0$, then $y = 0$, in which case, $f_y(x,0) = 0$ for $x = 0$. So, the only critical point of f is $(0,0)$, which does not lie in the given area.

Now, we want to find the minimum and maximum values on the boundary. We can describe the boundary of the area using 3 line segments: $x = y$ for $1 \leq x \leq 4$, $x = 4$ for $1 \leq y \leq 4$, and $y = 1$ for $1 \leq x \leq 4$.

On the segment $x = y$ for $1 \leq x \leq 4$, we get $f(x,y) = xe^x - e^x$. So, $f'(x) = e^x + xe^x - e^x = xe^x = 0$ only for $x = 0$. However, $x = 0$ is not in the domain of interest $1 \leq x \leq 4$. So, the maximum and minimum values may only occur at endpoints for $x = 1$ and $x = 4$. If $x = 1$, then $y = x = 1$, and $f(1,1) = 0$. If $x = 4$, then $y = x = 4$, and $f(4,4) = 3e^4$.

On the segment $x = 4$ for $1 \leq y \leq 4$, we get $f(4,y) = ye^4 - e^y$. So, $f'(y) = e^4 - e^y = 0$ only for $y = 4$. So, the minimum and maximum values may occur at $y = 4$ or $y = 1$. Evaluate f at those points, we get $f(4,1) = e^4 - e$, and $f(4,4) = 3e^4$.

On the segment $y = 1$ for $1 \leq x \leq 4$, we get $f(x,1) = e^x - e$. So, $f'(x) = e^x \neq 0$ for any values of x. So, the minimum and maximum values may occur at $x = 4$ or $x = 1$. Evaluate f at those points, we get $f(4,1) = e^4 - e$, and $f(1,1) = 0$.

Comparing the values of f at the points found above, we can conclude that f attains a maximum value of $3e^4$ at $(4,4)$, and a minimum value of 0 at $(1,1)$.

7. 10 marks Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Solution: Using Lagrange Multipliers method, we want to solve the system:

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x,y) &= 0 \end{aligned}$$

Compute the gradient of f and g, we get:

$$\begin{aligned} 5 &= \lambda 2x \\ -3 &= \lambda 2y \\ x^2 + y^2 &= 136. \end{aligned}$$

The first two equations give $x = \frac{5}{2\lambda}$ and $y = \frac{-3}{2\lambda}$. Substitute into the third equation, we get:

$$\begin{aligned} \frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} &= 136 \\ 34 &= 544\lambda^2 \\ \lambda &= \pm\frac{1}{4}. \end{aligned}$$

For $\lambda = \frac{1}{4}$, we get the point $x = 10$, $y = -6$, and $f(10,-6) = 68$. For $\lambda = \frac{-1}{4}$, we get the point $x = -10$, $y = 6$, and $f(-10,6) = -68$. Thus, the function f attains a maximum value of 68 at the point $(10,-6)$ and a minimum value of -68 at the point $(-10,6)$ with respect to the constraint $x^2 + y^2 = 136$.

8. Let $f(x) = 2x + 1$.

(a) 5 marks Write down the left Riemann sum for $\int_1^5 f(x)\,dx$.

Solution: For a regular partition into n sub-intervals, we have: $\Delta x = \frac{4}{n}$, and $\overline{x_k} = 1 + (k-1)\frac{4}{n}$. So, the left Riemann sum for $\int_1^5 f(x)\,dx$ is:

$$\sum_{k=1}^{n} \Delta x f(\overline{x_k}) = \sum_{k=1}^{n} \frac{4}{n}(2\overline{x_k} + 1) = \sum_{k=1}^{n} \frac{4}{n}\left(3 + (k-1)\frac{8}{n}\right)$$

(b) 10 marks Compute the limit as $n \to \infty$ of the Riemann-sum expression found in part (a) and thus evaluate $\int_1^5 f(x)\,dx$.

Solution: Simplify the Riemann sum expression found in part (a), we get:

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{12}{n} + \frac{32}{n^2}(k-1)\right) &= \sum_{k=1}^{n}\frac{12}{n} + \sum_{k=1}^{n}\frac{32}{n^2}(k-1)\\ &= n\left(\frac{12}{n}\right) + \frac{32}{n^2}\left(\sum_{k=1}^{n-1} k\right)\\ &= 12 + \frac{32}{n^2}\left(\frac{n(n-1)}{2}\right) = 12 + 16 - \frac{16}{n} = 28 - \frac{16}{n}.\end{aligned}$$

Taking limit as $n \to \infty$, we get:

$$\lim_{n\to\infty}(28 - \frac{16}{n}) = 28.$$

Thus, $\int_1^5 f(x)\,dx = 28$.